CSE421: Introduction to Algorithms	September 25, 2018
Homework 1	
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Please see https://courses.cs.washington.edu/courses/cse421/18au/grading.html for general guidelines about Homework problems. Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details.

1. If m is the first man on the woman w's preference list and w is the first woman on m's preference list, does it have to be the case that m and w must be matched to each other in every stable matching?

2. (Kleinberg and Tardos, Chapter 1, Problem 4) Gale and Shapley published their paper on the stable marriage problem in 1962; but a version of their algorithm had already been in use for ten years by the National Resident Matching Program, for the problem of assigning medical residents to hospitals.

   Basically, the situation was the following. There were m hospitals, each with a certain number of available positions for hiring residents. There were n medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there were more students graduating than there were slots available in the m hospitals.

   The interest, naturally, was in finding a way of assigning each student to at most one hospital, in such a way that all available positions in all hospitals were filled. (Since we are assuming a surplus of students, there would be some students who do not get assigned to any hospital.) We say that an assignment of students to hospitals is stable if neither of the following situations arises.

   - **First type of instability:** There are students s and s', and a hospital h, so that s is assigned to h, and s' is assigned to no hospital, and h prefers s' to s.
   - **Second type of instability:** There are students s and s', and hospitals h and h', so that
     - s is assigned to h, and
     - s' is assigned to h', and
     - h prefers s' to s, and s' prefers h to h'.

   So we basically have the stable marriage problem from class, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

   Show that there is always a stable assignment of students to hospitals, and give an efficient algorithm to find one.

3. (Kleinberg and Tardos, Chapter 1, Problem 7) Here is the setup. There are n input wires and n output wires, each directed from a source to a terminus. Each input wire meets each

output wire in exactly one distinct point, at a special piece of hardware called a junction box. Points on the wire are naturally ordered in the direction from source to terminus; for two distinct points x and y on the same wire, we say that x is upstream from y if x is closer to the source than y, and otherwise we say x is downstream from y. The order in which one input wire meets the output wires is not necessarily the same as the order in which another input wire meets the output wires. (And similarly for the orders in which output wires meet input wires.) Figure 1.8 gives an example of such a collection of input and output wires.

Now, here's the switching component of this situation. Each input wire is carrying a distinct data stream, and this data stream must be switched onto one of the output wires. If the stream of Input i is switched onto Output j, at junction box B, then this stream passes through all junction boxes upstream from B on Input i, then through B, then through all junction boxes downstream from B on Output j. It does not matter which input data stream gets switched onto which output wire, but each input data stream must be switched onto a different output wire. Furthermoreand this is the tricky constraintno two data streams can pass through the same junction box following the switching operation.

Finally, here's the problem. Show that for any specified pattern in which the input wires and output wires meet each other (each pair meeting exactly once), a valid switching of the data streams can always be foundone in which each input data stream is switched onto a different output, and no two of the resulting streams pass through the same junction box. Additionally, give an algorithm to find such a valid switching.

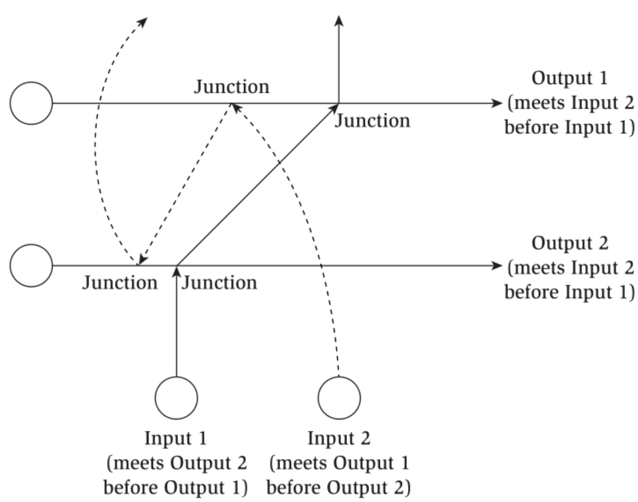

Figure 1.8 An example with two input wires and two output wires. Input 1 has its junction with Output 2 upstream from its junction with Output 1; Input 2 has its junction with Output 1 upstream from its junction with Output 2. A valid solution is to switch the data stream of Input 1 onto Output 2, and the data stream of Input 2 onto Output 1. On the other hand, if the stream of Input 1 were switched onto Output 1, and the stream of Input 2 were switched onto Output 2, then both streams would pass through the junction box at the meeting of Input 1 and Output 2—and this is not allowed.

4. **Extra Credit:** Prove that Gale-Shapley algorithm takes at most $n(n-1)+1$ iterations. Also, prove that this is the best possible bound. Namely, for every n, give an instance such that GS algorithm takes exactly $n(n-1)+1$ iterations.