CSE 421

Max Flow Min Cut, Bipartite Matching

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Residual Graph

Original edge: e = (u, v) $\in$ E.

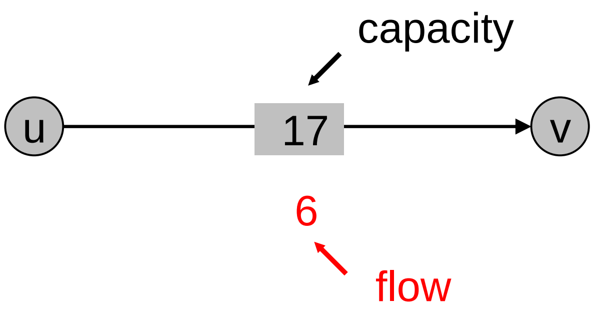
- Flow f(e), capacity c(e).

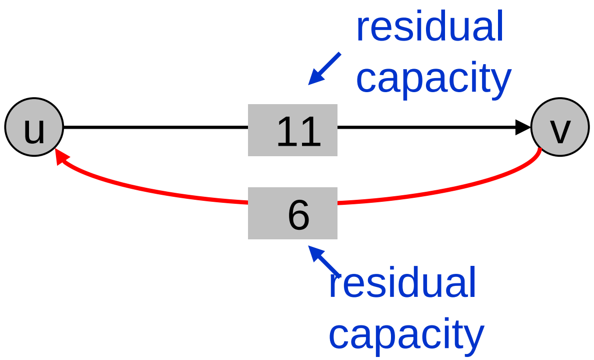
Residual edge.

- "Undo" flow sent.
- e = (u, v) and e^R = (v, u).

Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^R \in E \end{cases}$$





Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b
      else       f(eR) ← f(e) - b
   }
   return f
}
```

Smallest capacity edge on P

Forward edge

Reverse edge

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Gf ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update Gf
   }
   return f
}
```

Max Flow Min Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max s-t flow is equal to the value of the min s-t cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

(i) There exists a cut (A, B) such that v(f) = cap(A, B).

(ii) Flow f is a max flow.

(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along that path.

Pf of Max Flow Min Cut Theorem

(iii) => (i)

No augmenting path for f => there is a cut (A,B): v(f)=cap(A,B)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B)
\end{aligned}
$$

Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations, if f^* is optimal flow.

Pf. Each augmentation increase value by at least 1.

Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

Pf. Since algorithm terminates, theorem follows from invariant.

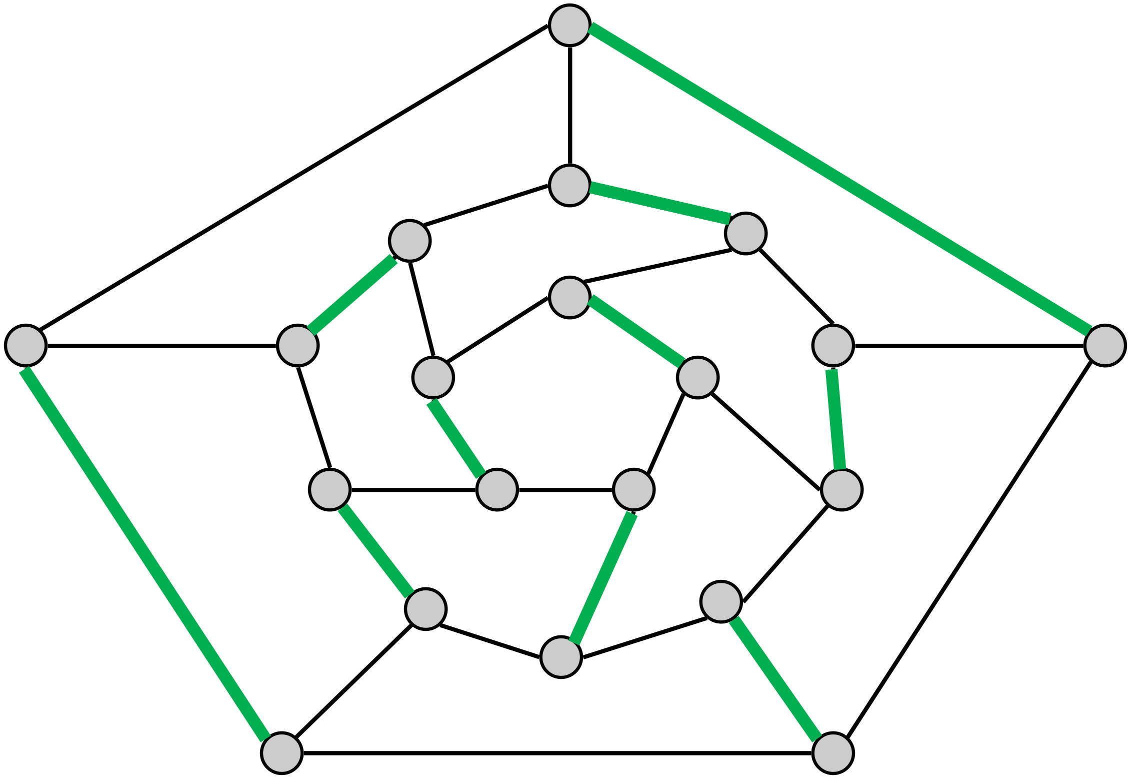
Applications of Max Flow:
Bipartite Matching

