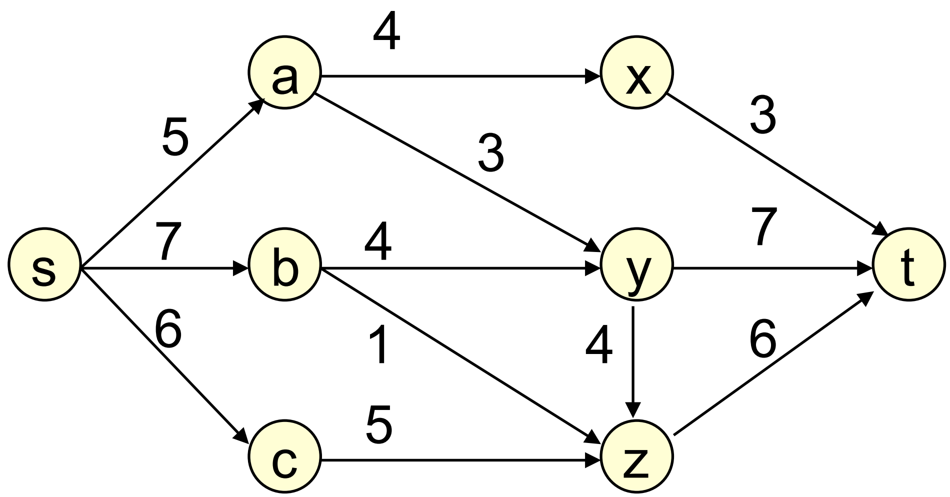
CSE 421
Introduction to Algorithms

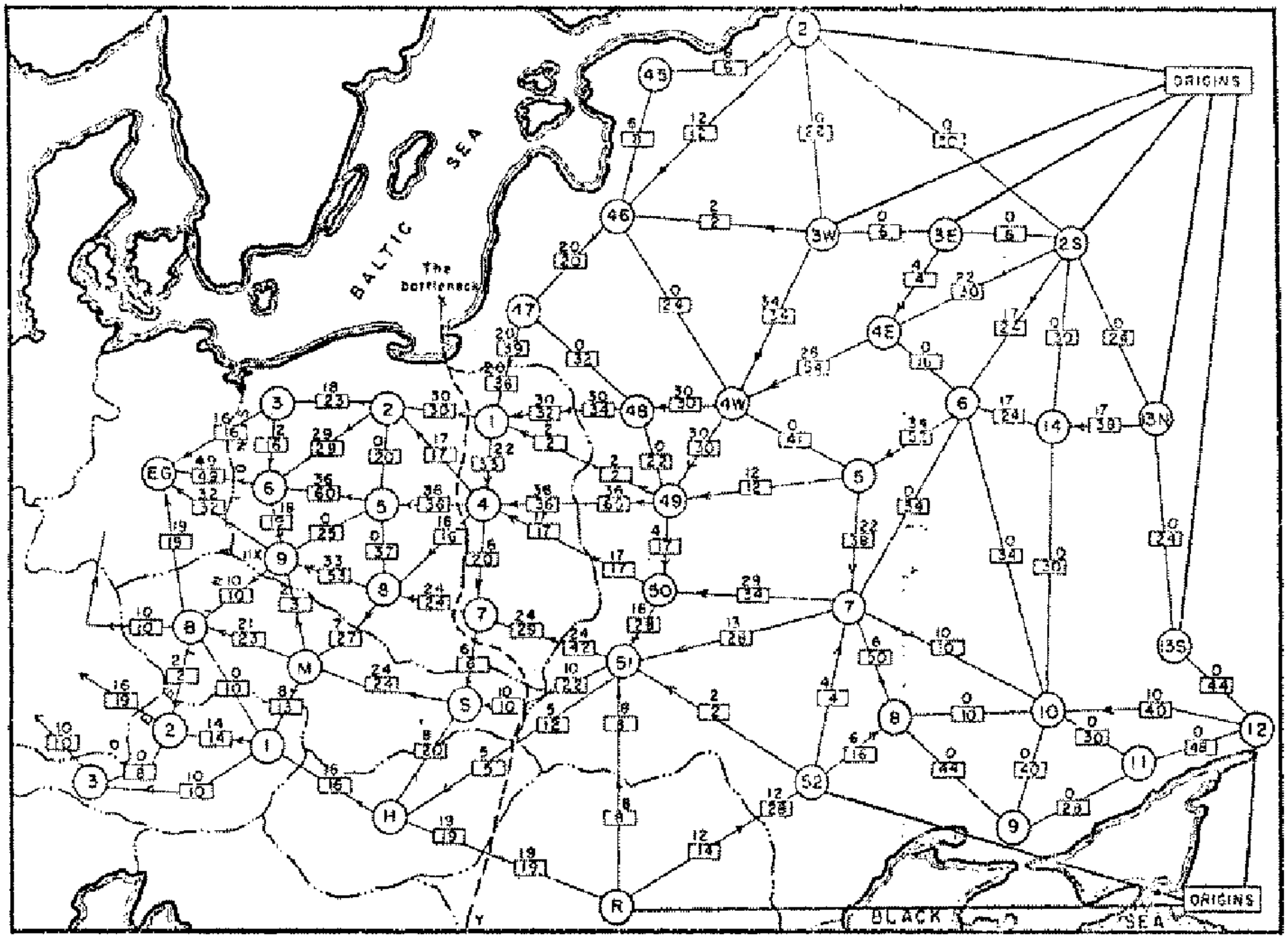
The Network Flow Problem

The Network Flow Problem



How much stuff can flow from *s* to *t*?

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems*.
Alexander Schrijver in Math Programming, 91: 3, 2002.

Net Flow: Formal Definition

Given:

A digraph $G = (V,E)$

Two vertices s,t in V
(s = source, t = sink)

