CSE 421

Greedy Algorithms / Minimum Spanning Tree

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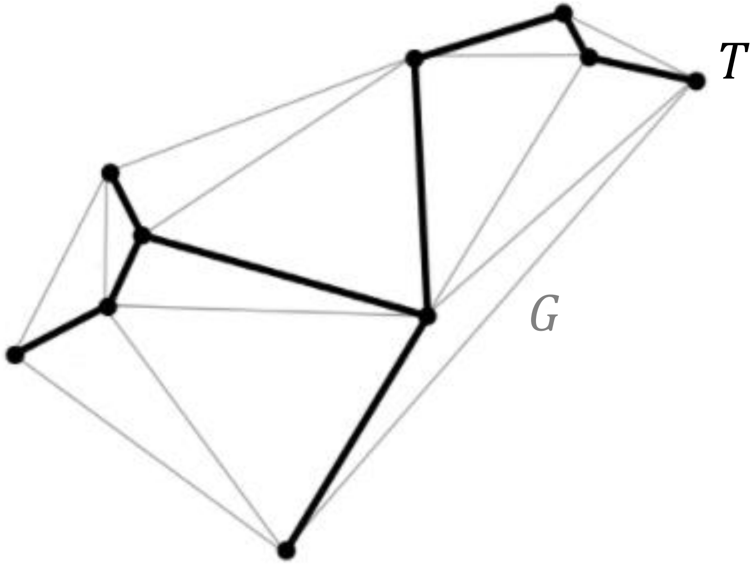
Spanning Tree

Given a connected undirected graph $G = (V, E)$.

We call T is a spanning tree of G if

- All edges in T are from E.
- T includes all of the vertices of G.



Why spanning tree?

Many problems is easy for tree.

General framework:

- Approximate the graph by a tree.
- Solve the problem on a tree.

We have covered different tree:

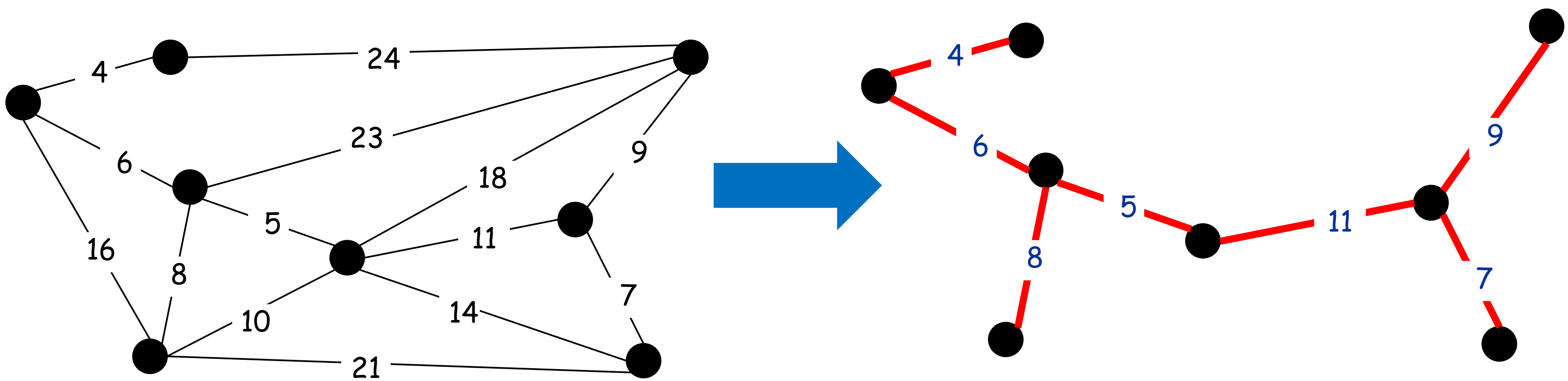
- BFS tree / Dijkstra tree
  - Remember all shortest paths from s.
- DFS tree
  - Compute lowest common ancestor.

There are many other different trees depending on applications.

Minimum Spanning Tree (MST)

Given a connected undirected graph G with weights $c_e \geq 0$.