Maximum Matching Problem

Given an undirected graph G = (V, E).

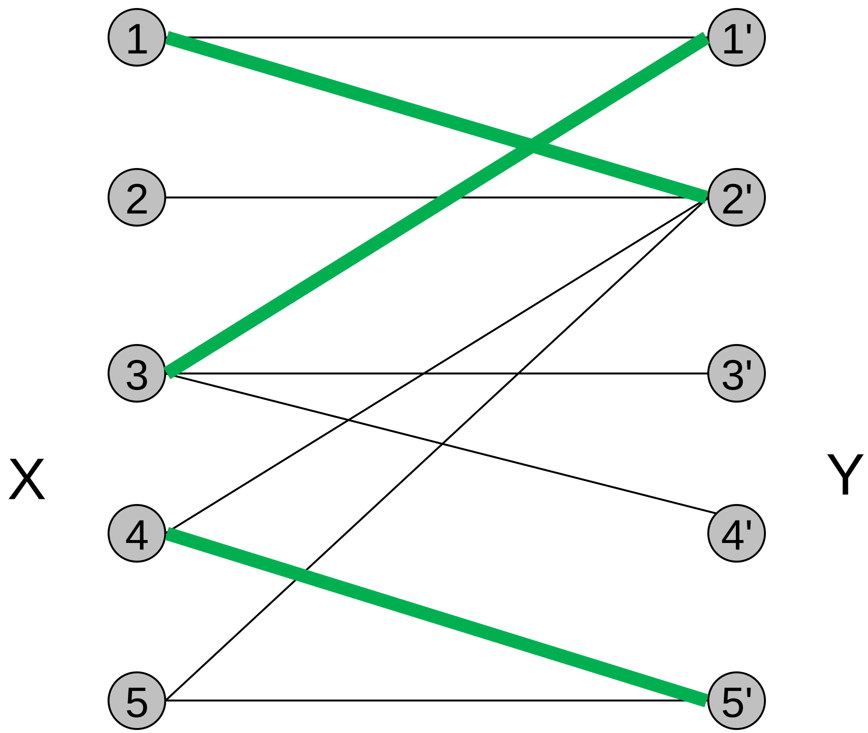
A set $M \subseteq E$ is a matching if each node appears in at most edge in M.

Goal: find a matching with largest cardinality.



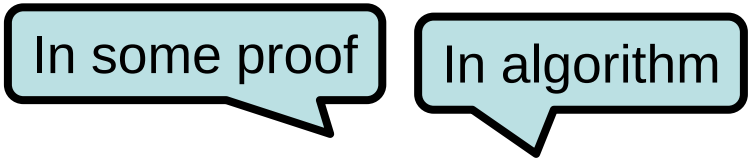
Bipartite Matching Problem

Given an undirected bipartite graph $G = (X \cup Y, E)$
A set $M \subseteq E$ is a matching if each node appears in at most edge in M.
Goal: find a matching with largest cardinality.

