CSE 421

Divide and Conquer

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Master Theorem

Suppose $T(n) = a\,T\left(\frac{n}{b}\right) + cn^k$ for all $n > b$. Then,

- If $a < b^k$ then $T(n) = \Theta\left(n^k\right)$
- If $a = b^k$ then $T(n) = \Theta\left(n^k \log n\right)$
- If $a > b^k$ then $T(n) = \Theta\left(n^{\log_b a}\right)$

Works even if it is $\left\lceil \frac{n}{b} \right\rceil$ instead of $\frac{n}{b}$.

We also need $a \geq 1, b > 1, k \geq 0$ and $T(n) = O(1)$ for $n \leq b$.

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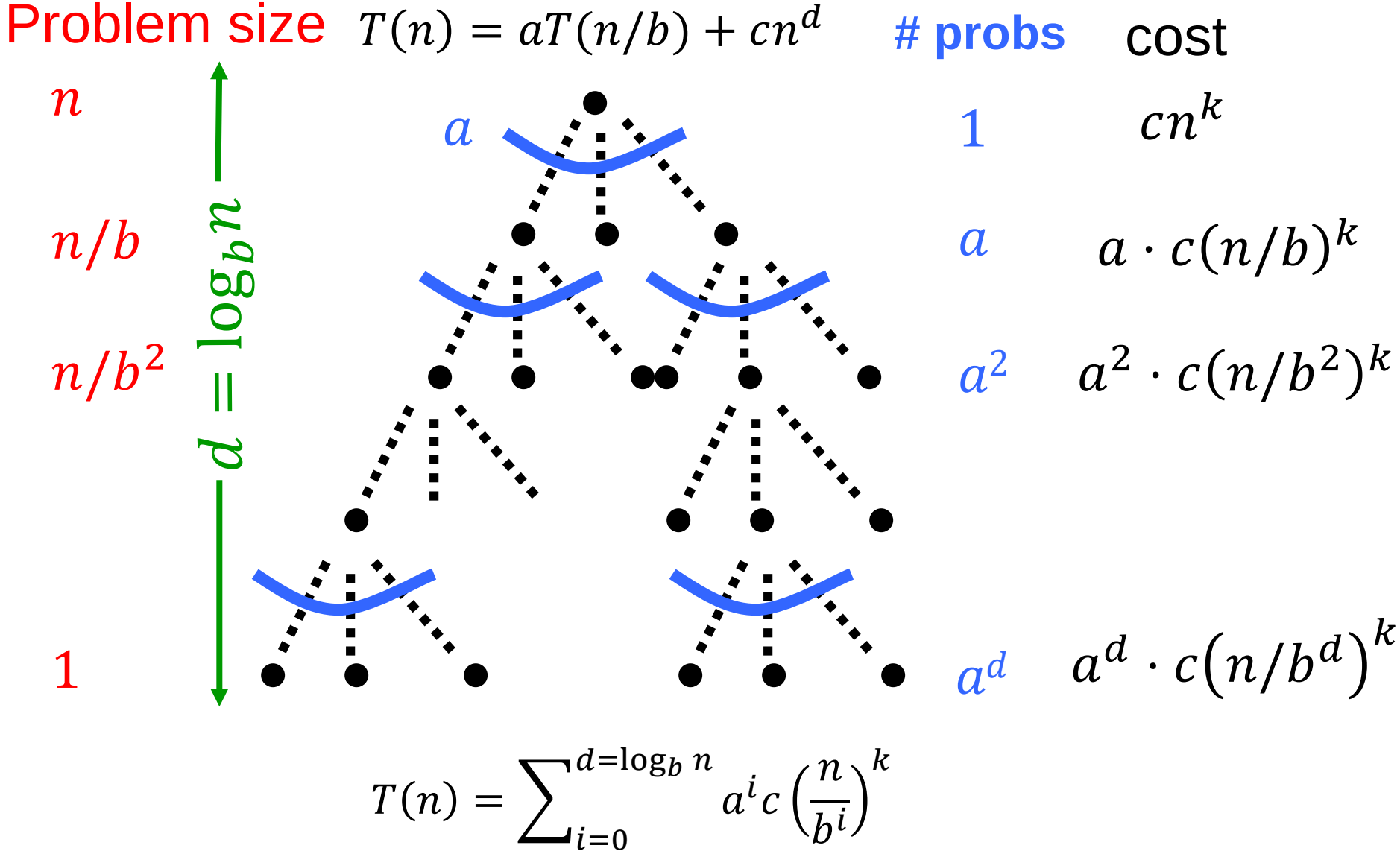
Example: For mergesort algorithm we have