An MST T is a spanning tree whose sum of edge weights is minimized.



$$G = (V, E)$$

$$c(T) = \sum_{e \in T} c_e = 50$$

Applications: Network design, …

Kruskal's Algorithm [1956]

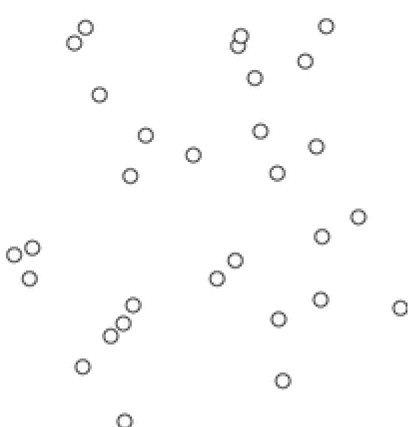
Kruskal(G, c) {
    Sort edges weights so that $c_1 \leq c_2 \leq \cdots \leq c_m$.
    $T \leftarrow \emptyset$

    foreach $(u \in V)$ make a set containing singleton $\{u\}$

    for $i = 1$ to m
        Let $(u, v) = e_i$
        if (u and v are in different sets) {
            $T \leftarrow T \cup \{e_i\}$
            merge the sets containing u and v
        }
    return T
}

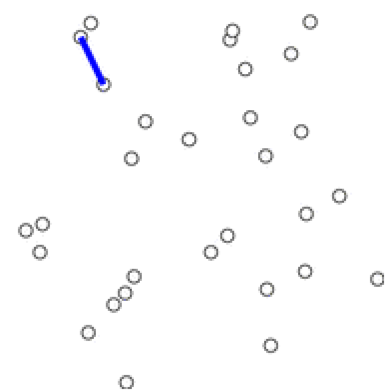
Question:
How to find maximum spanning tree?

Kruskal



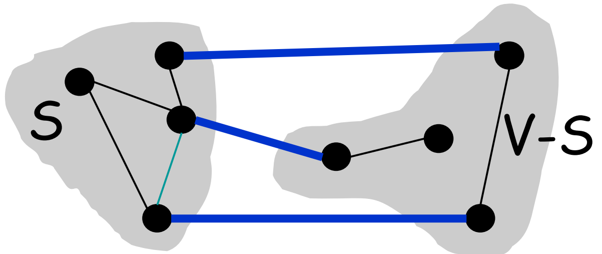
Sort edges weight.
Add edges whenever it does not create cycle.

Prim



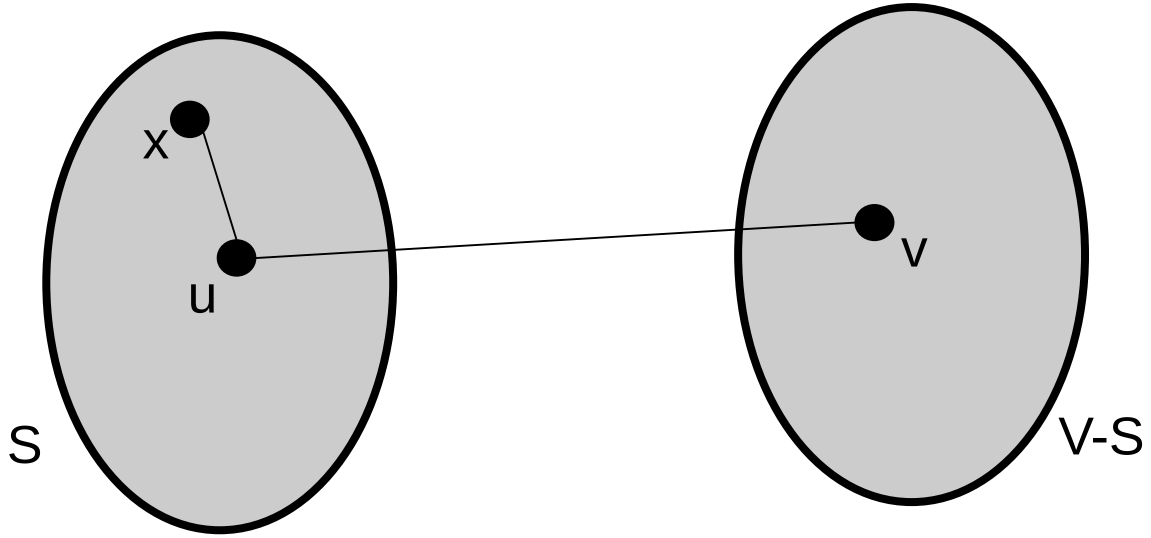
add the cheapest edge from the tree to another vertex.

Cuts



In a graph $G = (V, E)$, a cut is a bipartition of V into disjoint sets $S, V - S$ for some $S \subseteq V$. We write it as $(S, V - S)$.