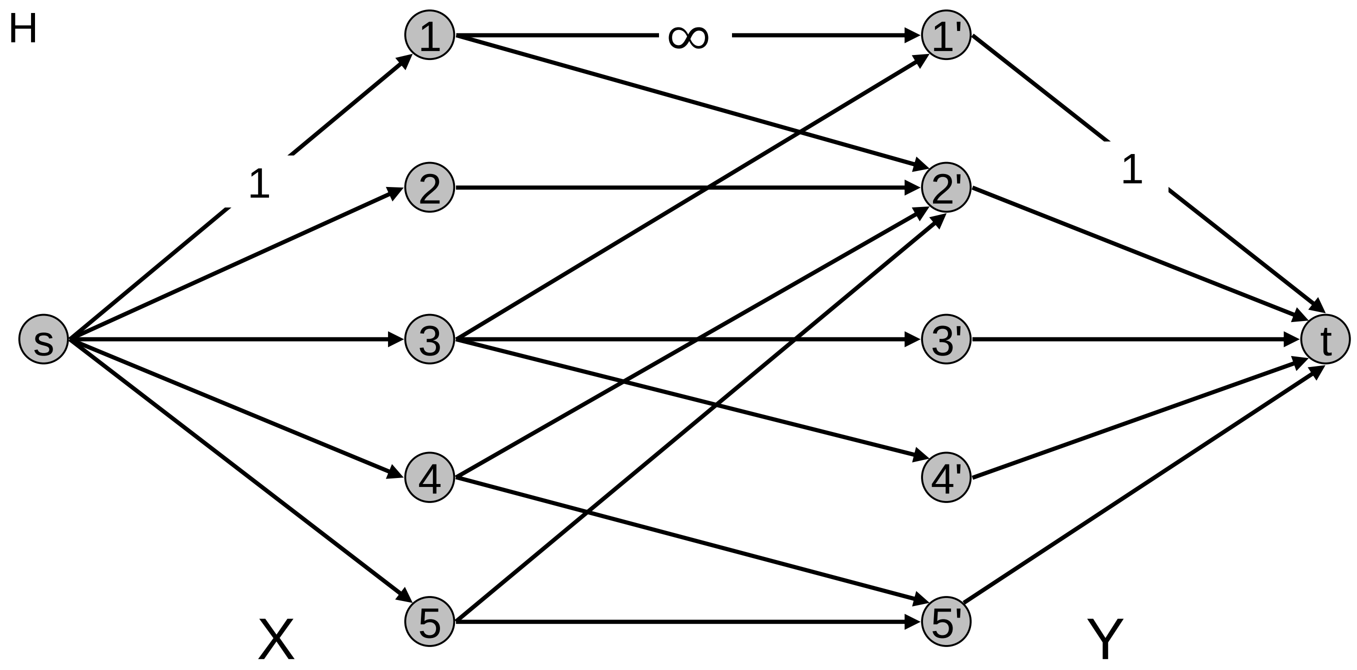


Bipartite Matching using Max Flow

Create digraph H as follows:



- Orient all edges from X to Y, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.



Bipartite Matching: Proof of Correctness

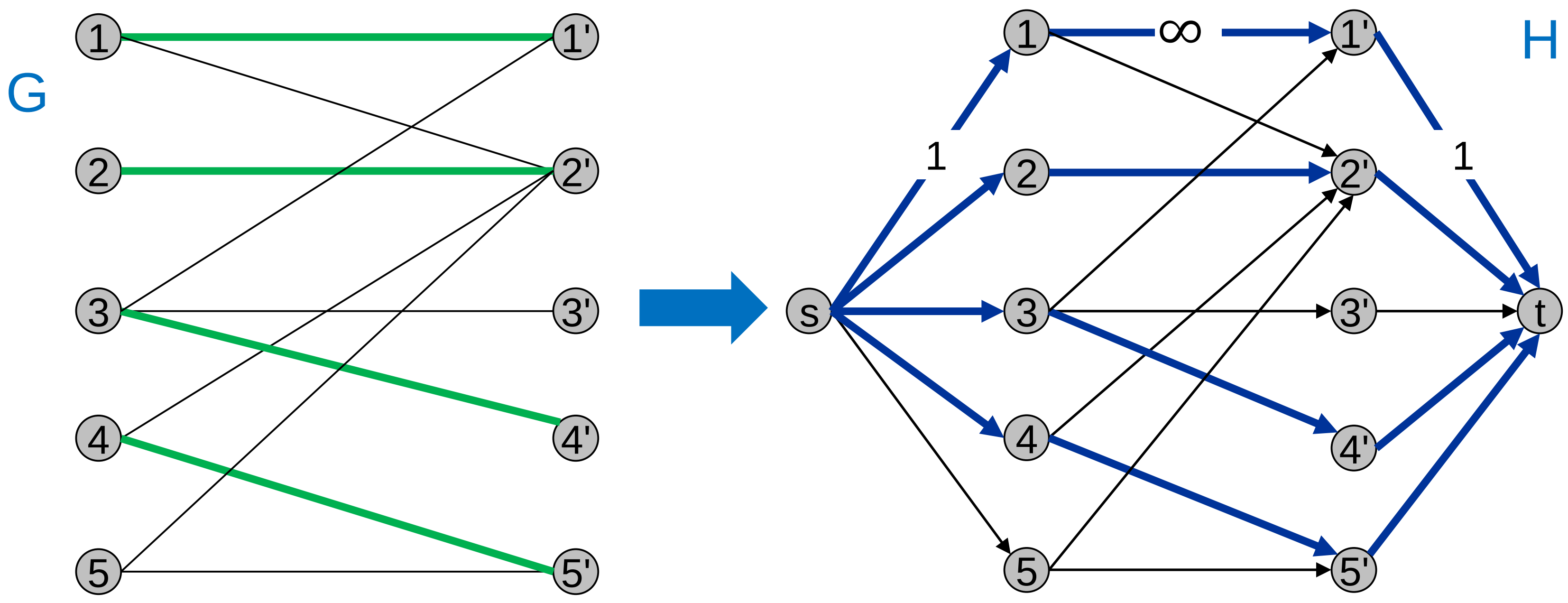
Thm. Max cardinality matching in G = value of max flow in H.

