CSE 421

Approximation Algorithms:
Set Cover

(substituting for Yin Tat Lee)

Approximation Algorithms

How to deal with NP-complete problems

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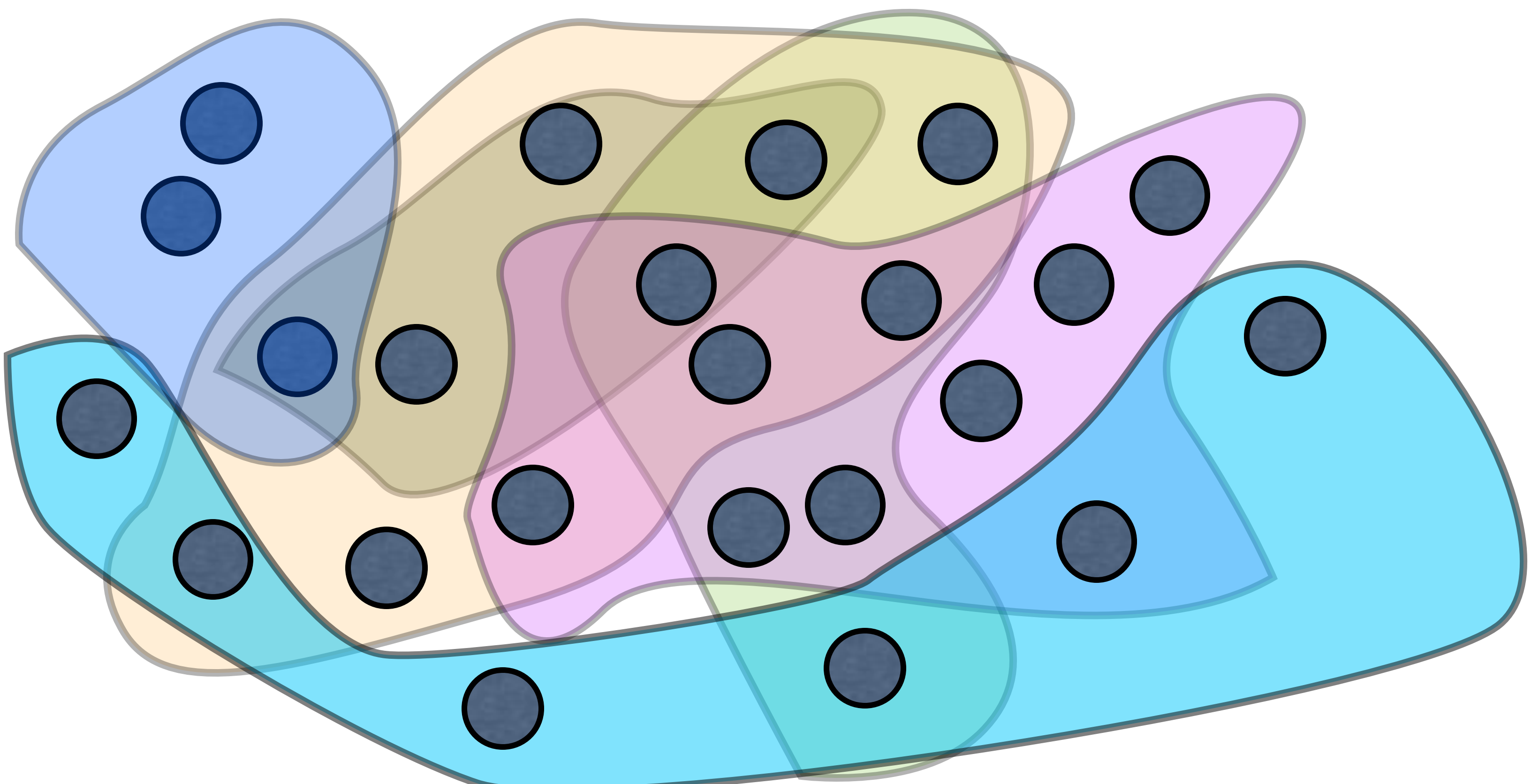
Getting a guarantee on the approximation ratio is inherently tricky! Many methods to do so: we'll see one example today.

The Set Cover Problem

Problem Statement: Given a set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, find the smallest collection of these sets whose union is U.



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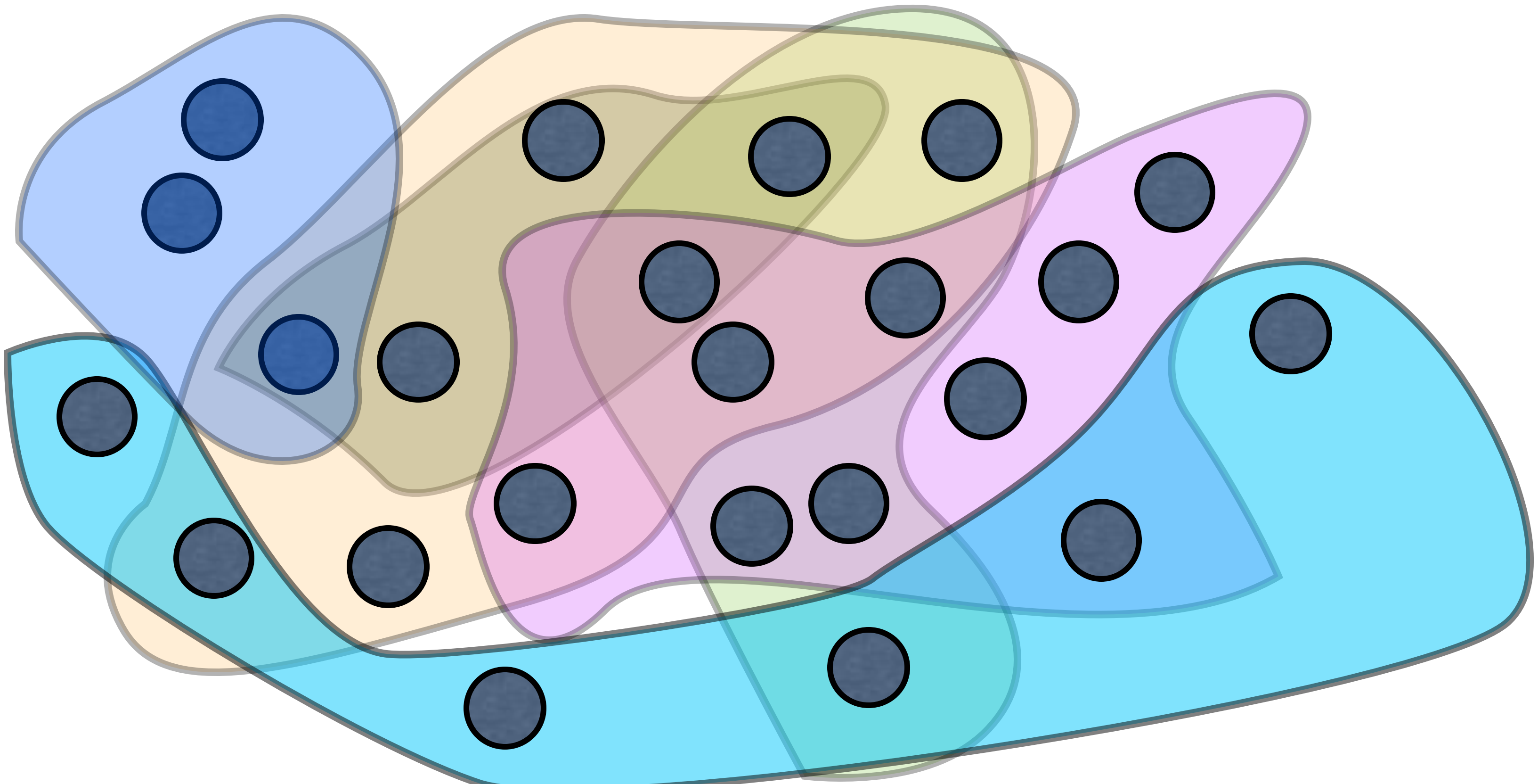

