Cut Property

Let \( e \) be the min cost edge in a cut \( (S, V-S) \).
Let \( T^* \) be any MST. Then \( e \in T^* \).

Wrong Proof: (by contradiction)

Suppose \( e \in T^* \).
\[ \exists \ F \in T^* \ st \ F \notin (S, V-S) . \]
By def \( e \), \( C_F < C_e \).
So, \( T' = T^* - \{F\} + \{e\} \) has a lower cost.

Proof: (by contradiction)

Suppose \( e \in T^* \). Add \( e \) into \( T^* \).
Get a cycle \( C \) which cross \( (S, V-S) \).

Let \( g \in C \) s.t. \( g \neq e \).
\[ g \in (S, V-S) \]

Let \( T = T^* - \{g\} + \{e\} \).
Since \( C_e < C_g \), so, \( T \) has smaller cost.
\( T \) is spanning tree.
$T$ is spanning tree
- $T$ covers $n$ vertices. √
- $T$ has $n-1$ edge √

$T$ is connected

Since both ends of $e$ is covered by rest of $G$.

Cycle property
Let $f$ be max cost edge in cycle $C$.
Then $f \not\in T^*$ for any MST $T^*$.

Proof by contradiction.
Suppose $f \in T^*$.
Consider removing $f$ from $T^*$
It creates 2 connected components
($S$ is the connected components containing $u$).
Let $T = T^* - e + f$
Again $T$ is spanning tree.
Since $c_e < c_f$, so cost $T < cost T^*$ (contradict).
$T$ is a tree $\Rightarrow |T| = n-1$

$T$ is connected

$T$ covers every vertex.