A *capacity* $c(u,v) \geq 0$
for each $(u,v) \in E$
(and $c(u,v) = 0$ for all non-edges (u,v))

(technically, not quite the same definition as in the book…)

Find:

A *flow function* f: $V \times V \to R$ s.t., for all u,v:

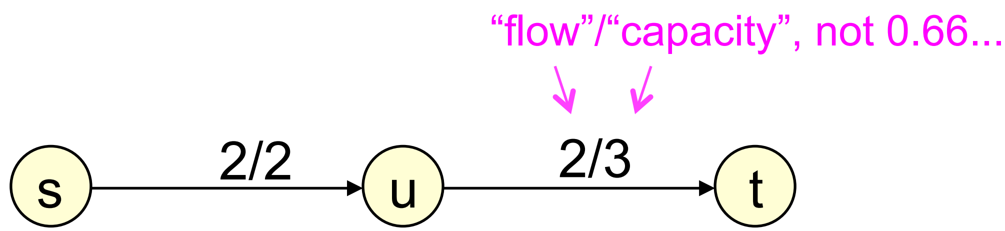
- $f(u,v) \leq c(u,v)$ [Capacity Constraint]
- $f(u,v) = -f(v,u)$ [Skew Symmetry]
- if $u \neq s,t$, $f(u,V) = 0$ [Flow Conservation]

Maximizing total flow $|f| = f(s,V)$

Notation:

