CSE421: Introduction to Algorithms

Homework 7

Due: November 14, 2018

1. Input: We are given a set of n food items, each item i has a_i units of vitamin A, b_i units of vitamin B, c_i units of vitamin C, and k_i units of calories. We are also given the target intake units of vitamin A, B, and C, denoted by α, β, and γ respectively. Assume that all a_i, b_i, c_i and α, β, γ are **non-negative integers**.

   Output: A subset $S \subseteq \{1, 2, \cdots, n\}$ of food items with minimum total calories that fulfills the target intakes of each vitamin. More formally, your algorithm should return a subset S that minimizes $\sum_{i \in S} k_i$ satisfying the constraints $\sum_{i \in S} a_i \geq \alpha$, $\sum_{i \in S} b_i \geq \beta$ and $\sum_{i \in S} c_i \geq \gamma$. You can assume that such a set always exists.

   Give an algorithm to solve this problem. Note that your algorithm should return an optimal subset, not just the optimal value. The time complexity should depend polynomially on n, α, β, γ.

2. Consider the Gerrymandering problem:[1] Suppose we have a set of n precincts $P_1, P_2, \cdots, P_n$, each containing m registered voters. We're supposed to divide these precincts into two districts, each consisting of $n/2$ of the precincts. Now, for each precinct, we have information on how many voters are registered to each of two political parties. We'll say that the set of precincts is susceptible to gerrymandering if it is possible to perform the division into two districts in such a way that the same party holds a majority in both districts.

   Give an algorithm to determine whether a given set of precincts is susceptible to gerrymandering; the running time of your algorithm should be polynomial in n and m

   Example: Suppose we have $n = 4$ precincts, and the following information on registered voters: Precinct 1 has 55 voters for party A, 45 voters for party B; Precinct 2 has 43 for A, 57 for B; Precinct 3 has 60 for A, 40 for B; Precinct 4 has 47 for A, 53 for B.

   This set of precincts is susceptible since, if we grouped precincts 1 and 4 into one district, and precincts 2 and 3 into the other, then party A would have a majority in both districts.

3. You are given a rectangular piece of cloth with dimensions $X \times Y$, where X and Y are positive integers, and a list of n products that can be made using the cloth. For each product i you know that a rectangle of cloth of dimensions $a_i \times b_i$ is needed and that the selling price of the product is c_i. Assume the a_i, b_i and c_i are all positive integers. You have a machine that can cut any rectangular piece of cloth into two pieces either horizontally or vertically. Design an algorithm that runs in time that is polynomial in X, Y, n and determines the best return on the $X \times Y$ piece of cloth, that is, a strategy for cutting the cloth so that the products made from the resulting pieces give the maximum sum of selling prices. You are free to make as many copies of a given product as you wish, or none, if desired.

4. **Extra Credit:** Let $T(n)$ be a function such that $T(n) = T(n-1) + T(\lfloor \frac{n}{2} \rfloor) + O(n^{2018})$ and that $T(1) = 1$. Show that $T(n) = 2^{\Theta(\log n^2)}$.

[1] Now, you have an excuse to watch John Oliver (https://www.youtube.com/watch?v=A-4dIImaodQ) while doing homework.