CSE 421

NP-Completeness

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Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP$, $A \leq_p 3$-SAT.
(See CSE 431 for the proof)

- So, 3-SAT is the hardest problem in NP.

What does this say about other problems of interest? Like Independent set, Vertex Cover, …

Fact: If $A \leq_p B$ and $B \leq_p C$ then, $A \leq_p C$

Pf: Just compose the reductions from A to B and B to C

So, if we prove $3$-SAT $\leq_p$ Independent set, then Independent Set, Clique, Vertex cover, Set cover are all NP-complete

$3$-SAT $\leq_p$ Independent Set $\leq_p$ Vertex Cover $\leq_p$ Set Cover
Steps to Proving
Problem B is NP-complete

• Show B is NP-hard:
  State: ”Reduction is from NP-hard Problem A”
  Show what the map f is
  Argue that f is polynomial time
  Argue correctness: two directions Yes for A implies Yes for B and vice versa.

• Show B is in NP
  State what hint/certificate is and why it works
  Argue that it is polynomial-time to check.
Is NP-complete as bad as it gets?

• NO! **NP**-complete problems are frequently encountered, but there are worse:

  Some problems provably require exponential time.

  • Ex: Does $M$ halt on input $x$ in $2^{|x|}$ steps?

Some require $2^n$, $2^{2^n}$, $2^{2^{2^n}}$, ... steps

And some are just plain uncomputable
3-SAT $\leq_p$ Independent Set

Map a 3-CNF to $(G,k)$. Say $k$ is number of clauses
- Create a vertex for each literal
- Joint two literals if
  - They belong to the same clause (blue edges)
  - The literals are negations, e.g., $x_i, \overline{x_i}$ (red edges)
- Set $k$ be the # of clauses.

\[
\begin{align*}
(x_1 \lor \overline{x_3} \lor x_4) \land (x_2 \lor \overline{x_4} \lor x_3) \land (x_2 \lor \overline{x_1} \lor x_3)
\end{align*}
\]

Polynomial-Time Reduction
Correctness of 3-SAT $\leq_p$ Indep Set

F satisfiable $\Rightarrow$ An independent of size $k$
Given a satisfying assignment, Choose one node from each clause where the literal is satisfied

$$(x_1 \lor \overline{x_3} \lor x_4) \land (x_2 \lor \overline{x_4} \lor x_3) \land (x_2 \lor \overline{x_1} \lor x_3)$$

Satisfying assignment: $x_1 = T, x_2 = F, x_3 = T, x_4 = F$

• S has exactly one node per clause $\Rightarrow$ No blue edges between S
• S follows a truth-assignment $\Rightarrow$ No red edges between S
• S has one node per clause $\Rightarrow$ $|S|=k$
Correctness of $3$-SAT $\leq_p$ Indep Set

An independent set of size $k \Rightarrow$ A satisfying assignment
Given an independent set $S$ of size $k$.
$S$ has exactly one vertex per clause (because of blue edges)
$S$ does not have $x_i, \overline{x_i}$ (because of red edges)
So, $S$ gives a satisfying assignment

Satisfying assignment: $x_1 = F, x_2 = ?, x_3 = T, x_4 = T$
$\overline{x_1} \lor \overline{x_2} \lor \overline{x_3} \lor \overline{x_4} \land (x_2 \lor \overline{x_4} \lor x_3) \land (x_2 \lor \overline{x_1} \lor x_3)$
Yet another example of NP completeness

Prove that Super Mario Bros is NP-complete.

What do we need to show?
• The problem is in NP.
• Some NP complete problem is easier than Super Mario.

Approach:
• $3\text{SAT} \leq_P \text{Super Mario}$
Yet another example of NP completeness

Given a 3SAT, we need to create a level.

We ignore the following issues:
• Need to consider the “crossing” coz the level is 2-D.
• Assume Mario can go both left or right.
Question 1: How to create this part?
Question 2: How to create this part?

Figure 11: Clause gadget for Super Mario Bros.
So, what you need to prove?

- If the 3SAT is satisfiable, then indeed the level is solvable. Usually, this part is easy. This is basically due to the design of your reduction.

- If the level is solvable, then the 3SAT is satisfiable. This part usually requires more argument. Need to prove no tricky way to solve the problem without solving the 3SAT.
More NP-completeness

• Subset-Sum problem
  (Decision version of Knapsack)
  • Given $n$ integers $w_1, \ldots, w_n$ and integer $W$
  • Is there a subset of the $n$ input integers that adds up to exactly $W$?

• $O(nW)$ solution from dynamic programming but if $W$ and each $w_i$ can be $n$ bits long then this is exponential time
3-SAT $\leq_p$ Subset-Sum

- Given a 3-CNF formula with $m$ clauses and $n$ variables
- Will create $2m + 2n$ numbers that are $m + n$ digits long
  - Two numbers for each variable $x_i$
    - $t_i$ and $f_i$ (corresponding to $x_i$ being true or $x_i$ being false)
  - Two extra numbers for each clause
    - $u_j$ and $v_j$ (filler variables to handle number of false literals in clause $C_j$)
### 3-SAT $\leq_p \text{Subset-Sum}$

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<th>$i$</th>
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<td>$u_2 = v_2$</td>
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</tbody>
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$C_3 = (x_1 \lor \neg x_2 \lor x_5)$

$W$ | 1 | 1 | 1 | 1 | ... | 1 | 3 | 3 | 3 | 3 | ... | 3 |
Graph Colorability

- **Defn:** Given a graph $G = (V, E)$, and an integer $k$, a $k$-coloring of $G$ is
  an assignment of up to $k$ different colors to the vertices of $G$ so that the endpoints of each edge have different colors.

- **3-Color:** Given a graph $G = (V, E)$, does $G$ have a 3-coloring?

- **Claim:** 3-Color is NP-complete

- **Proof:** 3-Color is in NP:
  Hint is an assignment of red, green, blue to the vertices of $G$
  Easy to check that each edge is colored correctly
3-SAT $\leq_p$ 3-Color

• Reduction:
  We want to map a 3-CNF formula $F$ to a graph $G$ so that
  • $G$ is 3-colorable iff $F$ is satisfiable
3-SAT $\leq_p$ 3-Color

Base Triangle
3-SAT $\leq_p$ 3-Color

Variable Part:
in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)
3-SAT $\leq_p$ 3-Color

Clause Part:
Add one 6 vertex gadget per clause connecting its ‘outer vertices’ to the literals in the clause.
Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph.
3-SAT $\leq_p$ 3-Color

Any 3-coloring of the graph colors each gadget triangle using each color.
Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget
Any 3-coloring of the graph has T at the other end of the blue edge connected to the F.