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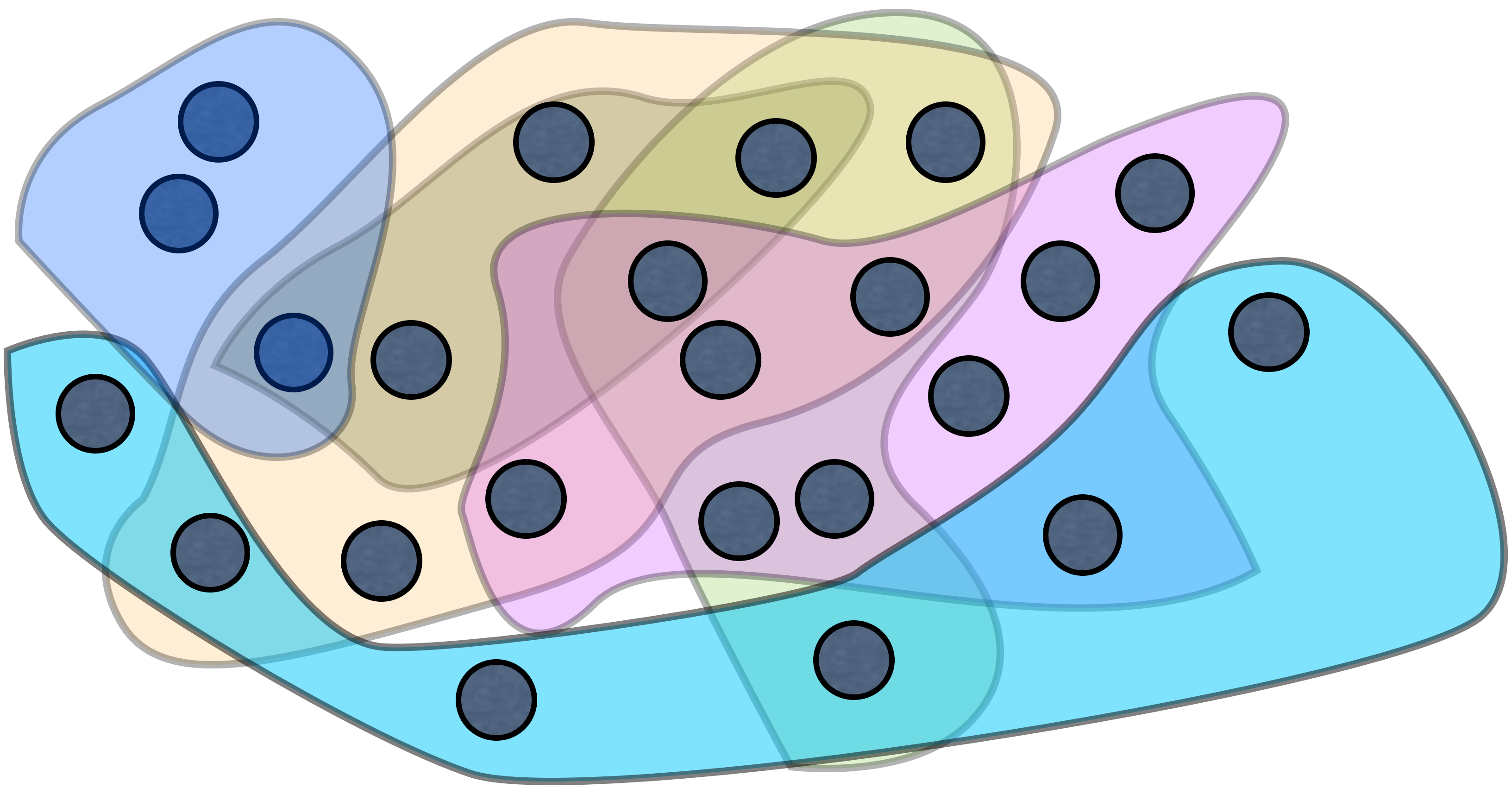
Applications: Company wants to hire people such that all required skills covered;



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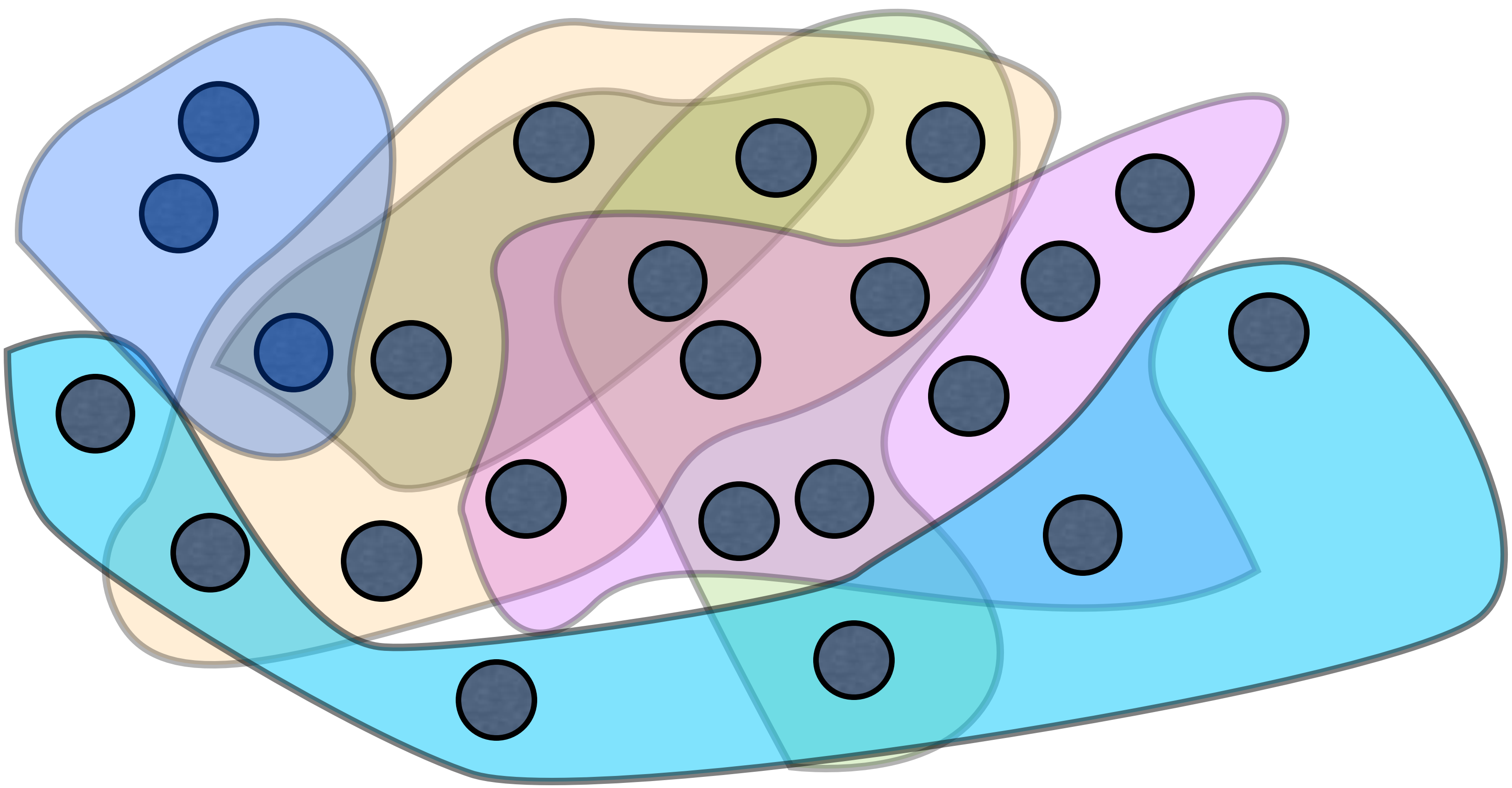