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n).$$

So, $k = 1, a = b^k$ and $T(n) = \Theta(n \log n)$

Proving Master Theorem



$$T(n) = \sum_{i=0}^{d=\log_b n} a^i c \left(\frac{n}{b^i}\right)^k$$

Master Theorem

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# of problems increases slower than the decreases of cost. First term dominates.

# of problems increases faster than the decreases of cost Last term dominates.

A Useful Identity

Theorem: $1 + x + x^2 + \cdots + x^d = \frac{x^{d+1}-1}{x-1}$

Proof: Let $S = 1 + x + x^2 + \cdots + x^d$

Then, $xS = x + x^2 + \cdots + x^{d+1}$

So, $xS - S = x^{d+1} - 1$

i.e., $S(x-1) = x^{d+1} - 1$

Therefore, $S = \frac{x^{d+1}-1}{x-1}$

Corollary:

$$1 + x + x^2 + \cdots + x^d = \begin{cases} O_x(1) & \text{if } x < 1 \\ d+1 & \text{if } x = 1 \\ O_x\left(x^{d+1}\right) & \text{if } x > 1 \end{cases}$$

O_x means the hidden constant depends on x

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k$

Corollary:

$$1 + x + x^2 + \cdots + x^d = \begin{cases} \Theta_x(1) & \text{if } x < 1 \\ \Theta(d) & \text{if } x = 1 \\ \Theta_x\left(x^{d+1}\right) & \text{if } x > 1 \end{cases}$$

Going back, we have

$$T(n) = \sum_{i=0}^{d=\log_b n} a^i c \left(\frac{n}{b^i}\right)^k = cn^k \sum_{i=0}^{d=\log_b n} \left(\frac{a}{b^k}\right)^i$$

Hence, we have

$$T(n) = \Theta\left(n^k\right) \begin{cases} 1 & \text{if } a < b^k \\ \log_b n & \text{if } a = b^k \\ \left(\frac{a}{b^k}\right)^{\log_b n} & \text{if } a > b^k \end{cases}$$

constant depends on a, b, c

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k$

$$T(n) = \Theta\left(n^k\right)\begin{cases} 1 & \text{if } a < b^k \\ \log_b n & \text{if } a = b^k \\ \left(\frac{a}{b^k}\right)^{\log_b n} & \text{if } a > b^k \end{cases}$$

For $a < b^k$, we simply have $T(n) = \Theta\left(n^k\right)$.

For $a = b^k$, we have $T(n) = \Theta\left(n^k \log_b n\right) = \Theta(n^k \log n)$.

For $a > b^k$, we have $T(n) = \Theta\left(n^k \left(\frac{a}{b^k}\right)^{\log_b n}\right) = \Theta(n^{\log_b a})$.

$b^{k \log_b n}$
$= \left(b^{\log_b n}\right)^k$
$= n^k$

$a^{\log_b n}$
$= (b^{\log_b a})^{\log_b n}$
$= (b^{\log_b n})^{\log_b a}$
$= n^{\log_b a}$

Median

Selecting k-th smallest

Problem: Given numbers $x_1, \ldots, x_n$ and an integer $1 \leq k \leq n$ output the k-th smallest number

$$\mathrm{Sel}(\{x_1, \ldots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time $O(n \log n)$ then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

An Idea

Choose a number w from $x_1, \dots, x_n$

Define

- $S_<(w) = \{x_i : x_i < w\}$
- $S_=(w) = \{x_i : x_i = w\}$
- $S_>(w) = \{x_i : x_i > w\}$

Can be computed in linear time

Solve the problem recursively as follows:

- If $k \le |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \le |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \le n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

How to choose w?