An edge $e = \{u, v\}$ is in the cut $(S, V - S)$ if exactly one of u, v is in S.

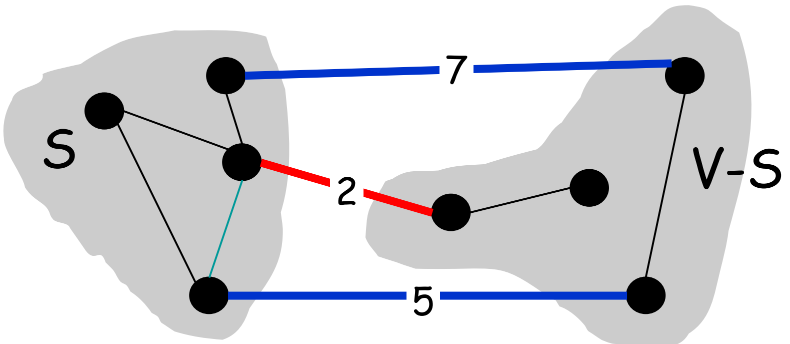


Properties of the OPT

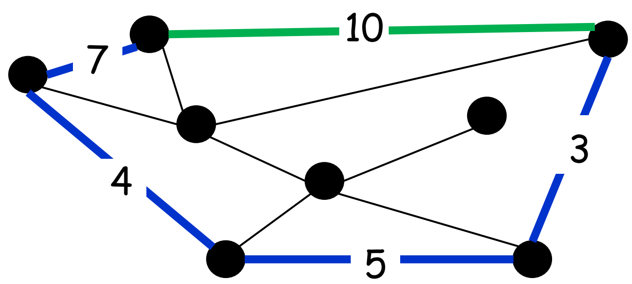
Simplifying assumption: All edge costs c_e are distinct.

Cut property: Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



red edge is in the MST

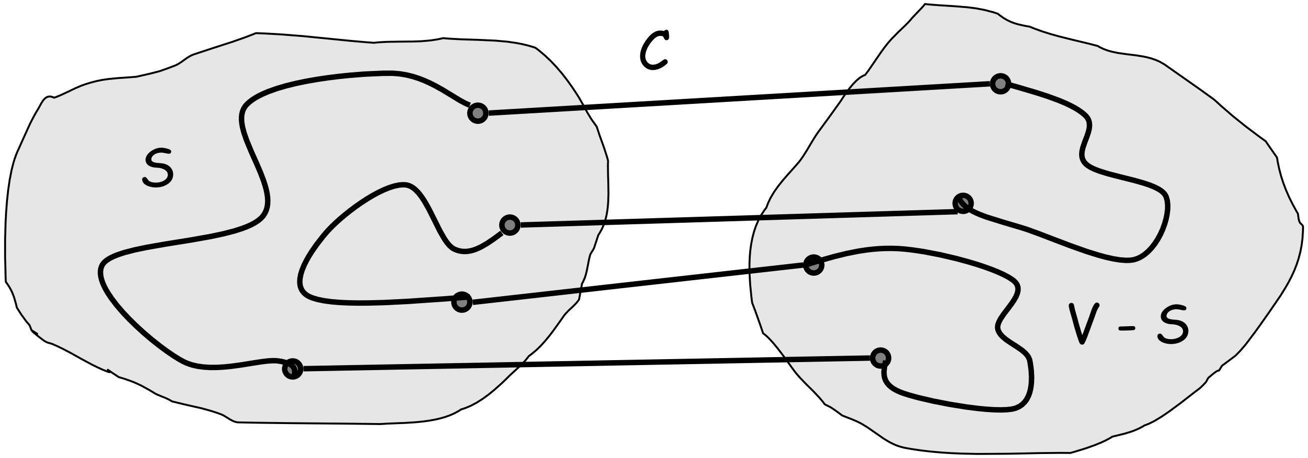


Green edge is not in the MST

Cycles and Cuts

Claim. A cycle crosses a cut (from S to $V - S$) an even number of times.

Proof. (by picture)



Every time the cycle crosses a cut, it goes from S to $V - S$ or from $V - S$ to S.

Cut Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the T^* contains e.

Proof. By contradiction

Suppose $e = \{u, v\}$ does not belong to T^*.

