Lecture 4

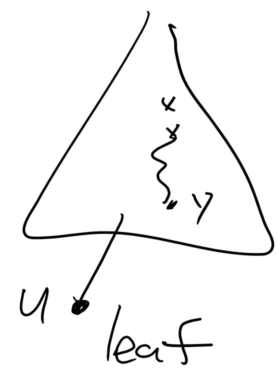
Wednesday, October 3, 2018 1:24 PM

If G is tree, . . . - - -

Lem If G has no cycle, $\exists v$ st $\deg(v) \le 1$ ✓

Thm Every tree with n vertices has exactly $n-1$ edges.

Proof



IH: Every tree with n vetices has exactly $n-1$ edges.

Base case $n=1$: single vertex. 0 edge

IS: Given a tree with n vertices.

Goal: show it has exactly $n-1$ edges.

By prev lemma, there is vertex u with $\deg u = 1$

Let $T' = T - \{u\}$.

Claim T' is tree.

- T' has no cycle

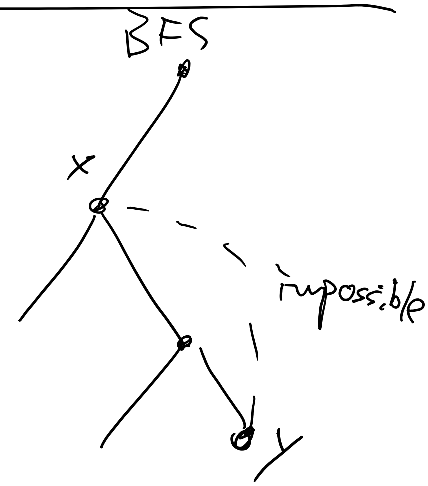
- T' is connected

Use IH, T' has $n-2$ edges
(because T' has $n-1$ vertices.)
So, T has $n-1$ edges.

Recall what induction:
Say we want to prove $P(n)$. $\forall n \in \{1, 2, \ldots, \}$
Need to prove $P(1)$ first. (Base case)
It suffices to show $P(1), P(2) \ldots, P(k)$ is true
$\Rightarrow P(k+1)$ is true

$P(k+1) \Rightarrow P(k)$ is not enough in general

<u>Lemma</u> All non-tree edge connect vertices at the same or adjacent levels of the tree



Proof: Fix any edge $\{x, y\}$
Say x is discovered first.

$$L(x) \le L(y)$$

Case 1. y is discovered by x

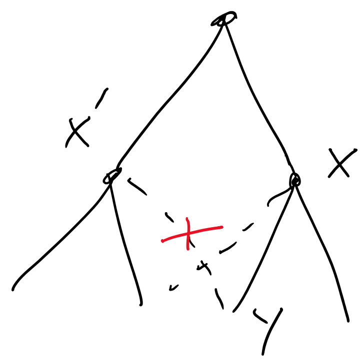
$$L(y) = L(x) + 1$$

$$L(y) = L(x) + 1$$

Case 2. y is discovered by x' (before x)

$$L(y) = L(x') + 1 \le L(x) + 1$$

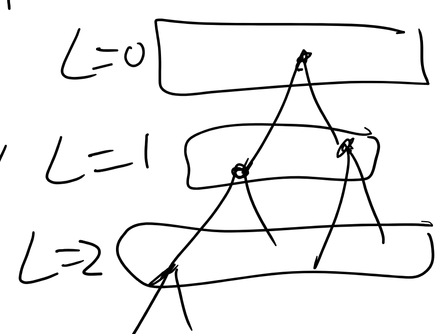
Hence $\quad L(x) \le L(y) \le L(x) + 1$



<u>Theorem</u> $L(v) = i$ iff shortest path distance between s and v is i (for BFS(s))

Proof:

Let $d(s,v)$ be the shortest path distance between s and v



Goal: $\quad d(s,v) = L(v) \quad \forall v$

Claim 1: $\quad d(s,v) \le L(v)$

(if level of v is i, distance $\le i$)

level $i \Rightarrow \exists$ path on BFS tree with i edges from s to v.

Claim 2: $L(v) \le d(s,v)$

Suppose $d(s,v) = i$

$s = v_0 \quad v_1 \quad v_2 \quad \cdots \quad v_i = v$

$L(s) = 0$ Want $L(v_i) \le i$

Recall every edge connects only to same or adjacent layer

$$L(v_0) = 0$$

$$\begin{cases} L(v_1) \le L(v_0) + 1 \\ L(v_2) \le L(v_1) + 1 \\ \vdots \\ L(v_i) \le L(v_{i-1}) + 1 \end{cases}$$

$$\Rightarrow L(v_i) \le i$$