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Example: A Flow Function



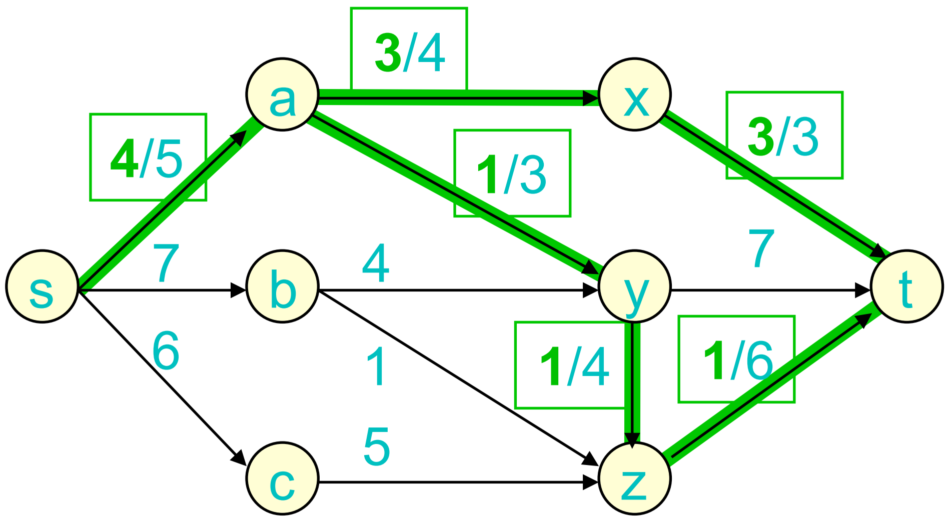
$$f(s,u) = f(u,t) = 2$$

$$f(u,s) = f(t,u) = -2$$ *(Why?)*

$$f(s,t) = -f(t,s) = 0$$ *(In every flow function for this G. Why?)*

$$f(u,V) = \sum_{v \in V} f(u,v) = f(u,s) + f(u,t) = -2 + 2 = 0$$

Example: A Flow Function



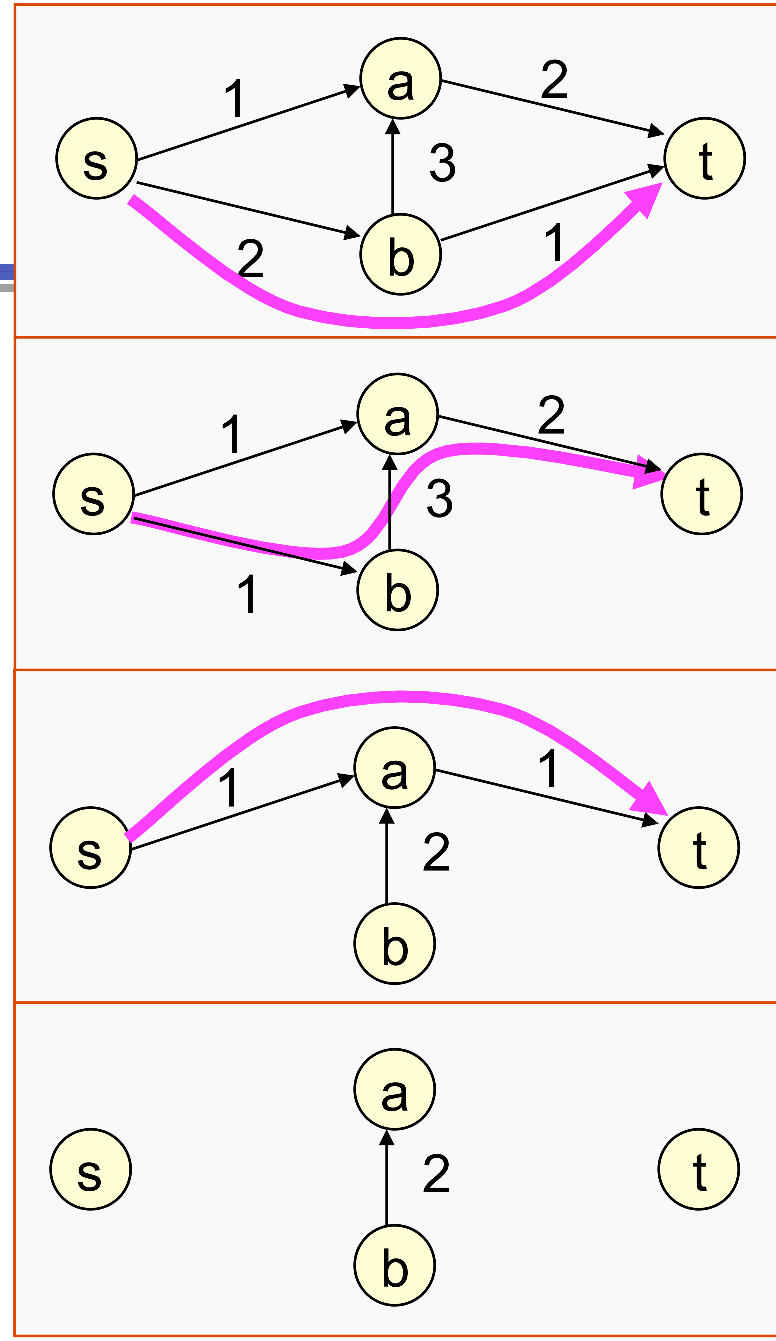
Not shown: $f(u,v)$ if ≤ 0

Note: max flow ≥ 4 since f is a flow, $|f| = 4$

Max Flow via a Greedy Alg?