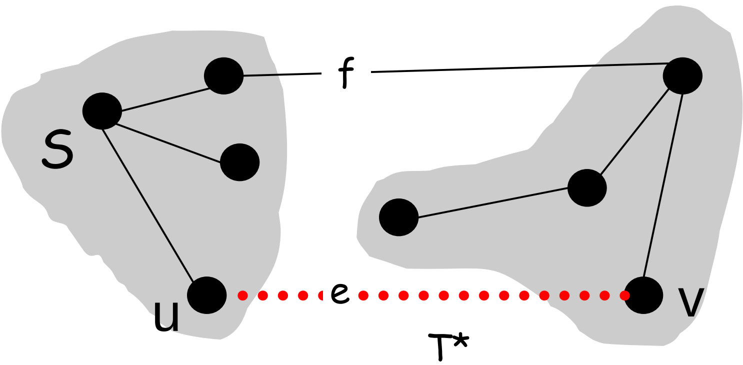
Adding e to T^* creates a cycle C in T^*. (coz all tree has $n - 1$ edges)

There is a path from u to v in T^* $\Rightarrow$ there exists another edge, say f, that leaves S.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Cycle Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T^* does not contain f.

Proof. By contradiction

Every connected graph has a spanning tree.
Hence it has at least $n-1$ edges.

Suppose f belongs to T^*.

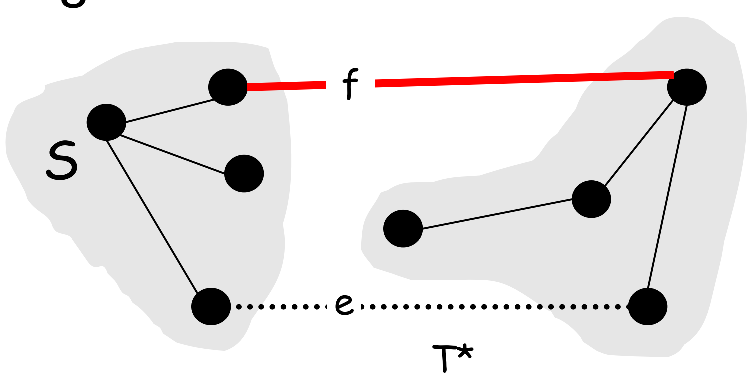
Deleting f from T^* cuts T^* into two connected components.

There exists another edge, say e, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Proof of Correctness (Kruskal)

Consider edges in ascending order of weight.