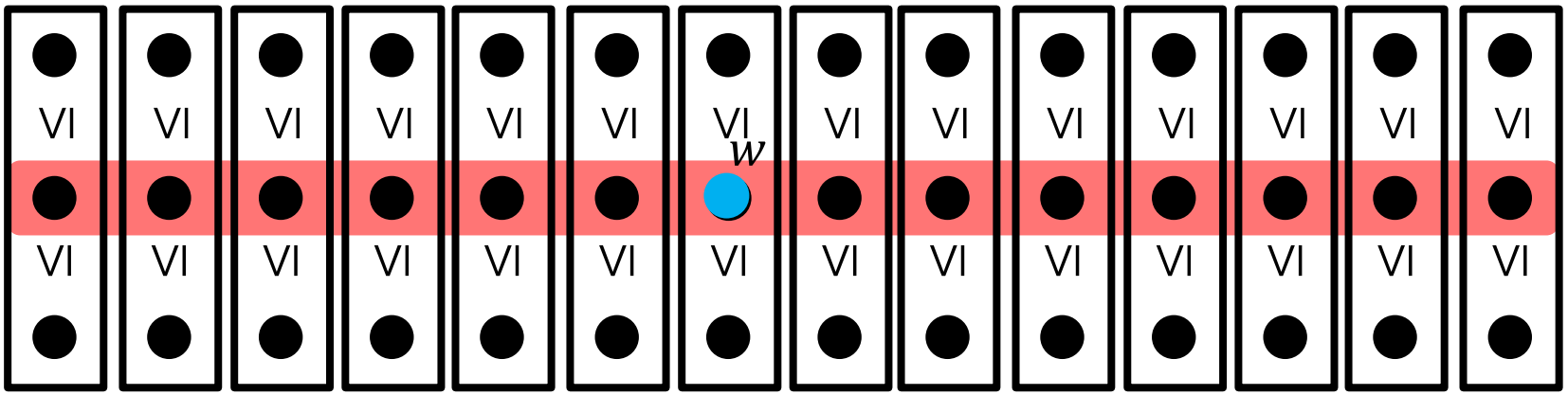
Suppose we choose w uniformly at random

similar to the pivot in quicksort.

Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.

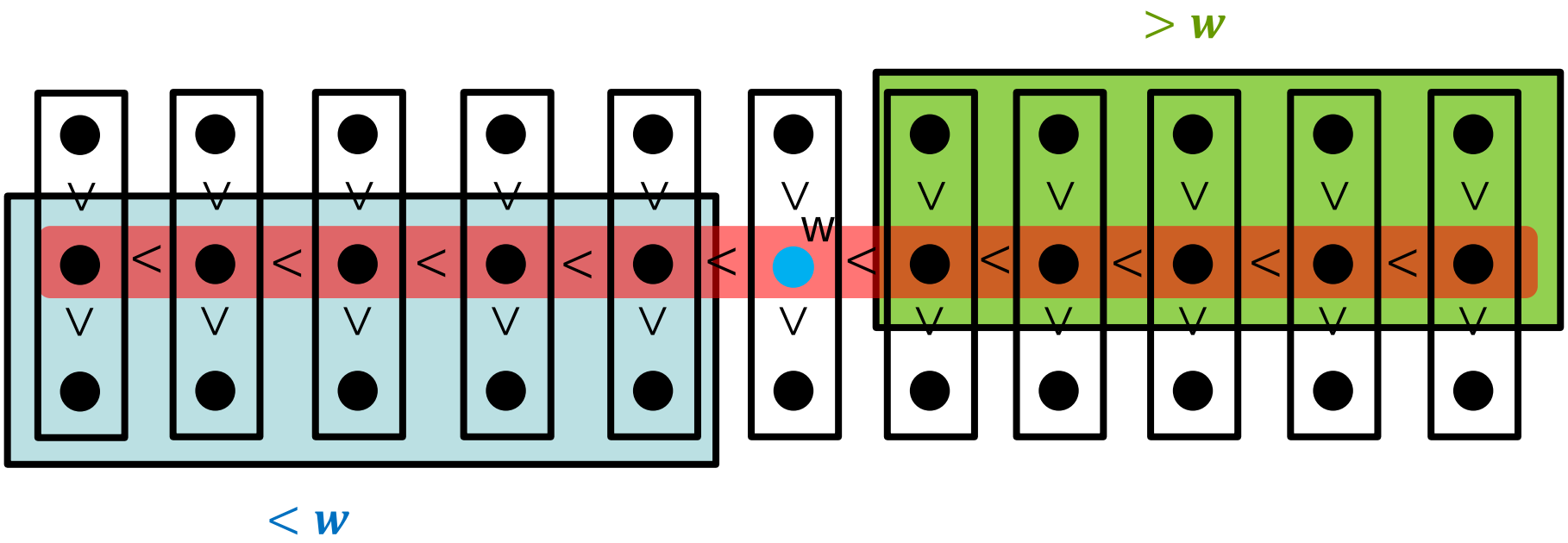
Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(midpoints, n/6)$