While there is an $s \rightarrow t$ path in G

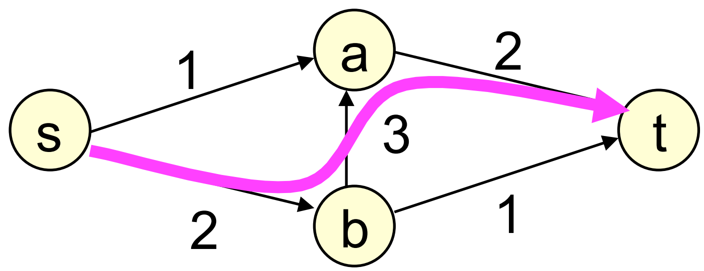
- Pick such a path, p
- Find c_p, the min capacity of any edge in p
- Subtract c_p from all capacities on p
- Delete edges of capacity 0

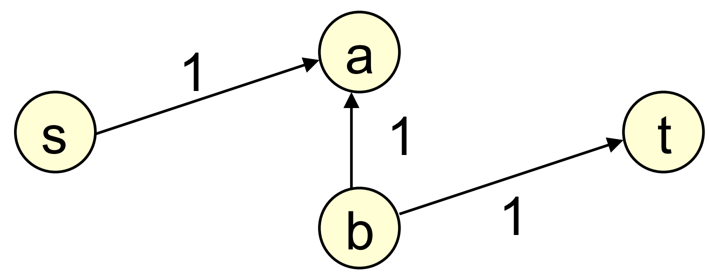


Max Flow via a Greedy Alg?

This does NOT always find a max flow:

If you pick $s \rightarrow b \rightarrow a \rightarrow t$ first,





Flow stuck at 2, but 3 possible (above).