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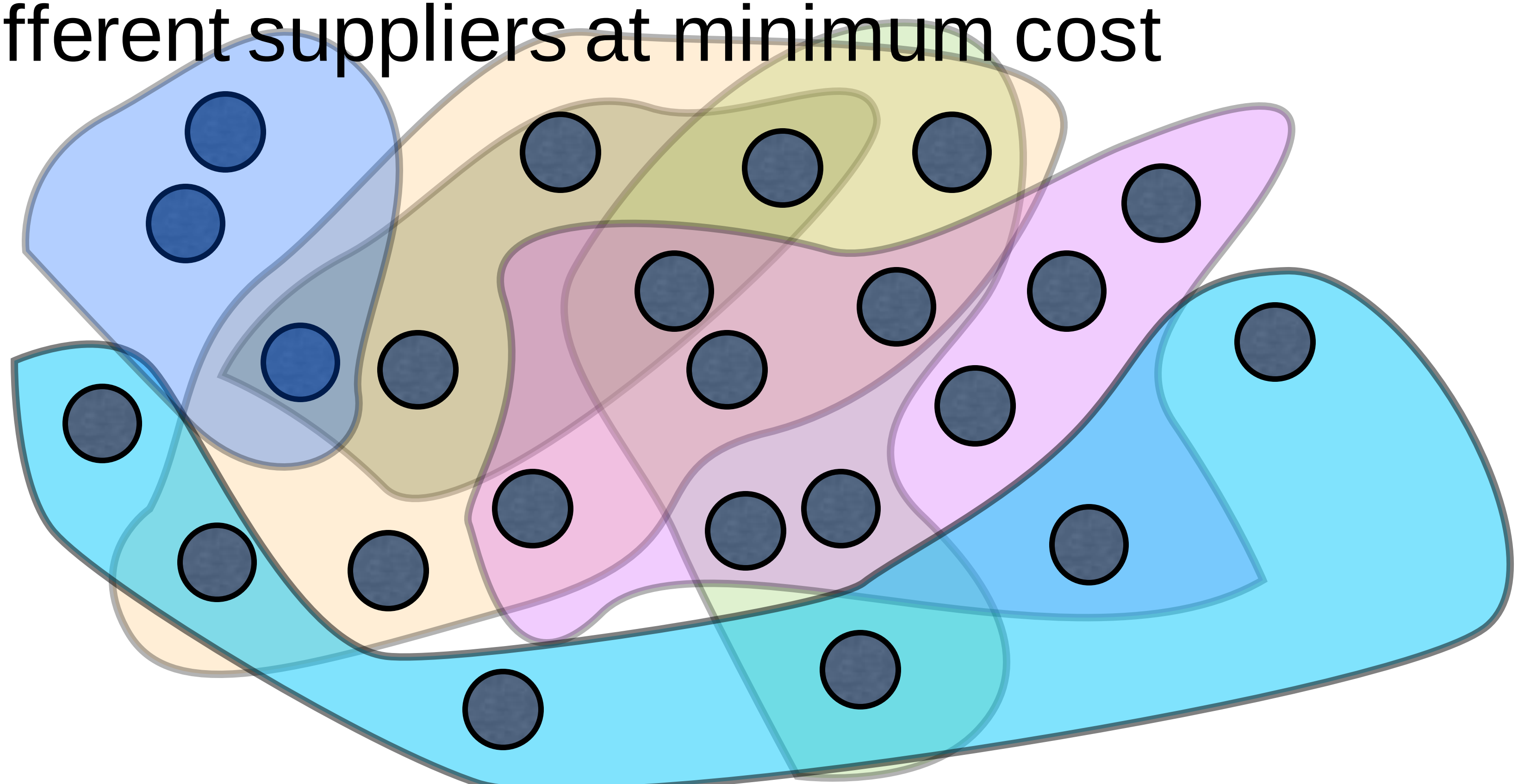
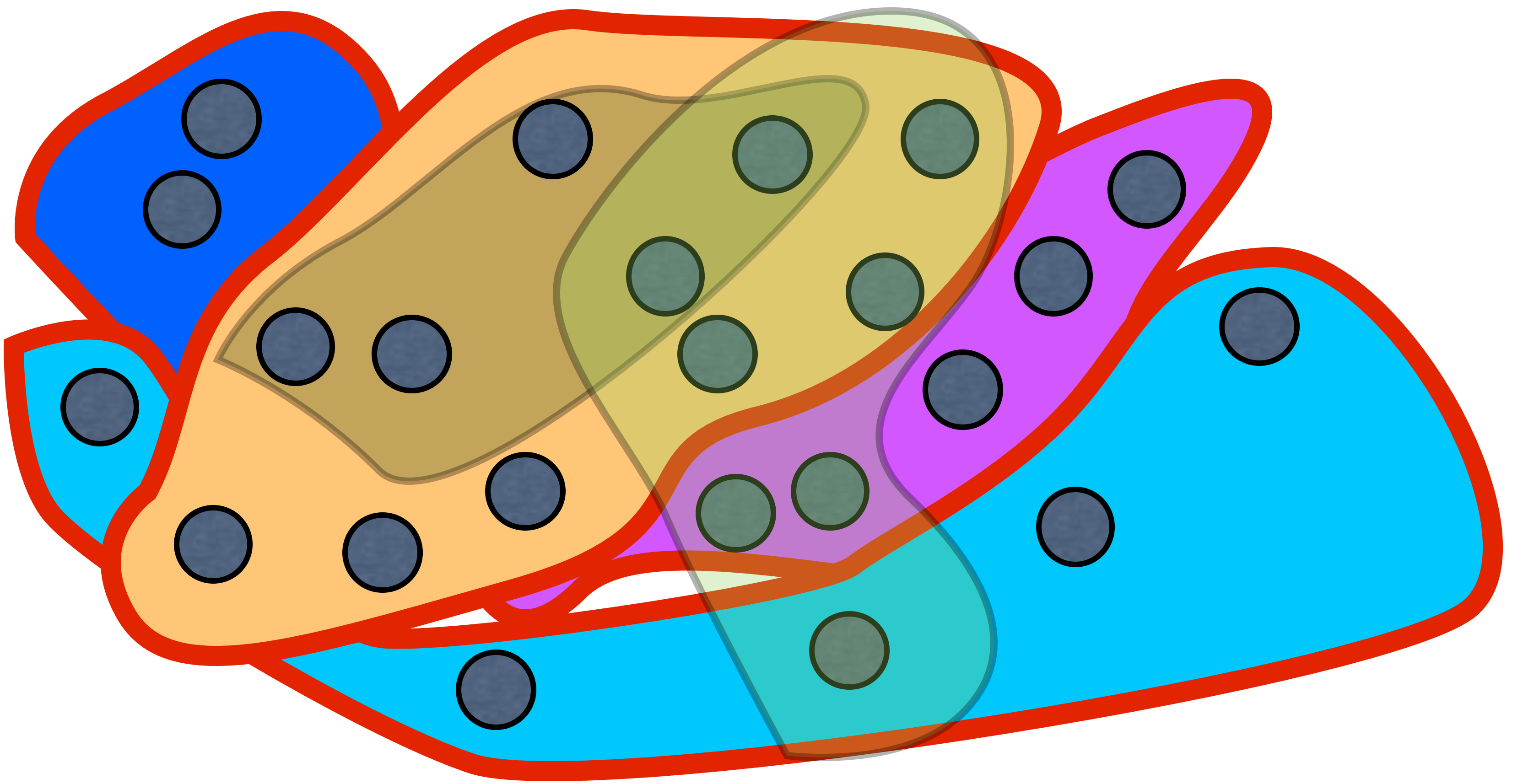
Applications : Company wants to hire people such that all required skills covered; “Fuzz” testing in software development; Manufacturer wants to get all items from different suppliers at minimum cost