Pf. $\leq$

Given max matching M of cardinality k.

Consider flow f that sends 1 unit along each of k edges of M.

f is a flow, and has cardinality k. ▪



Bipartite Matching: Proof of Correctness

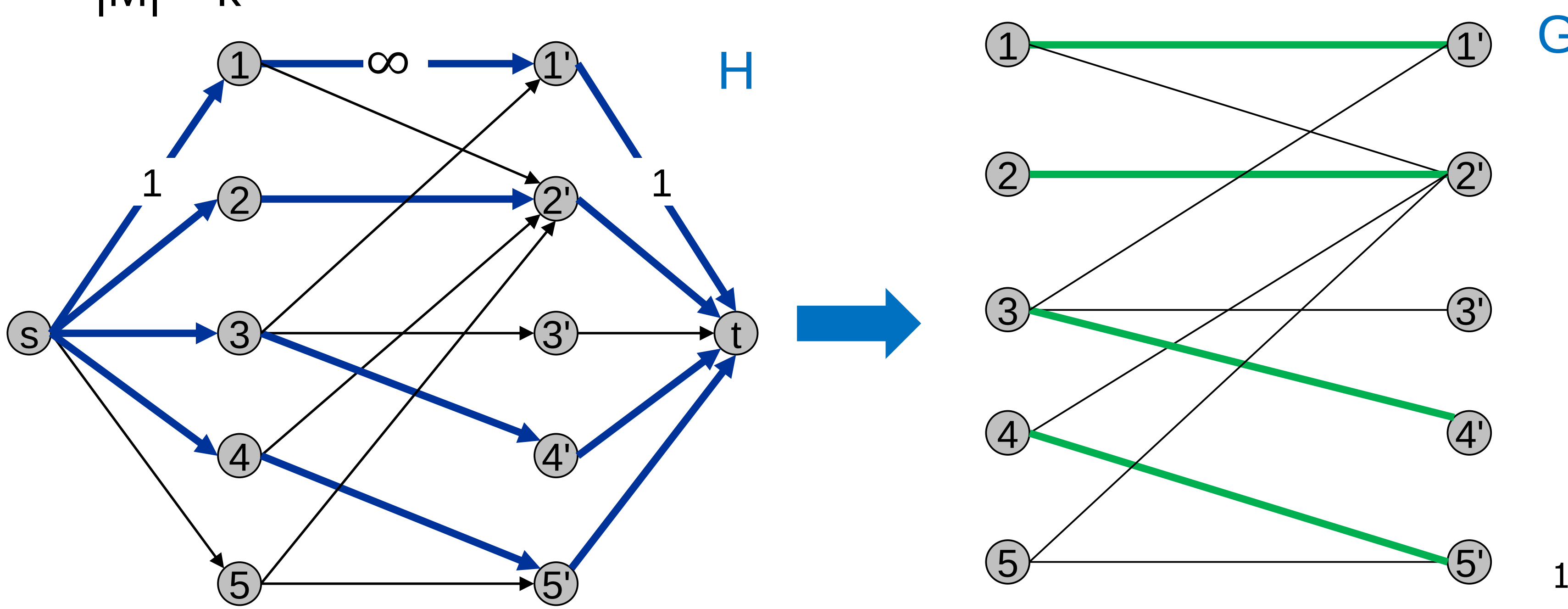
Thm. Max cardinality matching in G = value of max flow in H.

Pf. (of $\geq$) Let f be a max flow in H of value k.

