CSE 421

Linear Programs

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Linear System of Equations

In high school we learn Gaussian elimination algorithm to solve a system of linear equations

$$\begin{aligned} x_1 + x_3 &= 7 \\ 2x_2 + x_1 &= 5 \\ 3x_1 + 7x_2 - x_3 &= 1 \end{aligned}$$

We set $x_3 = 7 - x_1$ and we substitute in the following equations.

Then we substitute $x_2 = \frac{5-x_1}{2}$ in to the third equations.

The third equational uniquely defines x_1

Linear Programming

Optimize a linear function subject to linear inequalities

$$\begin{array}{ll} \max & 3x_1 + 4x_3 \\ s.t., & x_1 + x_2 \leq 5 \\ & x_3 - x_1 = 4 \\ & x_3 - x_2 \geq -5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

- We can have inequalities,
- We can have a linear objective functions

Applications of Linear Programming

Generalizes: Ax=b, 2-person zero-sum games, shortest path, max-flow, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- We can solve linear programming in polynomial time.
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: CPLEX, Gorubi, ….
- CPLEX can solve LPs with millions of variables/constraints in seconds

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alories and (h)appiness per pound:

	veggies	**meat**	**fruits**	**dairy**
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat an 0.2lb of fruits we will be $0.5\ h_m + 0.2\ h_f$ happy

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

- You should eat 1500 calaroies to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned} \max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\ s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \le 20 \\ & x_v c_v + x_m c_m + x_f c_f + x_d c_d \le 1500 \\ & x_v, x_m, x_f, x_d \ge 0 \end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

How to Design an LP?

- Define the set of variables
- Put constraints on your variables,
  - should they be nonnegative?
- Write down the constraints
  - If a constraint is not linear try to approximate it with a linear constraint
- Write down the objective function
  - If it is not linear approximation with a linear function
- Decide if it is a minimize/maximization problem

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$\begin{aligned}
\max \quad & \sum\nolimits_{e \text{ out of } s} x_e \\
s.t. \quad & \sum\nolimits_{e\ out\ of\ v} x_e = \sum\nolimits_{e\ in\ to\ v} x_e && \forall v \neq s,t \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}$$

Example 3: Min Cost Max Flow

Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$
for each gallon of water that we send through e.

Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathrm{E}} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e && \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}
$$

Example 4: Metabolic Network

Let v_i are the rate of different chemical reaction in your body.

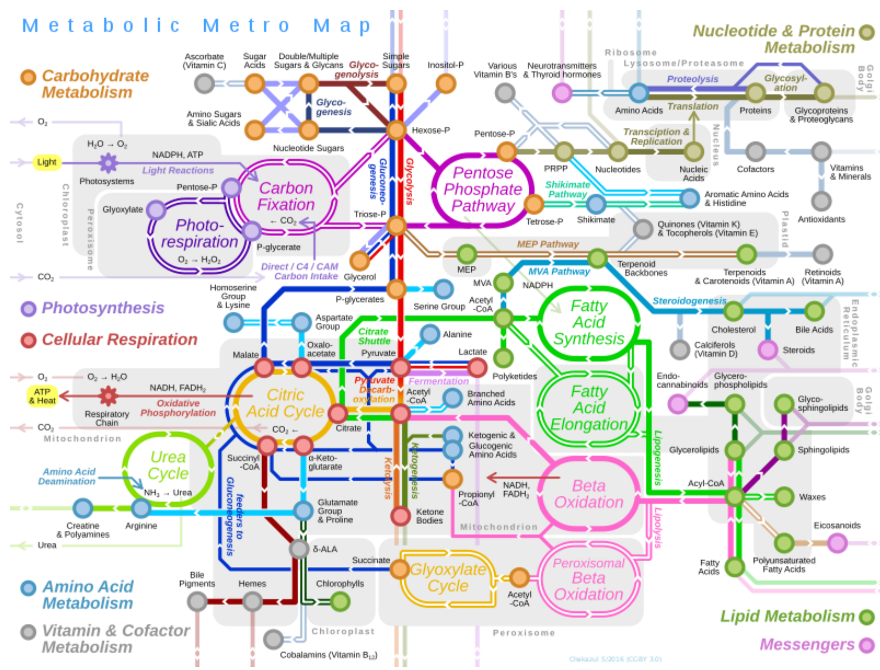
It satisfies mass conversation (translate to linear inequality).

It satisfies some upper and lower bound.

Optimizing certain function in your body is corresponding to solving a linear program!

How you find that LP? DNA!

https://vmh.uni.lu/#reconmap



Disclaimer: I suspect your biology is better than mine.

Summary (Linear Programming)

- Linear programming is one of the biggest advances in 20th century

- It is being used in many areas of science: Mechanics, Physics, Operations Research, and in CS: AI, Machine Learning, Theory, …

- Almost all problems that we talked can be solved with LPs, Why not use LPs?
  - In some sense, current fastest algorithm for maxflow is based on LP!
  - Maybe one day, I need to rewrite CSE 421.
  - ☹ But I need to able to teach the current 421 well first.

- There is rich theory of LP-duality which generalizes max-flow min-cut theorem