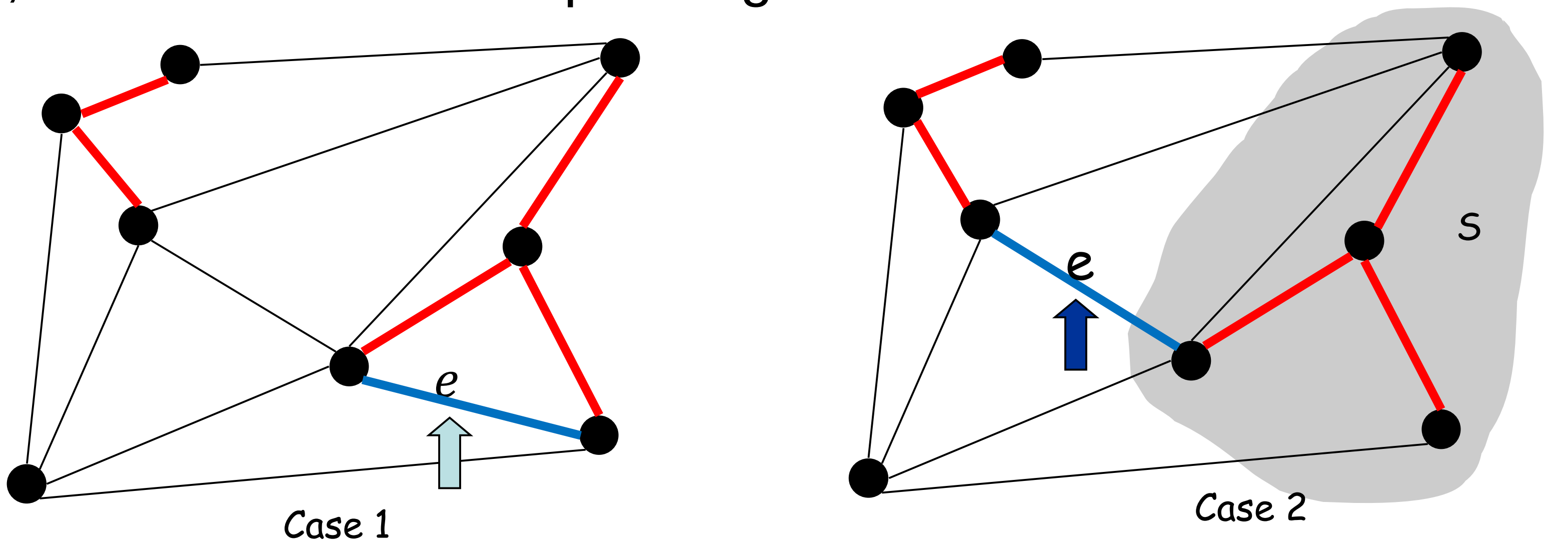
Case 1: adding e to T creates a cycle,

e is the maximum weight edge in that cycle.

cycle property show e is not in any minimum spanning tree.

Case 2: $e = (u, v)$ is the minimum weight edge in the cut S where S is the set of nodes in u's connected component.

So, e is in all minimum spanning tree.



This proves MST is unique if weights are distinct.

Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain a set for each connected component.
- O($m \log n$) for sorting and O($m \log n$) for union-find

```
Kruskal(G, c) {
   Sort edges weights so that c_1 ≤ c_2 ≤ ··· ≤ c_m.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

   for i = 1 to m
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         merge the sets containing u and v
      }
   return T
}
```

Union Find Data Structure

The goal is to have a data structure to tracks a set of elements partitioned into disjoint subsets.

Initially, we have {{1},{2},{3},{4},⋯}.

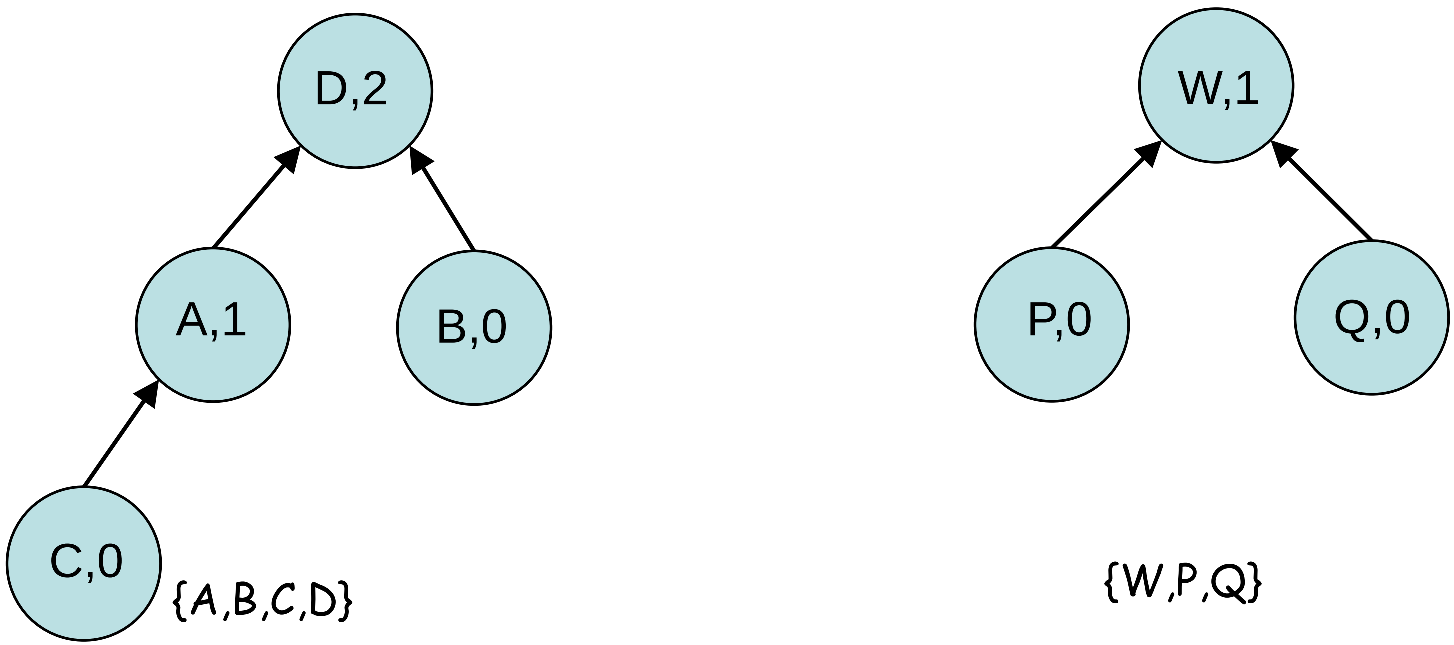
We need two operations:

- Union(S_1, S_2): Merge subsets S_1 and S_2 into one.
- Find(x): Output the subset S containing x.

