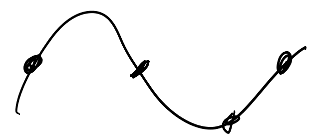
Fix $\boxed{y_i}$

$$\boxed{\begin{aligned} A(x) &= a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1} \\ B(x) &= b_0 + b_1 x + \cdots + b_{\frac{n}{2}-1} x^{\frac{n}{2}-1} \end{aligned}}$$

naive : $O(n^2)$ $\Big\downarrow$ evaluate

$$\boxed{(A(y_i)) \quad (B(y_i))} \xrightarrow[C(x) = A(x)\,B(x)]{O(n) \text{ time}} \boxed{C(y_i) = A(y_i)B(y_i)}$$

$\deg(A) \le \frac{n}{2}-1$ $\quad$ $\deg(B) \le \frac{n}{2}-1$



$\Big\uparrow$ interpolation
naive : n^2

$$\boxed{C(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}}$$

Vandermonde matrix, evaluation, interpolation

To evaluate on $\{y_i\}_{i=0}^{n-1}$ for $(n-1)$-deg poly $C(x)$

$$\underbrace{\begin{pmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^{n-1} \\ 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ & & \vdots & & \\ 1 & y_{n-1} & y_{n-1}^2 & \cdots & y_{n-1}^{n-1} \end{pmatrix}}_{\text{Vandermonde matrix}} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} C(y_0) \\ \vdots \\ C(y_{n-1}) \end{pmatrix}$$

evaluate Vc
interpolation $V^{-1}C$

$$\boxed{A(x), B(x)} \qquad\qquad \boxed{C(x)}$$

$\boxed{n \log n}$

$\Big\downarrow$ VA, VB $\qquad\qquad$ $\Big\uparrow$ $V^{-1}[VA \odot VB]$

$$\boxed{\text{values of } A, B \text{ on } \{y_i\}} \longrightarrow \boxed{(VA) \odot (VB)}$$

$\boxed{\text{values of } AB \text{ on } \{y_i\}} \longrightarrow \boxed{(VA)\odot(VB)}$

Lemma $\det V = \prod_{0 \le i \le j \le n-1} (y_i - y_j)$.

Cor V is invertible. Interpolation always exists and unique if y_i are distinct.

Fact One can compute Vc and $V^{-1}c$ in $O(n \log n)$ time. for any distinct y_i

---

Remember we can choose y_i

Key observation

$$C(y) = c_0 + c_1 y + c_2 y^2 + \cdots + c_{n-1} y^{n-1}$$

$$= c_0 + c_2 y^2 + \cdots + c_{n-2} y^{n-2} \leftarrow \text{"deg } \tfrac{n-1}{2}\text{"}$$

$$+ y(c_1 + c_3 y^2 + \cdots + c_{n-1} y^{n-2})$$

$$= C_{even}(y^2) + y\, C_{odd}(y^2)$$

$$C(-y) = C_{even}(y^2) - y\, C_{odd}(y^2)$$

$C_{even}(y) = c_0 + c_2 y + \cdots + c_{n-2} y^{\frac{n-2}{2}}$

Let $T(y_0, \ldots, y_{n-1})$ is the cost evaluating deg $n-1$ poly on points $(y_0, \ldots, y_{n-1})$

$$T(\pm y_0, \ldots, \pm y_{\frac{n}{2}-1}) = \underbrace{2}_{\times 2 \text{ \#problems}} T\underbrace{(y_0^2, \ldots, y_{\frac{n}{2}-1}^2)}_{\div 2 \text{ size}} + O(n)$$

÷2 size

$C(x) = 1 + 4x + 2x^2 + 3x^3$

$C_{even}(x) = 1 + 2x$

$C_{odd}(x) = 4 + 3x$

$C(x) = C_{even}(x^2) + x\,C_{odd}(x^2)$

Complex.

$$\begin{array}{ccc} -y_0^2 & \cdots & -y_{\frac{n}{4}-1}^2 \\ \| & & \| \\ y_{\frac{n}{4}}^2 & \cdots & y_{\frac{n}{2}-1}^2 \end{array}$$

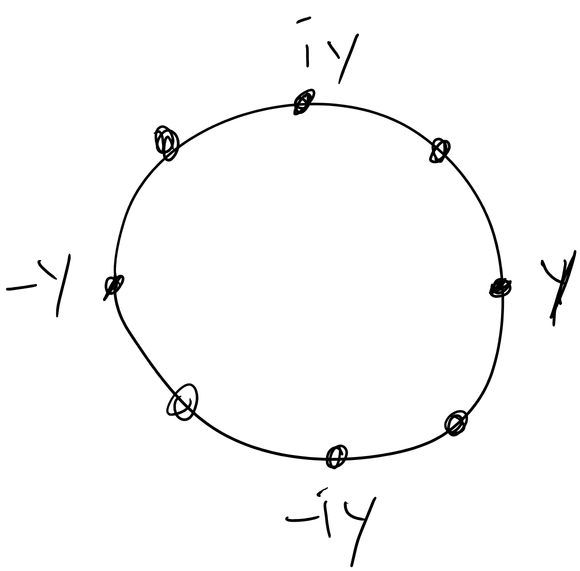
$y_{\frac{n}{4}+j} = i \cdot y_j \qquad j \in \{0, \dots, \frac{n}{4}-1\}$

$$T\left(\begin{array}{l} y_0, \dots, y_{\frac{n}{4}-1}, \\ -y_0, \dots, -y_{\frac{n}{4}-1}, \\ iy_0, \dots, iy_{\frac{n}{4}-1} \\ -iy_0, \dots, -iy_{\frac{n}{4}-1} \end{array}\right) = 2T\left(y_0^2, \dots, y_{\frac{n}{4}-1}^2, (iy_0)^2, \dots, (iy_{\frac{n}{4}-1})^2\right) + O(n)$$

$$= 2T(\pm y_0^2, \dots, \pm y_{\frac{n}{4}-1}^2) + O(n)$$

$$= 4T(y_0^4, \dots, y_{\frac{n}{4}-1}^4) + O(n)$$

$\vdots$



$T(\{e^{\frac{2\pi i}{n}}\}) = O(n \log n)$

$V^{-1}_{\{e^{\frac{2\pi i}{n}}\}} = \square \cdot V^{T}_{\{e^{\frac{2\pi i}{n}}\}}$