The Set Cover Problem

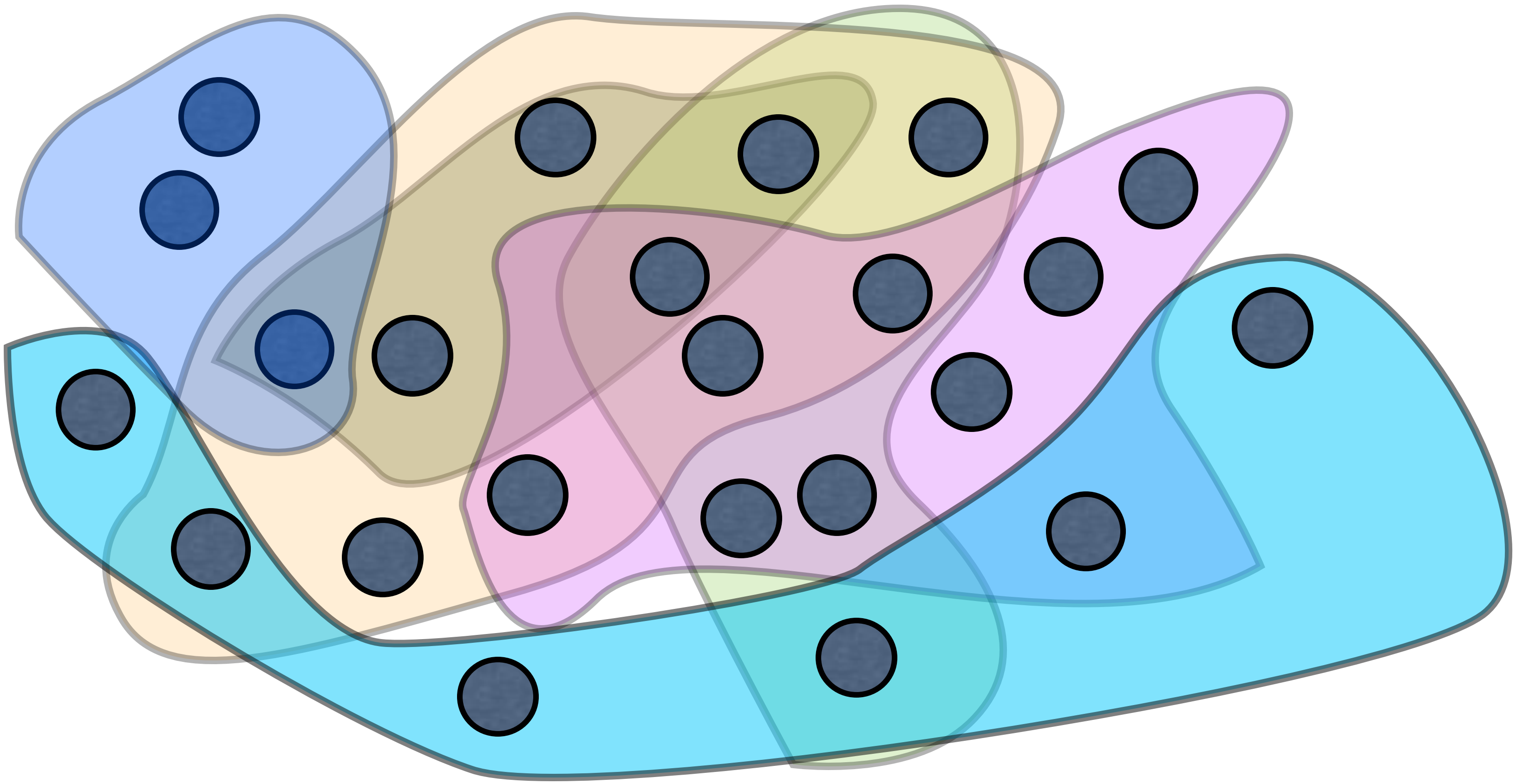
Example 1

Optimal set cover = 4



A Greedy Algorithm

Pick the set that maximizes the number of new elements covered



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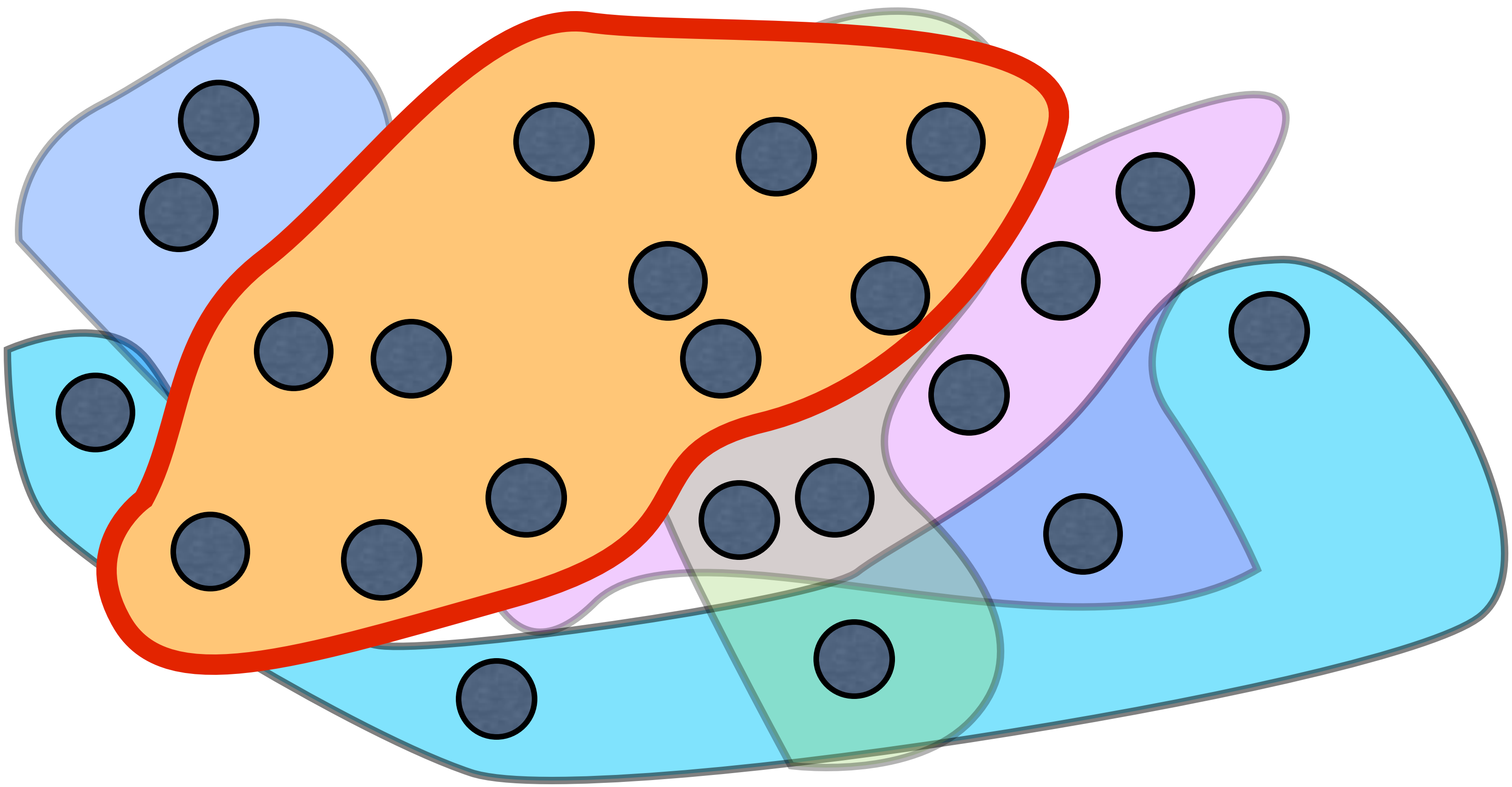

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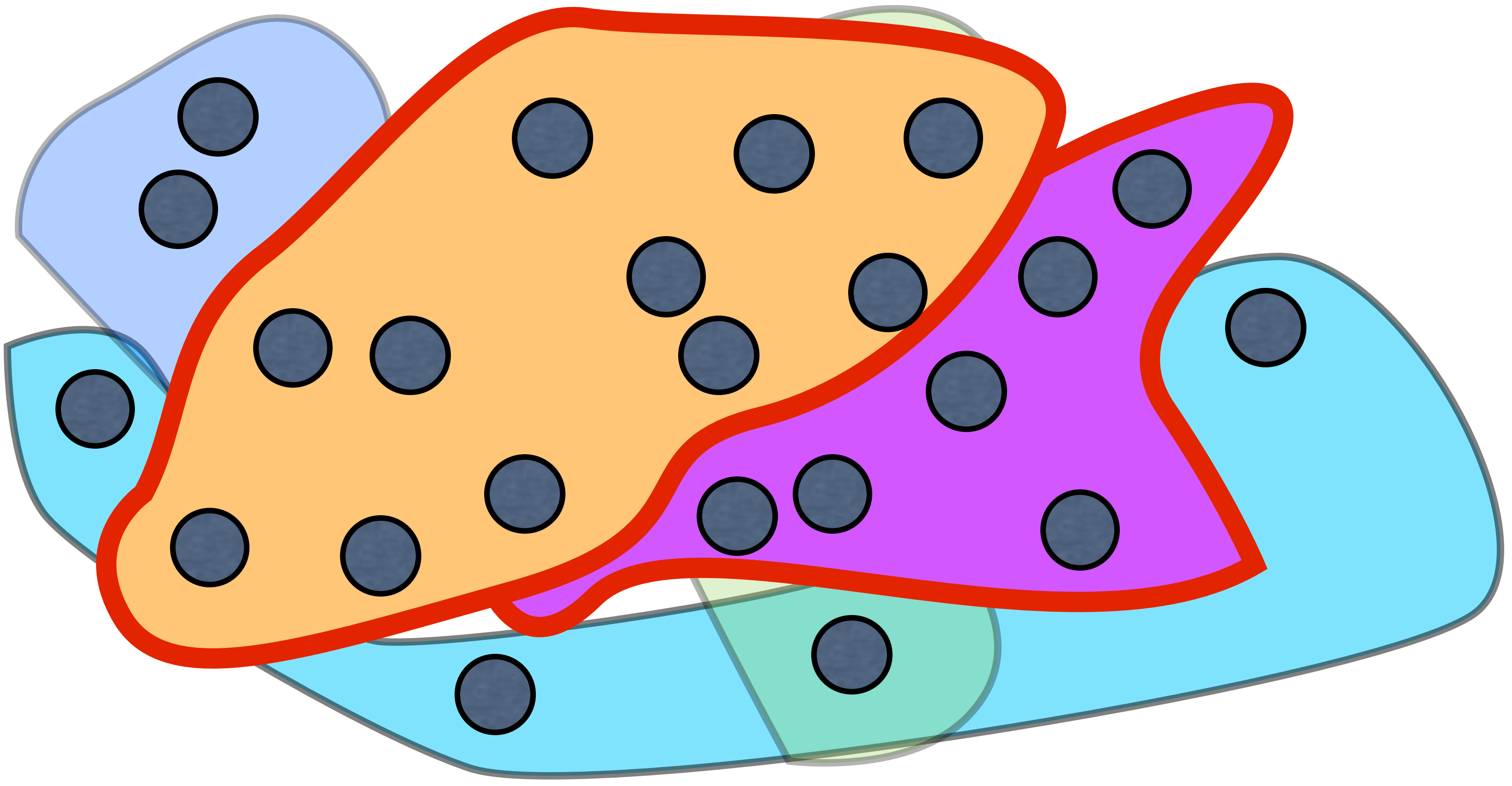

