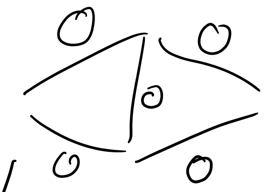
Lecture 21

Friday, November 16, 2018 11:28 AM

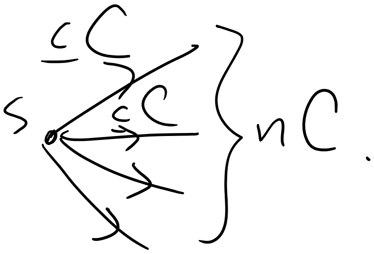
Lemma If the capacities are integral st $c(e) \le C$
Then algo runs in $O(mnC)$

Proof: flow out of source
increase every iteration by at least 1.



max flow $\le nC$.

So, there are at most nC iter.

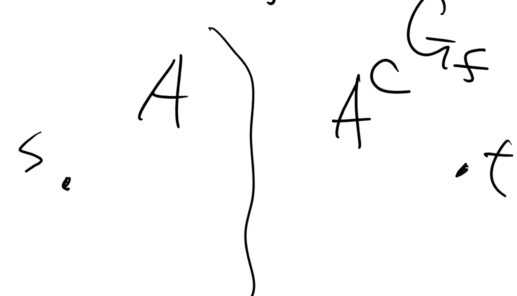


Theorem The algo is correct.

Proof: The output f is s-t flow.

There is no augment path for f.
(it means no path from s to t in G_f)

Let A be the connected component containing s.

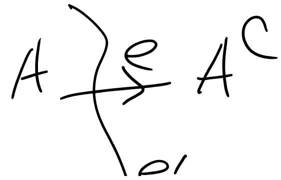


(A, A^c) is s-t cut because
$s \in A$ by def $t \notin A$ because no augmenting path.

$$v(f) = \underbrace{\sum_{e \text{ out of } A} f(e)}_{c(e)} - \underbrace{\sum_{e \text{ into } A} f(e)}_{0}$$

G

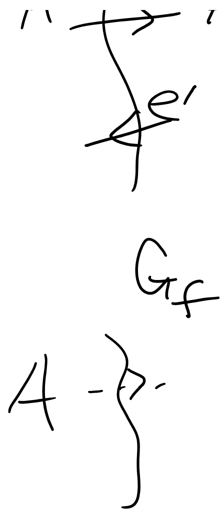
- $f(e) = c(e) \quad \forall e$ out of A



- $f(e) = c(e) \quad \forall\ e$ out of A
- $f(e') = 0 \quad \forall\ e'$ into A

$$= \sum_{e \text{ out of } A} c(e)$$

$$= \text{cap}(A, A^c)$$

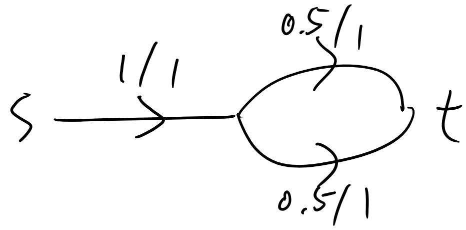


Then f is max s-t flow

(A, A^c) is min s-t cut

Corollary: Value of max s-t flow = Value of min s-t cut

Furthermore: if capacities are integral, there is an integral max-flow.

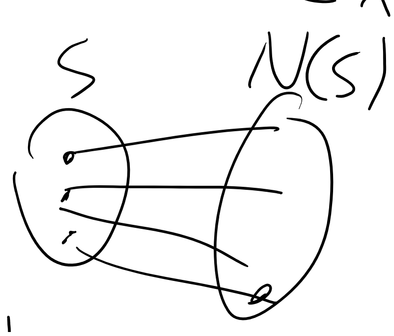


Marriage theorem

Given a bipartite graph $G = (X \cup Y, E)$ with $|X| = |Y|$.

Then G has a perfect matching iff $|N(S)| \geq |S|$ $\forall S \subset X$.