Find

Each set is represented as a tree of pointers, where every vertex is labeled with longest path ending at the vertex

To check whether A,Q are in same connected component, follow pointers and check if root is the same.

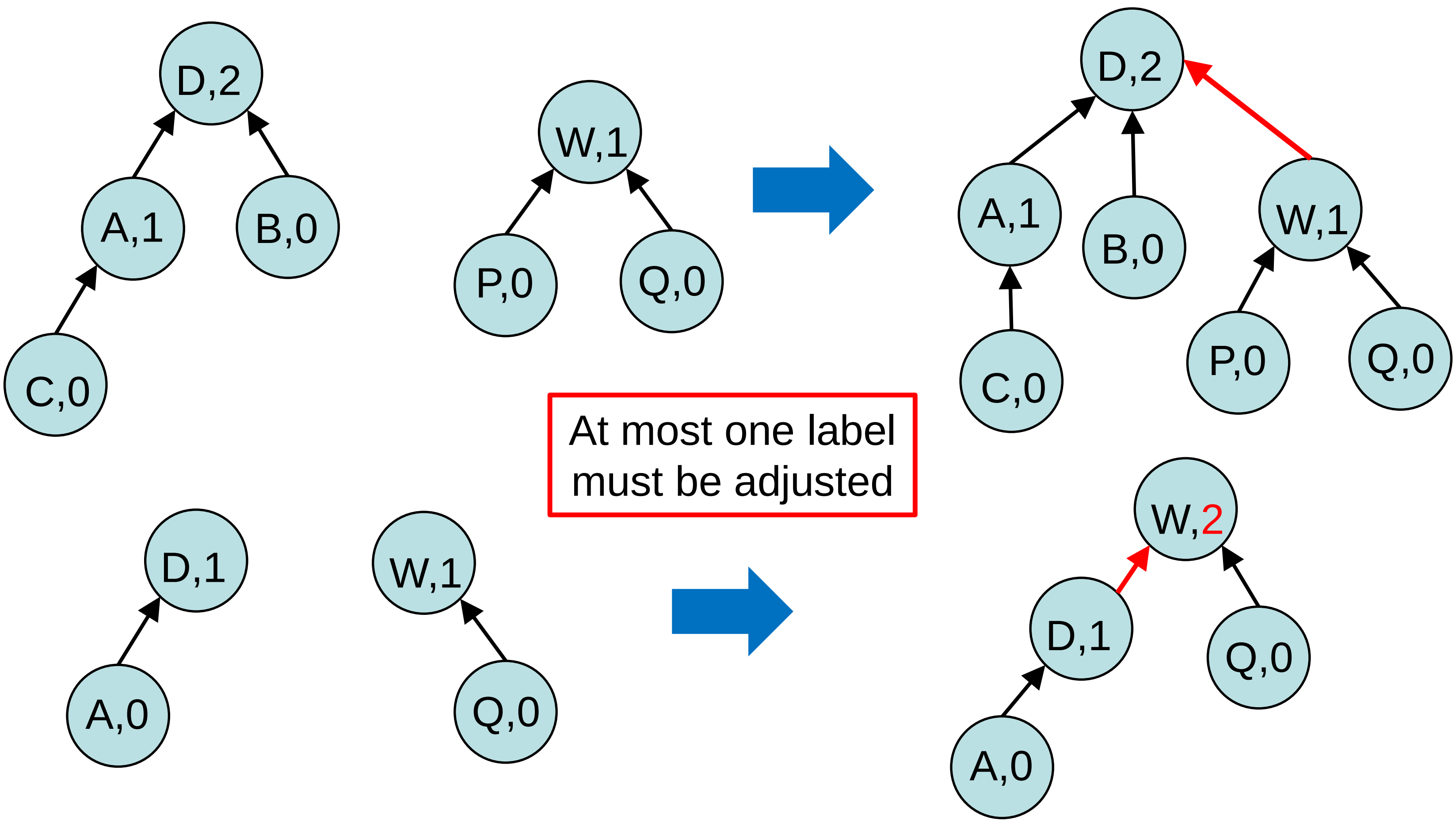


So, the cost of $Find(x)$ is at most height of the set.