Assume all numbers are distinct for simplicity.

How to lower bound $|S_<(w)|, |S_>(w)|$?



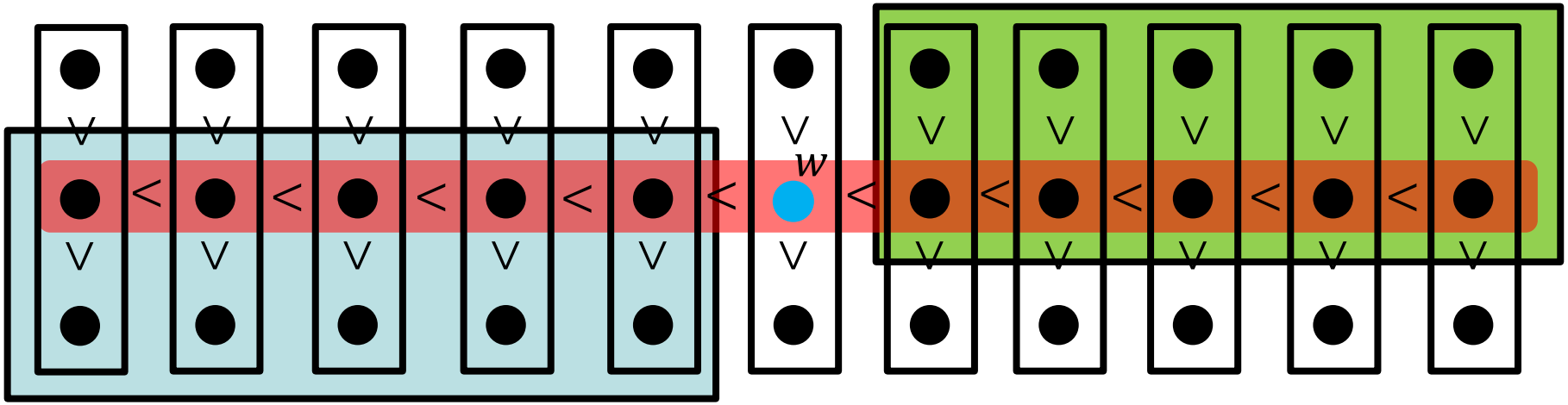
- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Assume all numbers are distinct for simplicity.

Asymptotic Running Time?