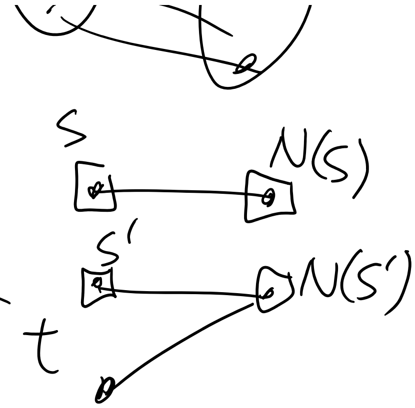
Proof "$\Rightarrow$"

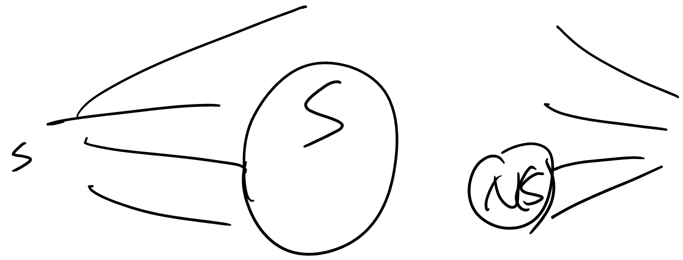


"$\Leftarrow$" Assume no perfect.

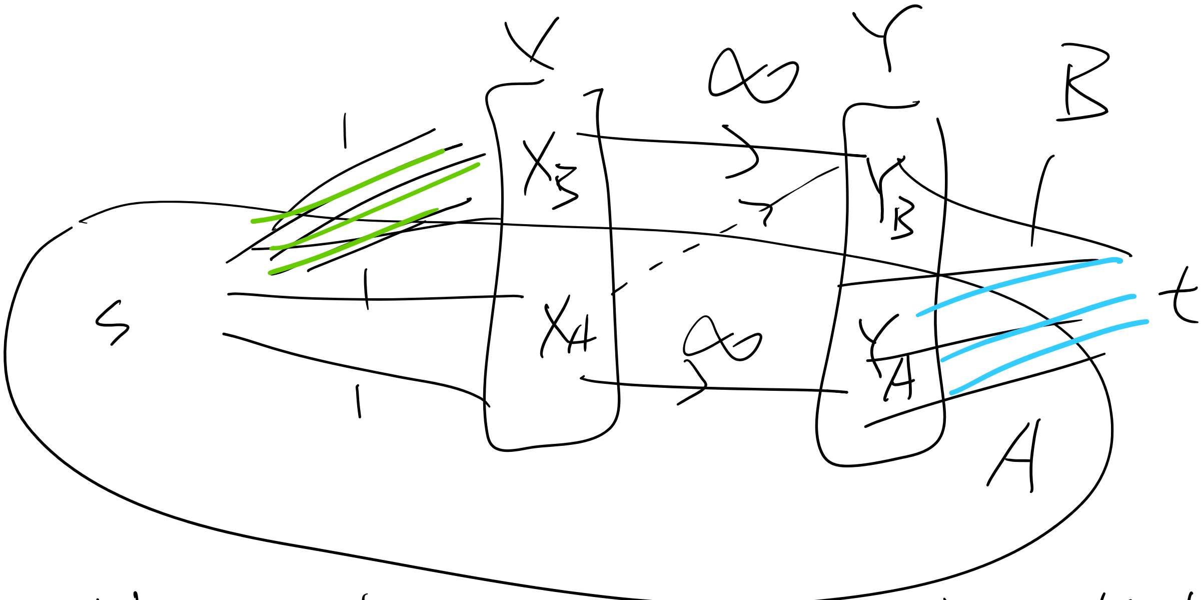
"⇐" Assume no perfect.

Then find set S st $|N(S)| < |S|$





Let (A,B) be min s-t cut for



No perfect matching $\Rightarrow v(f^*) < |X|$

$$= \mathrm{cap}(A,B) = \underbrace{|X_B|} + \underbrace{|Y_A|}$$

$$|N(X_A)| = |Y_A| = \mathrm{cap}(A,B) - |X_B|$$

$$< |X| - |X_B| = |X_A|$$