Union

You can get the same result by “Union by size”

Union: To merge two connected components, make the root with the smaller label point to the root with the bigger label (adjusting labels if necessary). Runs in $O(1)$ time



Depth vs Size: Correctness

Claim: If the label of a root is k, there are at least 2^k elements in the set.

Proof: By induction on k.

Base Case ($k = 0$): this is true. The set has size 1.

Inductive Step: If we merge roots with labels $k_1 > k_2$, the number of vertices only increases while the label (height) stays the same.

If $k_1 = k_2$, the merged tree has label (height) $k_1 + 1$,

and by induction, it has at least

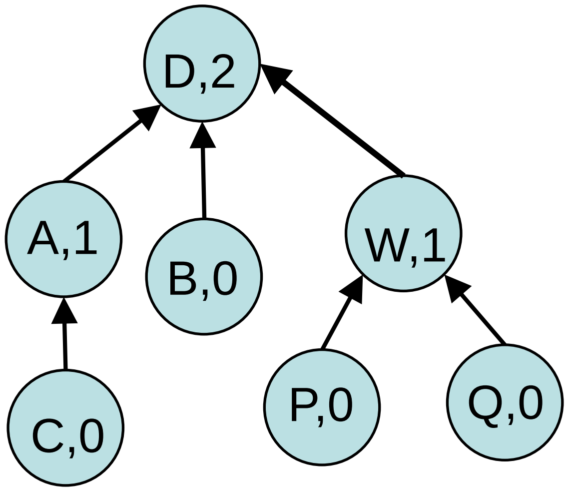
$$2^{k_1} + 2^{k_2} = 2^{k_1+1}$$

elements.

Depth vs Size

Claim: If the label of a root is k, there are at least 2^k elements in the set.

Therefore, the depth of any tree in algorithm is at most $\log_2 n$



So, we can check if u, v are in the same component in time $O(\log n)$

Kruskal's Algorithm with Union Find

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain a set for each connected component.
- O(m log n) for sorting and O(m log n) for union-find

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}
```

Find roots and compare

Merge at the roots

HW3 Extra

5. **Extra Credit:** Let $[n] = \{1, 2, 3, \cdots, n\}$ and $\mathcal{I}$ be a collection of subsets of $[n]$. We call any set $I \in \mathcal{I}$ is nice.

   We know that $\mathcal{I}$ satisfy two main axioms:

   (a) If $X \subset Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$. Namely, any subset of a nice set is nice.

   (b) If $X \in \mathcal{I}$, $Y \in \mathcal{I}$ and $|Y| > |X|$, then there exists $i \in Y \backslash X$ such that $X \cup \{i\} \in \mathcal{I}$. Namely, if X is nice and there exists a larger nice set Y, then X can be extended to a larger nice set by adding an element of $Y \backslash X$.

   The collection $\mathcal{I}$ may have exponentially size and is only defined implicitly. However, we assume that we can test if a set I is nice or not in polynomial time.

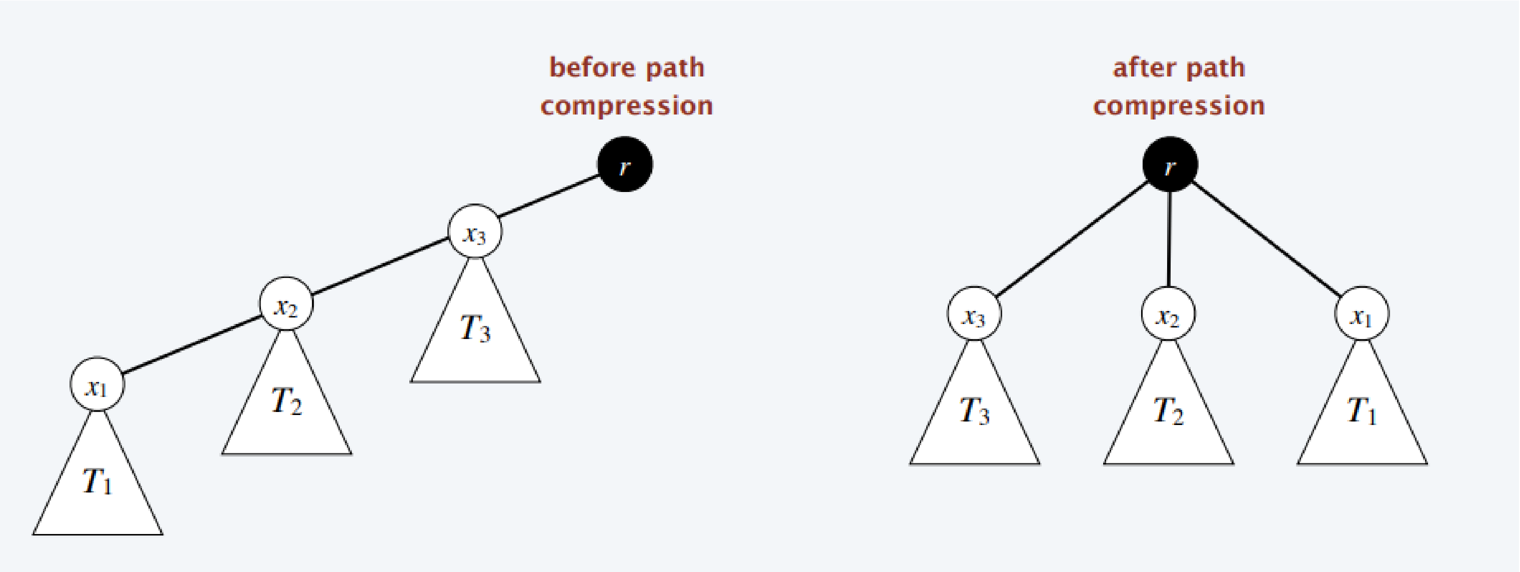
   Given a cost $c_1, c_2, c_3, \cdots, c_n$, design a greedy polynomial time algorithm to find a nice set X with maximum total cost $c(X) = \sum_{x \in X} c_x$.

- Let $[n]$ be all edges of G.
- Let I be the collection of subsets of edges without cycles.
- Any nice set $X \in I$ is a forest.
- Maximum $c(X)$ over nice set X is same as finding maximum spanning tree.
- The answer for this question is basically Kruskal’s Algorithm

Find: Path Compression

After finding the root r of the tree containing x,
Change the parent points of all nodes along the path to r directly.