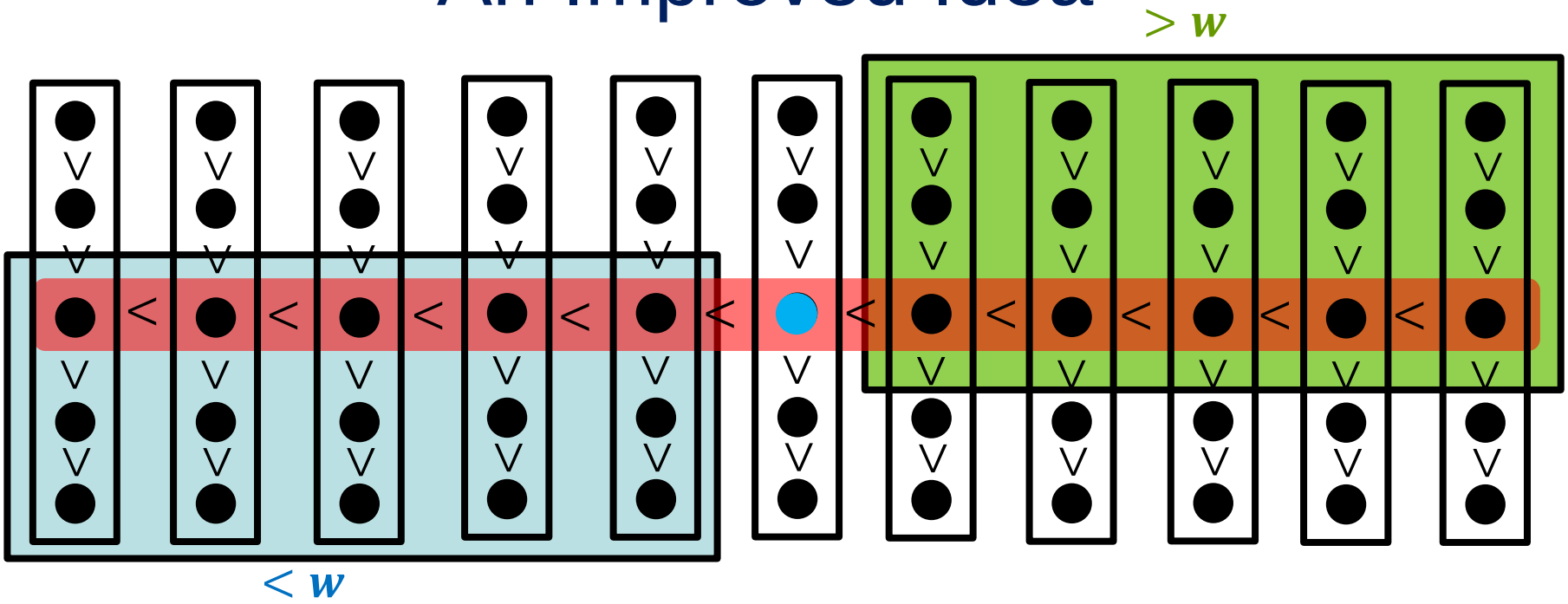
- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

$O(n \log n)$ again?
So, what is the point?

Where $\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(midpoints, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$$\frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

An Improved Idea

```
Sel(S, k) {
   n ← |S|
   If (n < ??) return ??
   Partition S into n/5 sets of size 5
   Sort each set of size 5 and let M be the set of medians, so |M|=n/5
   Let w=Sel(M,n/10)
   For i=1 to n{
     If x_i < w add x to S_<(w)
     If x_i > w add x to S_>(w)
     If x_i = w add x to S_=(w)
   }
   If (k ≤ |S_<(w)|)
     return Sel(S_<(w), k)
   else if (k ≤ |S_<(w)| + |S_=(w)|)
     return w;
   else
     return Sel(S_>(w), k − |S_<(w)| − |S_=(w)|)
}
```

We can maintain each set in an array

Integer Multiplication

Integer Arithmetic

Add: Given two n-bit integers a and b, compute $a + b$.

Add

	1	1	1	1	1	1	0	1
		1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

$O(n)$ bit operations.

Multiply: Given two n-bit integers a and b, compute $a \times b$. The "grade school" method:

Multiply

						1	1	0	1	0	1	0	1
					*	0	1	1	1	1	1	0	1
						1	1	0	1	0	1	0	1
					0	0	0	0	0	0	0	0	
				1	1	0	1	0	1	0	1		
			1	1	0	1	0	1	0	1			
		1	1	0	1	0	1	0	1				
	1	1	0	1	0	1	0	1					
0	0	0	0	0	0	0	0						
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

$O(n^2)$ bit operations.

Divide and Conquer

Let x, y be two n-bit integers

Write $x = 2^{n/2}x_1 + x_0$ and $y = 2^{n/2}y_1 + y_0$

where x_0, x_1, y_0, y_1 are all $n/2$-bit integers.

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

Therefore,

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

So,

$$T(n) = \Theta(n^2).$$

We only need 3 values $x_1y_1, x_0y_0, x_1y_0 + x_0y_1$
Can we find all 3 by only 3 multiplication?

