

CSE 421 LECTURE 1

Theorem 1. *Gale-Shapley algorithm outputs a stable matching.*

Proof.

Termination:

- Let S be the set of proposed (m, w) pair.
 - Each step, there is a man m proposed to some woman w .
 - Furthermore, m never proposed to w before.
- Hence, the size of S is increased exactly by 1.
- Since $|S| \leq n^2$, there are at most n^2 steps.

Correctness: Before the proof, recalled that I observed the following in the lecture

(Obs 1) man proposed in decreasing order of preference.

(Obs 2) woman's partner get better and better and unmatched once matched.

- Perfectness
 - Terminates because there is no (free man hasn't proposed to every woman)
 - Case 1: No free man
 - * Every man is matched.
 - * If some woman is not matched,
 - Pigeonhole principle said some woman is matched to two man. (Contradiction)
 - * Hence, every woman is matched.
 - Case 2: There is free man, but he proposed to every woman
 - * Every woman has been proposed.
 - * (Obs 2) shows every woman is matched.
 - * Same argument as before, every man is matched.
 - In both cases, it is a perfect matching.
- Stability
 - Consider any unmatched pair (m, w)
 - Case 1: m never proposed to w .
 - * (Obs 1) shows m prefer his current partner than w .
 - Case 2: m proposed to w .
 - * (Obs 2) shows w prefer her current partner than m
 - In both cases, (m, w) is stable.

□