```
Find(x) {
    If x ≠ parent(x)
        parent(x) ← Find(parent(x))
    return parent(x)
}
```



Union by height/size + Path Compression

In 1972, Fischer proved this it takes $O(n \log \log n)$ time to do $O(n)$ find and $n - 1$ union.

EFFICIENCY OF EQUIVALENCE ALGORITHMS

Michael J. Fischer

In 1973, Hopcroft-Ullman improve the bound to $O(n \log^* n)$.

SET MERGING ALGORITHMS*

J. E. HOPCROFT† AND J. D. ULLMAN‡

In 1975, Tarjan improved the bound to $O(n\,\alpha(n))$.

Efficiency of a Good But Not Linear Set Union Algorithm

ROBERT ENDRE TARJAN

University of California, Berkeley, California

Finally, in 1989, Fredman-Saks proved this is optimal.

The Cell Probe Complexity of Dynamic Data Structures

Michael L. Fredman [1]

Bellcore and
U.C. San Diego

Michael E. Saks [2]

U.C. San Diego,
Bellcore and
Rutgers University

What is $\alpha(n)$?

Inverse Ackermann function $\alpha(n)$.

Define $\alpha(n) = \min\{k : \alpha_k(n) \le 3\}$.
Any number n I can write down satisfies $\alpha(n) \le 3$.

Define

- $\alpha_1(n) = \lceil n/2 \rceil$.
- $\alpha_2(n) = \lceil \lg n \rceil$ = # of times we divide n by two until we reach 1.
- $\alpha_3(n) = \lg^* n$ = # of times we apply lg until we reach 1.
- $\alpha_4(n)$ = # of times we apply $\lg^*$ until we reach 1.

$$2 \uparrow 65536 = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{65536 \text{ times}}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...	2^{16}	...	2^{65536}	...	$2 \uparrow 65536$
$\alpha_1(n)$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	...	2^{15}	...	2^{65535}	...	*huge*
$\alpha_2(n)$	1	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	...	16	...	65536	...	$2 \uparrow 65535$
$\alpha_3(n)$	1	1	2	2	3	3	3	3	3	3	3	3	3	3	3	3	...	4	...	5	...	65536
$\alpha_4(n)$	1	1	2	2	3	3	3	3	3	3	3	3	3	3	3	3	...	3	...	3	...	4