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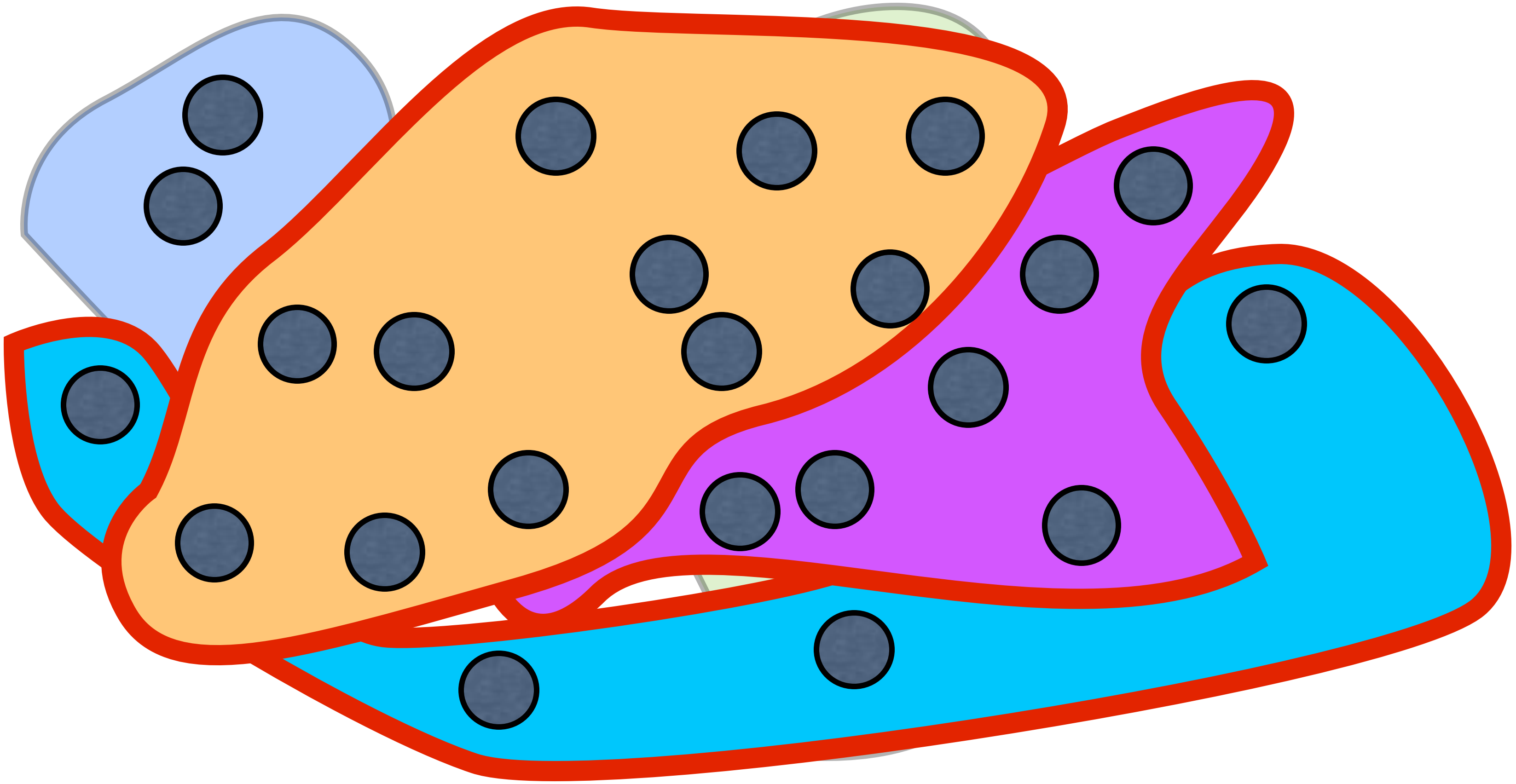

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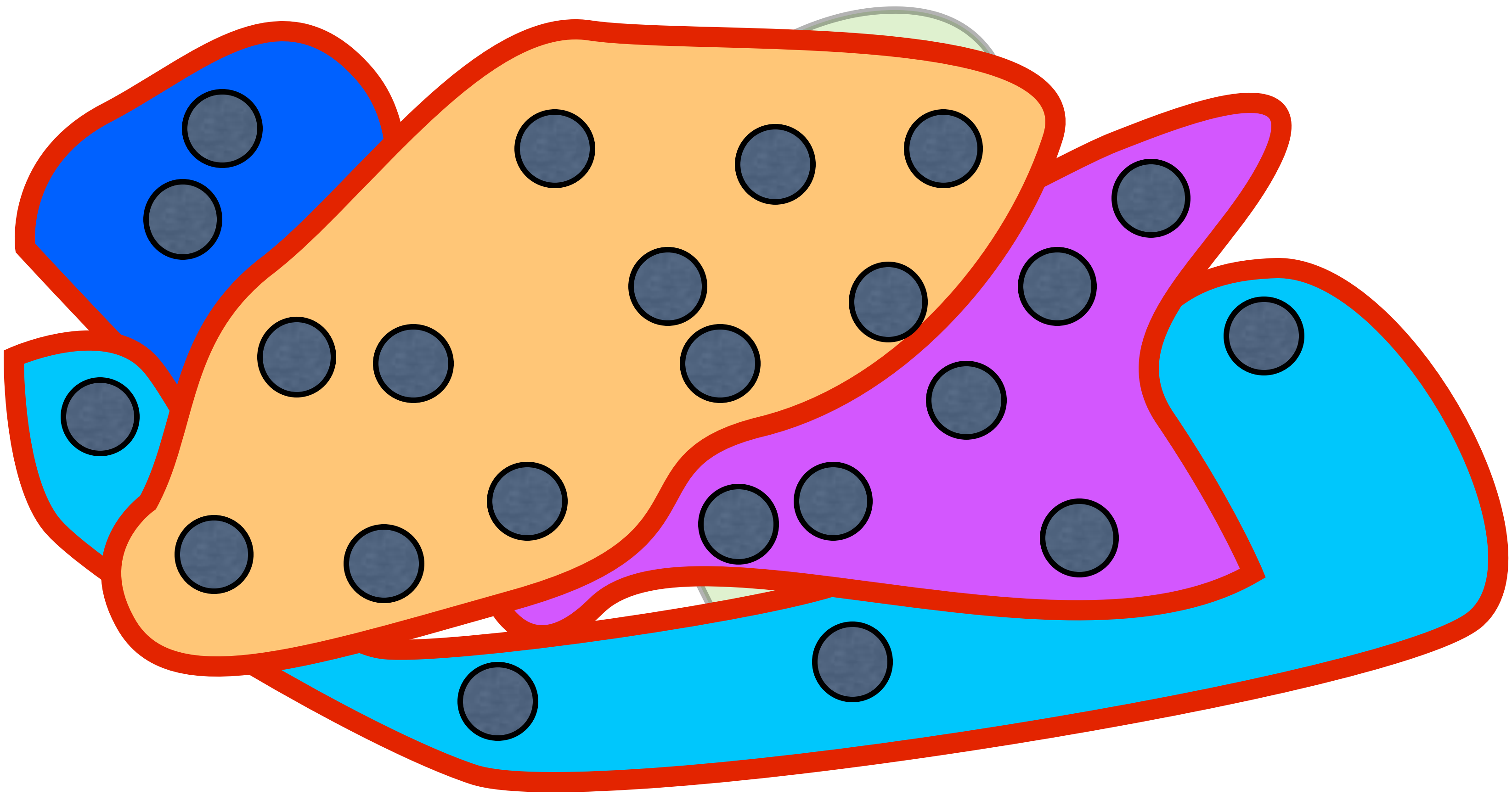