Key Trick: 4 multiplies at the price of 3

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\alpha = x_1 + x_0$$
$$\beta = y_1 + y_0$$
$$\alpha\beta = (x_1 + x_0)(y_1 + y_0)$$
$$= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$
$$(x_1 y_0 + x_0 y_1) = \alpha\beta - x_1 y_1 - x_0 y_0$$

Key Trick: 4 multiplies at the price of 3

Theorem [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585\ldots})$ bit operations.

$$x = 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= \underbrace{2^n \cdot x_1 y_1}_{A} + \underbrace{2^{n/2} \cdot (x_1 y_0 + x_0 y_1)}_{\alpha\beta - A - B} + \underbrace{x_0 y_0}_{B}$$

To multiply two n-bit integers:

- Add two $n/2$ bit integers.
- Multiply three $n/2$-bit integers.
- Add, subtract, and shift $n/2$-bit integers to obtain result.

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right) = O(n^{1.585\ldots})$$

Integer Multiplication (Summary)

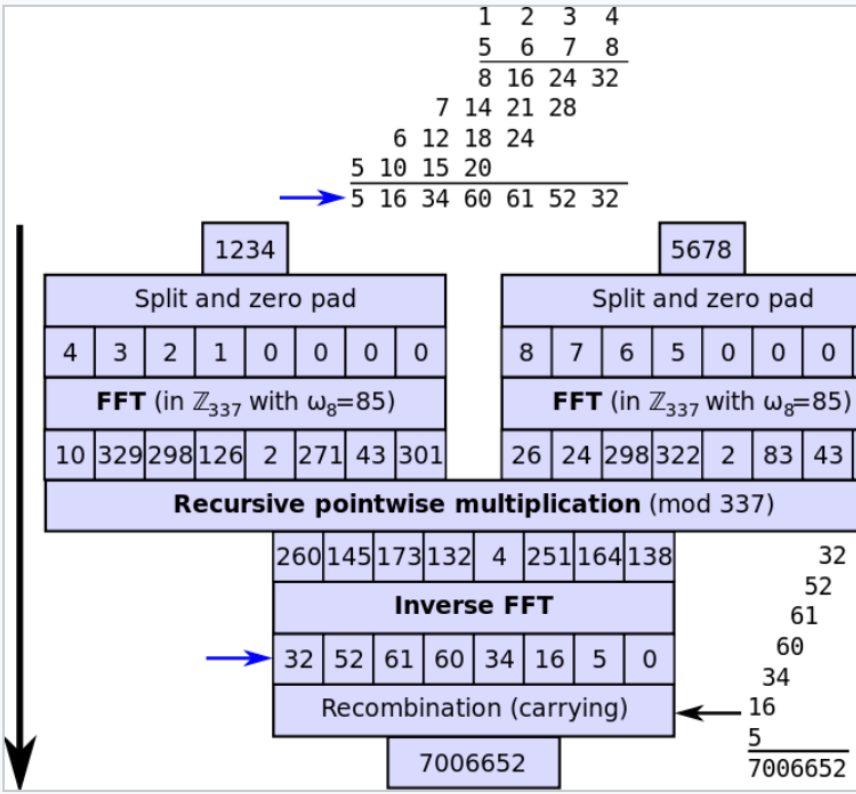
- Amusing exercise: generalize Karatsuba to do 5 size $n/3$ subproblems

  This gives $\Theta(n^{1.46\ldots})$ time algorithm

Date	Authors	Time complexity
<3000 BC	Unknown	$O(n^2)$
1962	Karatsuba	$O(n^{\log 3/\log 2})$
1963	Toom	$O(n\,2^{5\sqrt{\log n/\log 2}})$
1966	Schönhage	$O(n\,2^{\sqrt{2\log n/\log 2}}(\log n)^{3/2})$
1969	Knuth	$O(n\,2^{\sqrt{2\log n/\log 2}}\log n)$
1971	Schönhage–Strassen	$O(n\log n\log\log n)$
2007	Fürer	$O\left(n\log n\,2^{O(\log^* n)}\right)$
2014	Harvey-Hoeven-Lecerf	$O(n\log n\,8^{\log^* n})$

Still open problem.