Integrality theorem $\Rightarrow$ k is integral and we can assume f is 0-1.

Consider M = set of edges from X to Y with f(e) = 1.

- each node in X and Y participates in at most one edge in M
- |M| = k



Perfect Bipartite Matching

Perfect Bipartite Matching

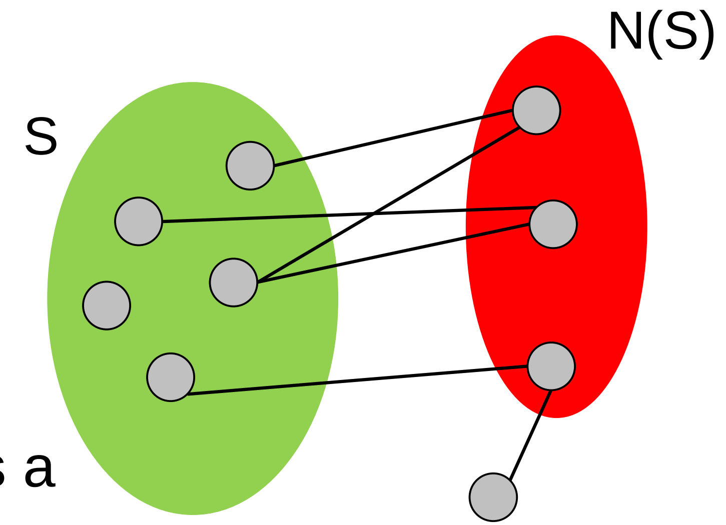
Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings:

- Clearly we must have $|X| = |Y|$.
- What other conditions are necessary?
- What conditions are sufficient?

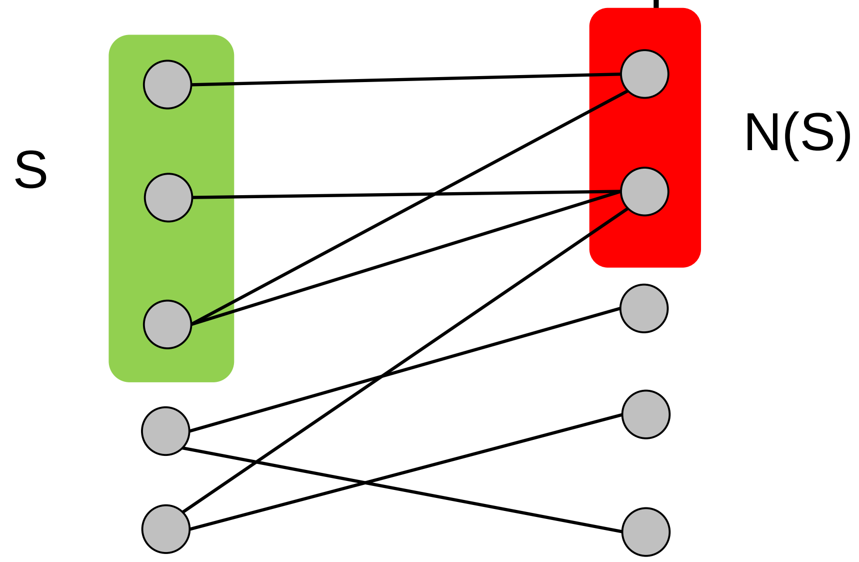
Perfect Bipartite Matching: N(S)



Def. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph G has a perfect matching, then |N(S)| ≥ |S| for all subsets $S \subseteq X$.

Pf. Each $v \in S$ has to be matched to a unique node in N(S).



Marriage Theorem

Thm: [Frobenius 1917, Hall 1935] Let $G = (X \cup Y, E)$ be a bipartite graph with |X| = |Y|.

Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Pf. $\Rightarrow$

This was the previous observation.

If |N(S)| < |S| for some S, then there is no perfect matching.

Marriage Theorem

Pf. $\exists S \subseteq X$ s.t., $|N(S)| < |S| \Leftarrow$ G does not a perfect matching

Formulate as a max-flow and let (A, B) be the min s-t cut

G has no perfect matching => $v(f^*) < |X|$. So, $cap(A, B) < |X|$

Define $X_A = X \cap A, X_B = X \cap B, Y_A = Y \cap A$

Then, $cap(A, B) \geq |X_B| + |Y_A|$

Since min-cut does not use ∞ edges, $N(X_A) \subseteq Y_A$

$|N(X_A)| \leq |Y_A| \leq cap(A, B) - |X_B| = cap(A, B) - |X| + |X_A| < |X_A|$