Greedy Algorithm

Example 2

$S_1 = \{1, 2, 3, 8, 9, 10\}$,

$S_2 = \{1, 2, 3, 4, 5\}$,

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Is this the optimal solution?

How well did Greedy do in this case?

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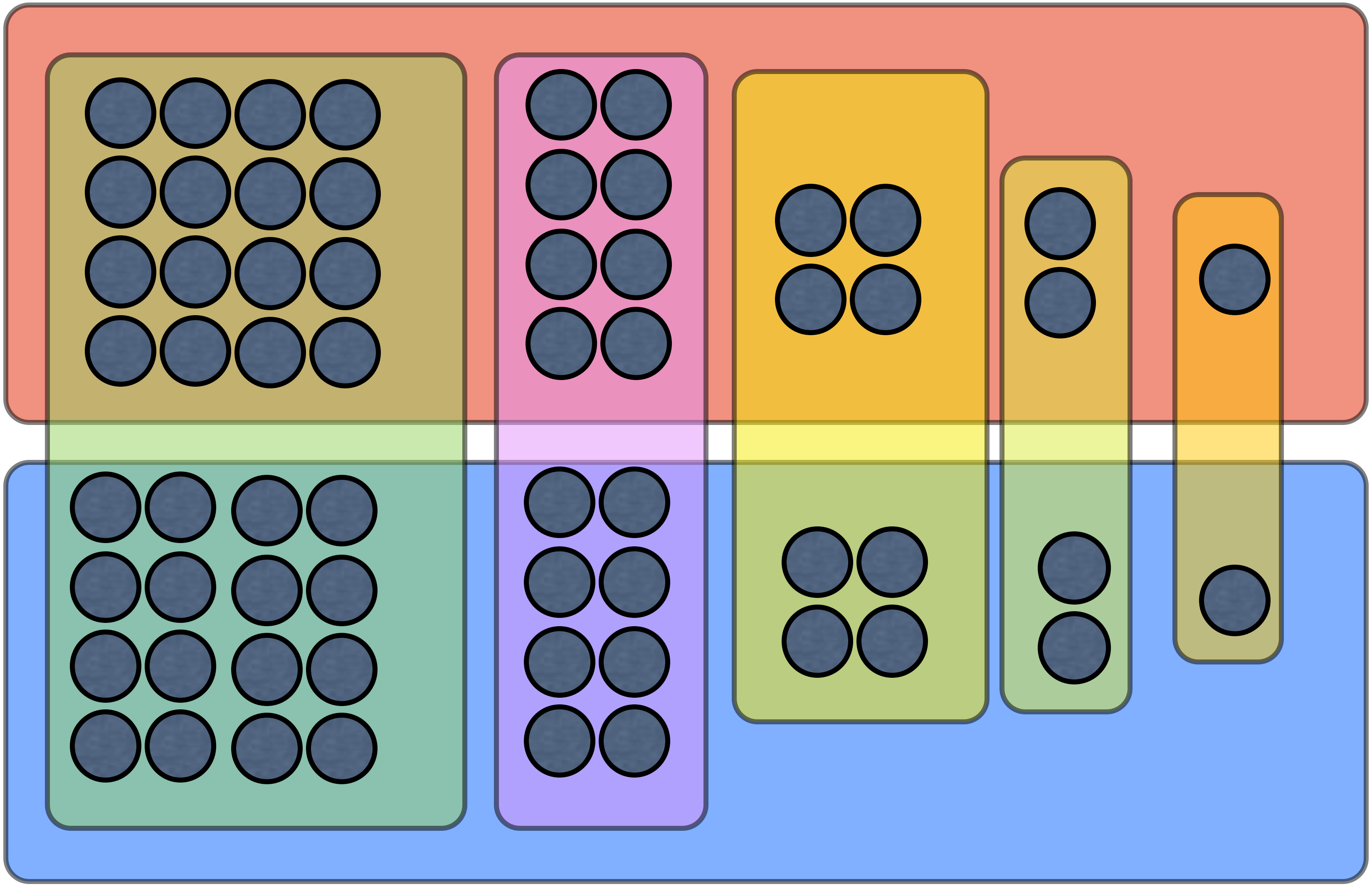
Greedy: 3

Optimum: 2

A Really Bad Example for Greedy



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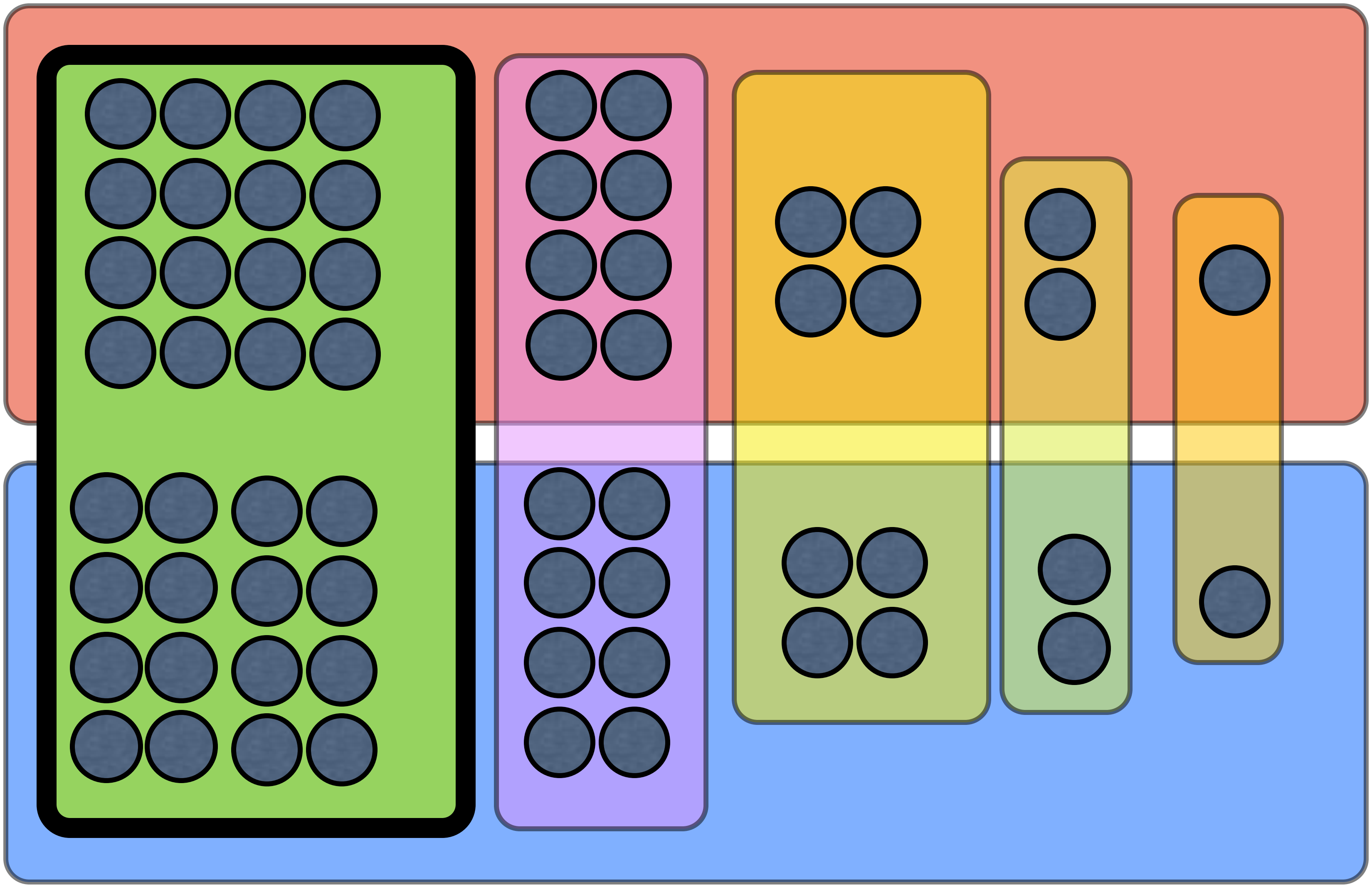