Integer Multiplication (Summary)



Demonstration of multiplying 1234 × 5678 = 7006652 using fast Fourier transforms (FFTs). Number-theoretic transforms in the integers modulo 337 are used, selecting 85 as an 8th root of unity. Base 10 is used in place of base 2^w for illustrative purposes.

Matrix Multiplication

Multiplying Matrices

Let A be an $n \times m$ matrix, B be an $m \times p$ matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

Then, $C = AB$ is an $n \times p$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj},$$

Question: Why matrix multiplication is defined in such way?

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \bullet \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right]$$

A_{11} A_{12} A_{21} A_{22} B_{11} B_{12} B_{21} B_{22}

$$= \left[\begin{array}{cc|cc} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ \hline a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{array}\right]$$

$A_{11}B_{11}+A_{12}B_{21}$ $A_{11}B_{12}+A_{12}B_{22}$ $A_{21}B_{11}+A_{22}B_{21}$ $A_{21}B_{12}+A_{22}B_{22}$

Simple Divide and Conquer

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$$

- $T(n) = 8T(n/2) + 4\left(\frac{n}{2}\right)^2 = 8T(n/2) + n^2$

  So, $T(n) = \Theta\left(n^{\log_2 8}\right) = \Theta(n^3)$

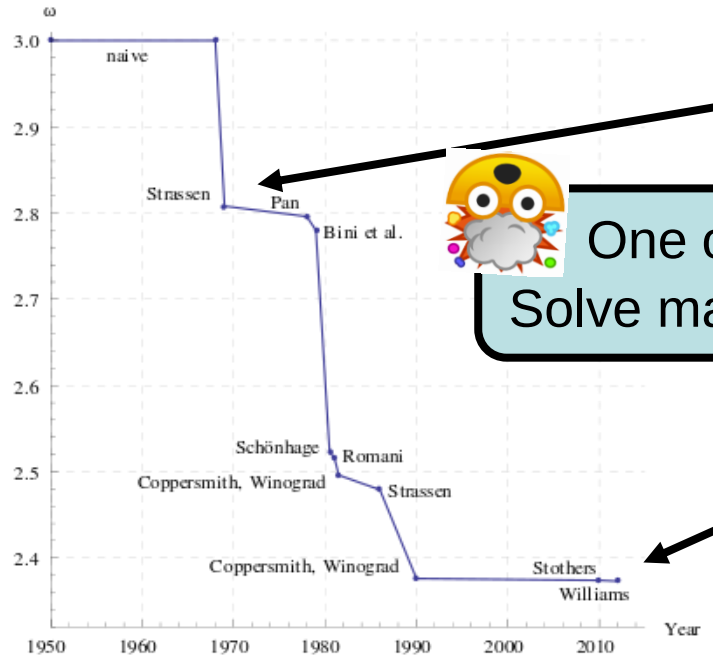
Strassen's Divide and Conquer Algorithm

- Strassen's algorithm

  Multiply 2×2 matrices using 7 instead of 8 multiplications (and 18 additions)

$$T(n) = 7T\left(\frac{n}{2}\right) + 18n$$

  Hence, we have $T(n) = O\left(n^{\log_2 7}\right)$.



Useful when $n \sim 500$.

One of the most important open problem:
Solve matrix multiplication in $O(n^2 \log^{O(1)} n)$ time

I am curious how large n need?

Strassen's Divide and Conquer Algorithm

Naive

$$\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$$
$$\mathbf{C}_{2,1} = \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{2,2} = \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}$$

Strassen

$$\mathbf{M}_1 := (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2})$$
$$\mathbf{M}_2 := (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1}$$
$$\mathbf{M}_3 := \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2})$$
$$\mathbf{M}_4 := \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1})$$
$$\mathbf{M}_5 := (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2}$$
$$\mathbf{M}_6 := (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2})$$
$$\mathbf{M}_7 := (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})$$

$$\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7$$
$$\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5$$
$$\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4$$
$$\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6$$

How did Strassen come up with his matrix multiplication method?

Stackexchange: I've been told no-one really knows, anything would be mainly speculation.