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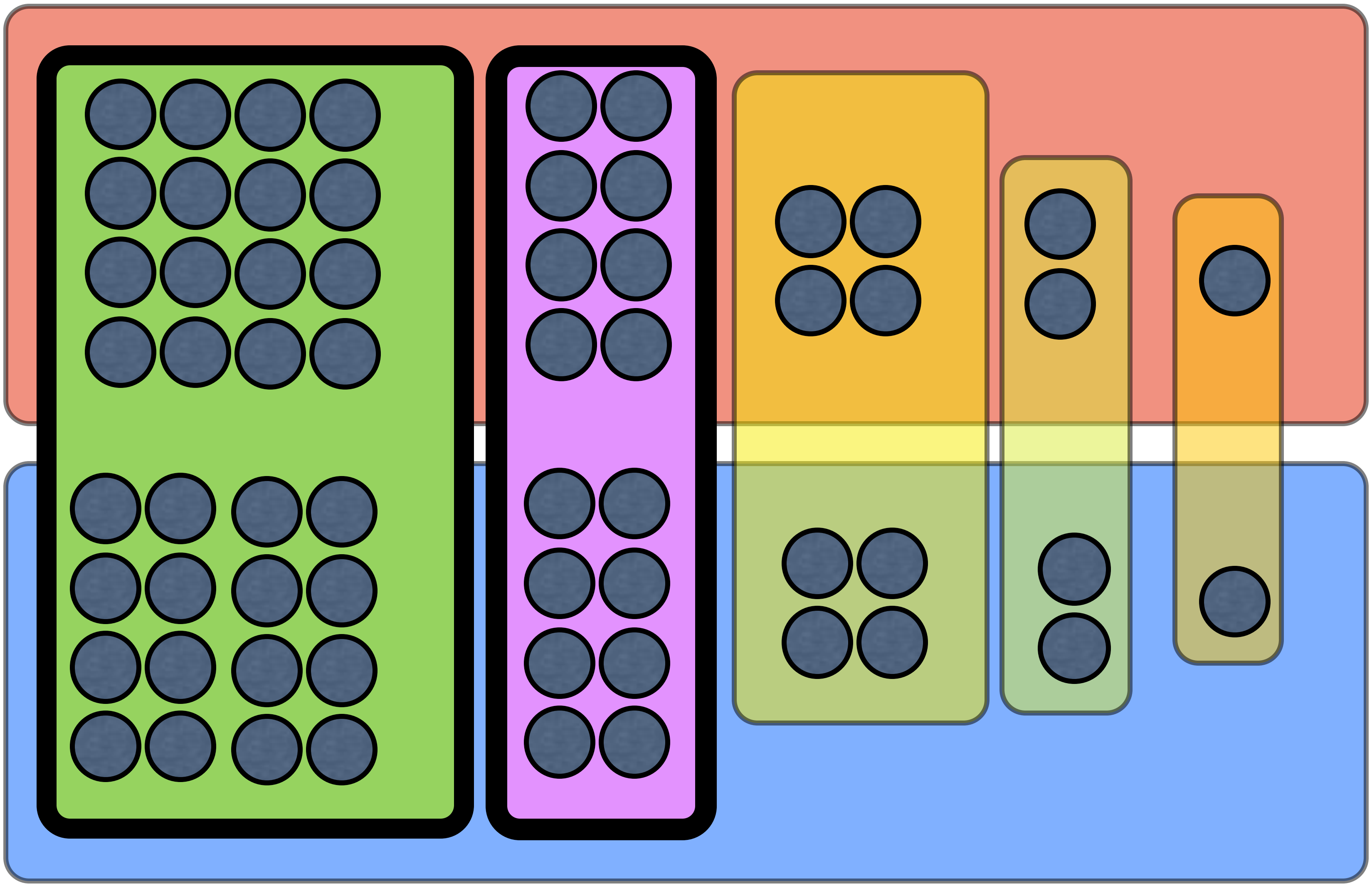

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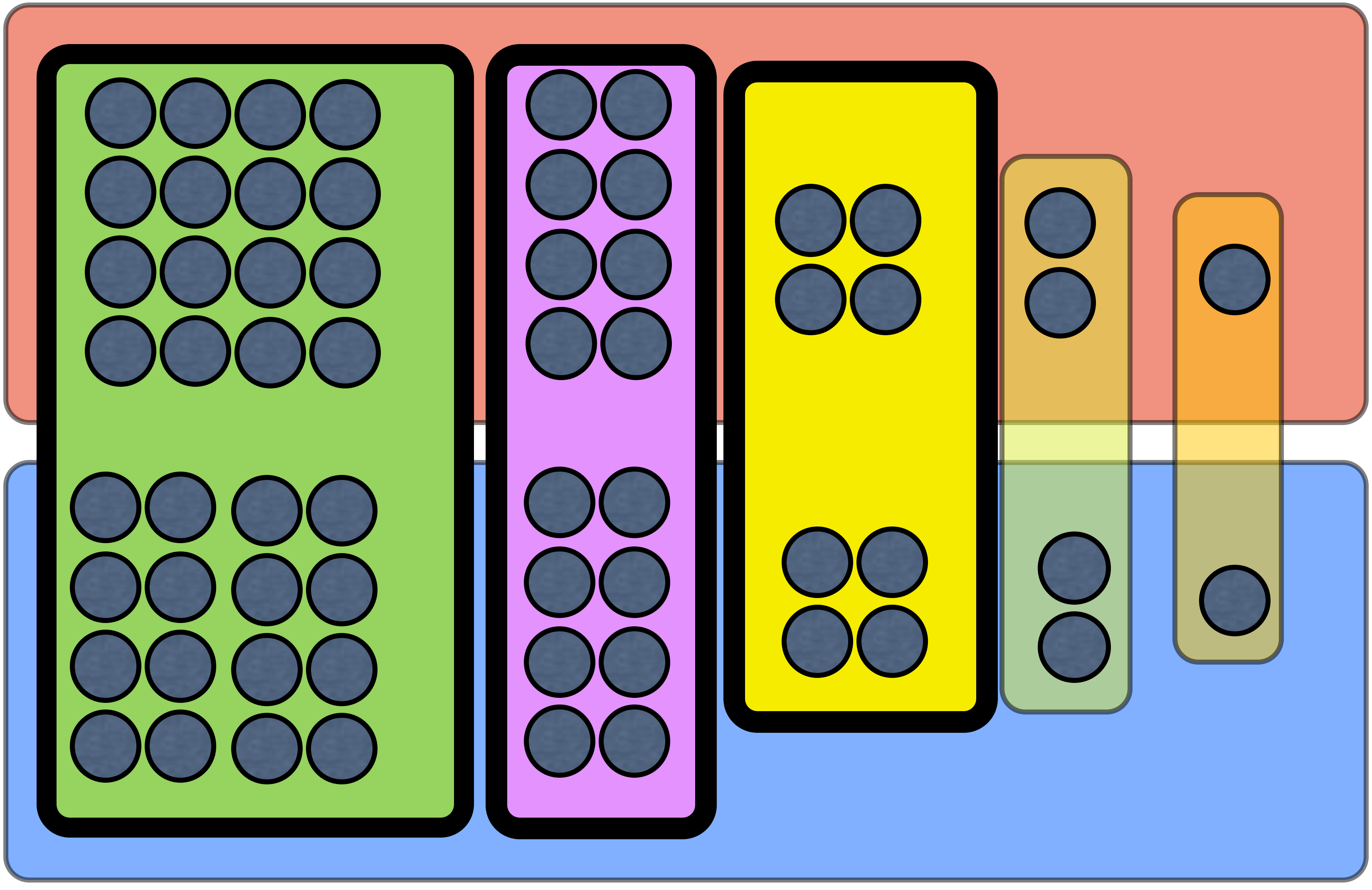

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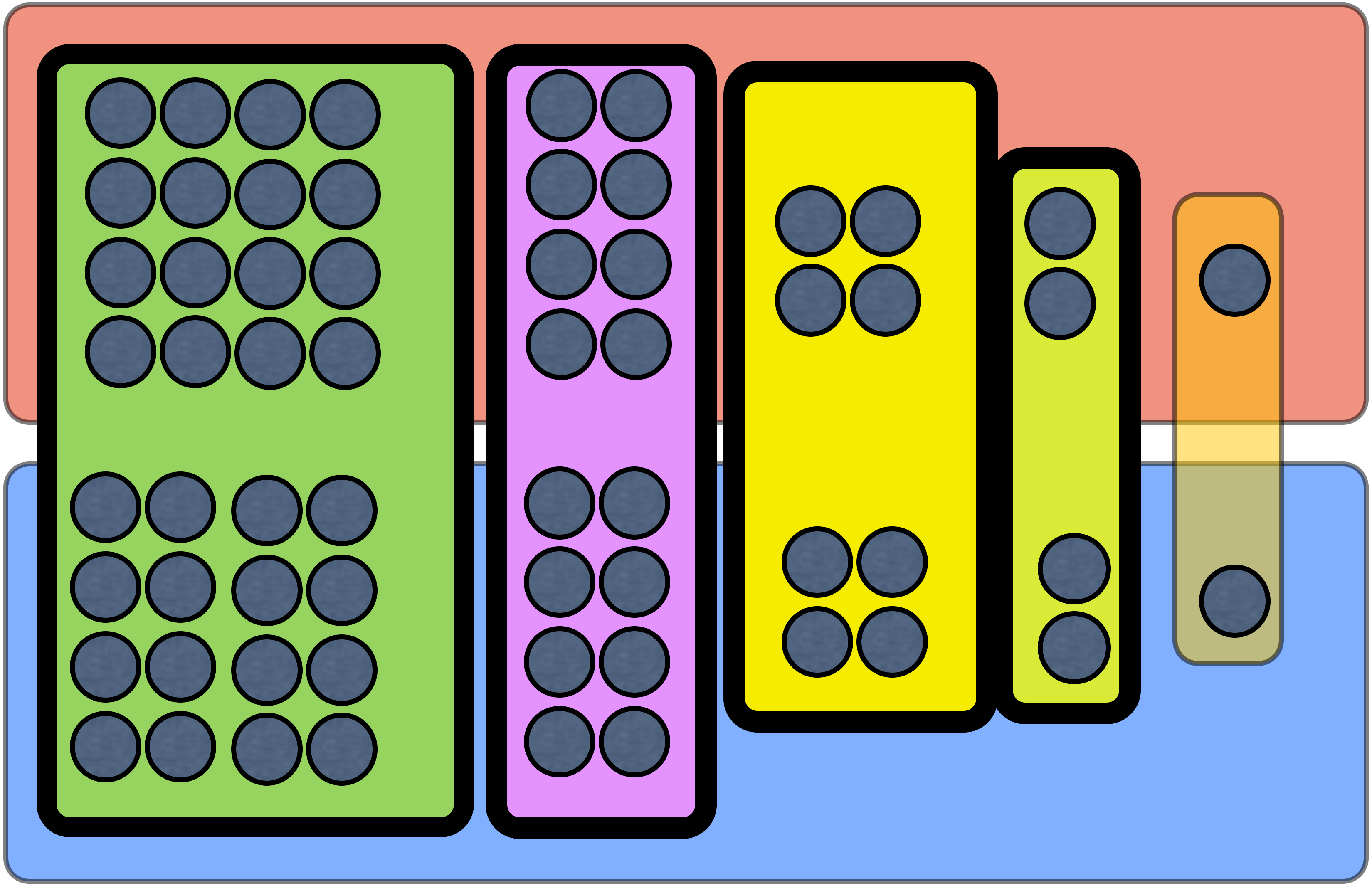

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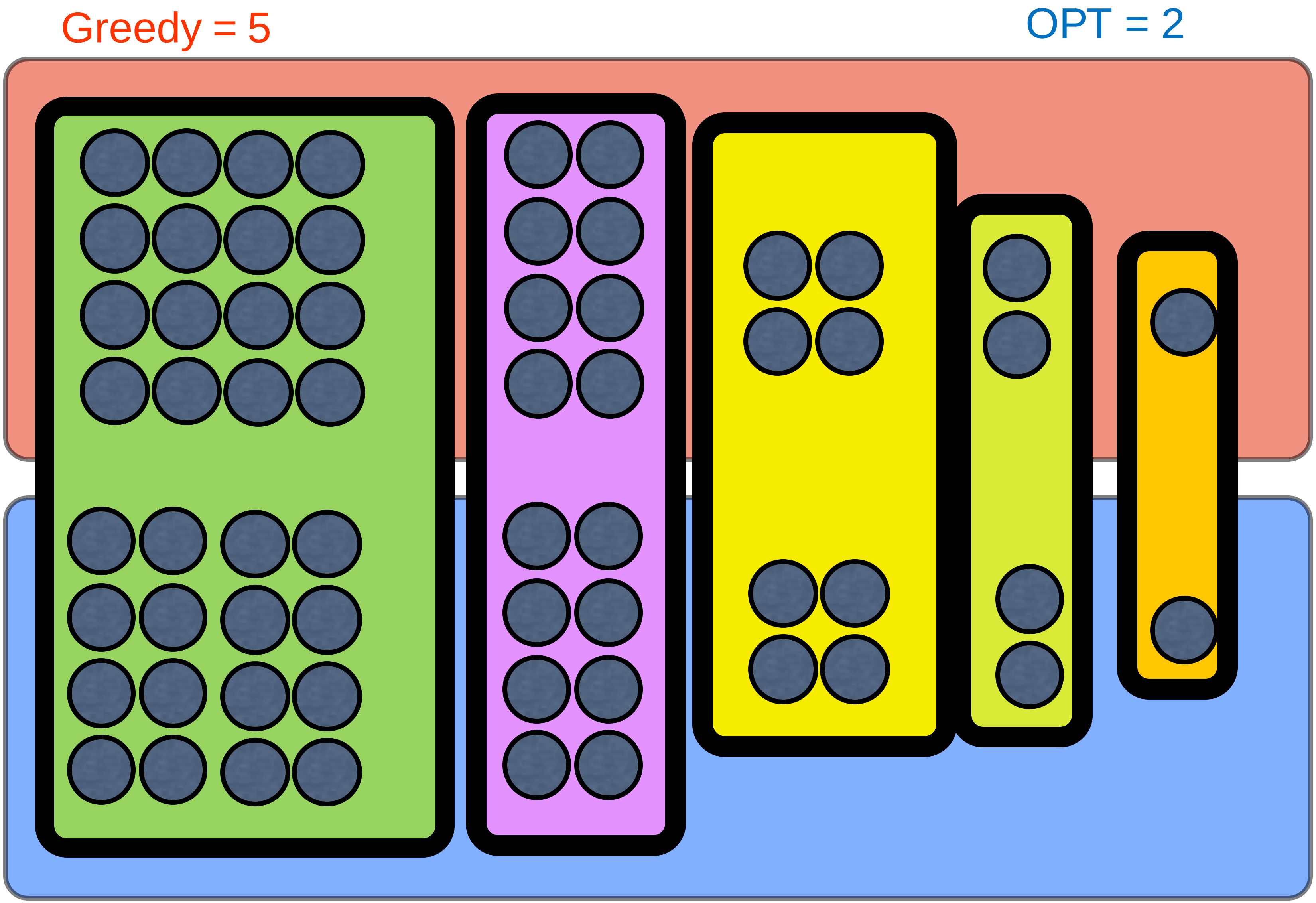

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Useful upper bound: for all real x, we have $1 + x \leq e^x$. (Lots of proofs exist for this; a simple one is using the Taylor expansion of e^x.)

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So after $\mathrm{i} = k \ln n$ steps, # uncovered elements < 1.

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- The best known approximation algorithm for vertex cover is the greedy.
  - It has been open for 40 years to obtain a polynomial time algorithm with approximation ratio better than 2
- There is a long list of problems for which we do not know the best approximation algorithms! Very active area of research (including in our Theory group!)