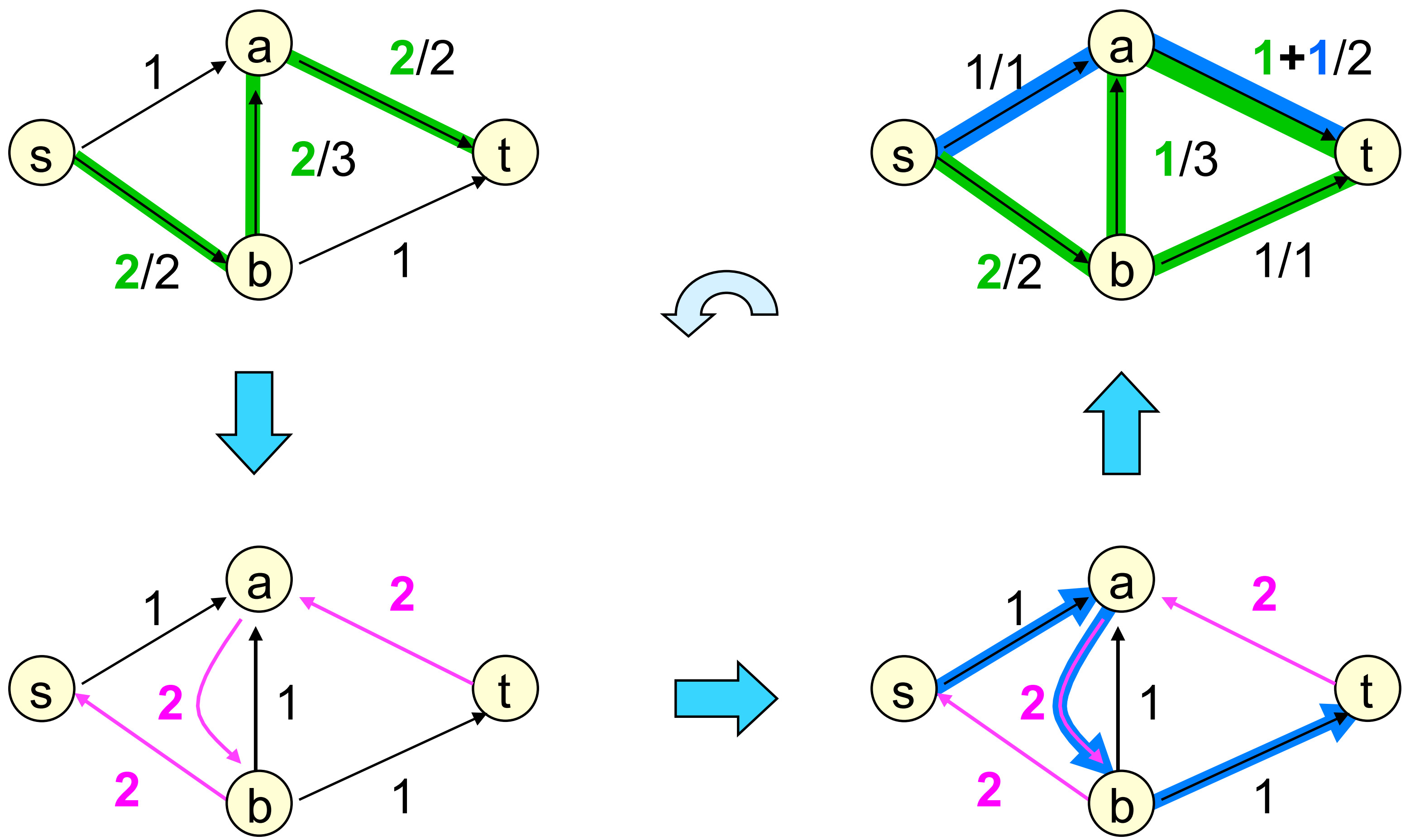
A Brief History of Flow

#	Year	Discoverer(s)	Bound
1	1951	Dantzig	$O(n^2 mC)$
2	1955	Ford & Fulkerson	$O(nmC)$
3	1970	Dinitz; Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2 m)$
5	1972	Edmonds & Karp; Dinitz	$O(m^2 \log C)$
6	1973	Dinitz;Gabow	$O(nm \log C)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2 \text{ sqrt}(m))$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg &Tarjan	$O(nm \log (n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log C)$
13	1987	Ahuja et al.	$O(nm \log(n \text{ sqrt}(\log C)/(m+2))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\varepsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\varepsilon} n)$
19	1994	King et al.	$O(nm(\log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log C)$; $O(n^{2/3} m \log(n^2/m) \log C)$
…	…	…	…

n = # of vertices
m= # of edges
C = Max capacity

Source: Goldberg & Rao, FOCS '97

Greed Revisited



Residual Capacity

The *residual capacity* (w.r.t. f) of (u,v) is
$c_f(u,v) = c(u,v) - f(u,v)$

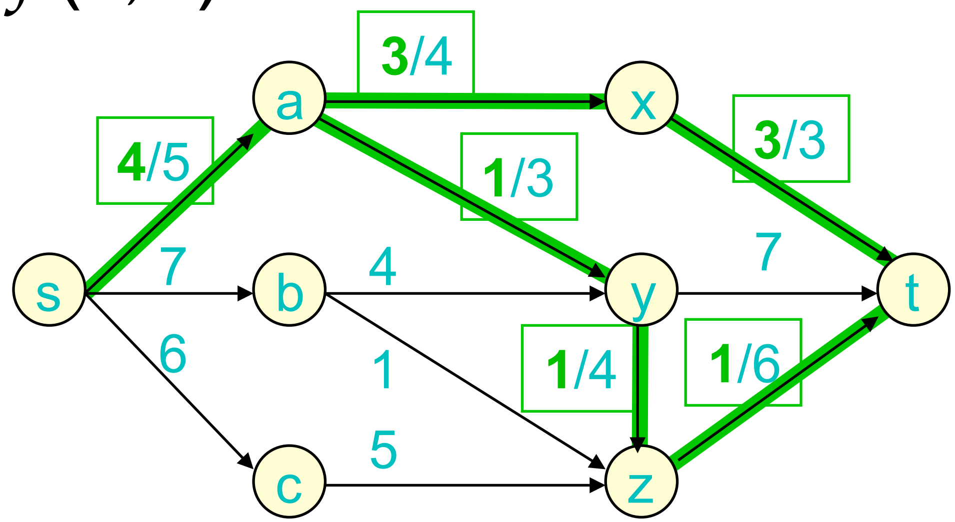
E.g.:

$c_f(s,b) = 7;$

$c_f(a,x) = 1;$

$c_f(x,a) = 3;$

$c_f(x,t) = 0$ (a saturated edge)



Residual Networks & Augmenting Paths

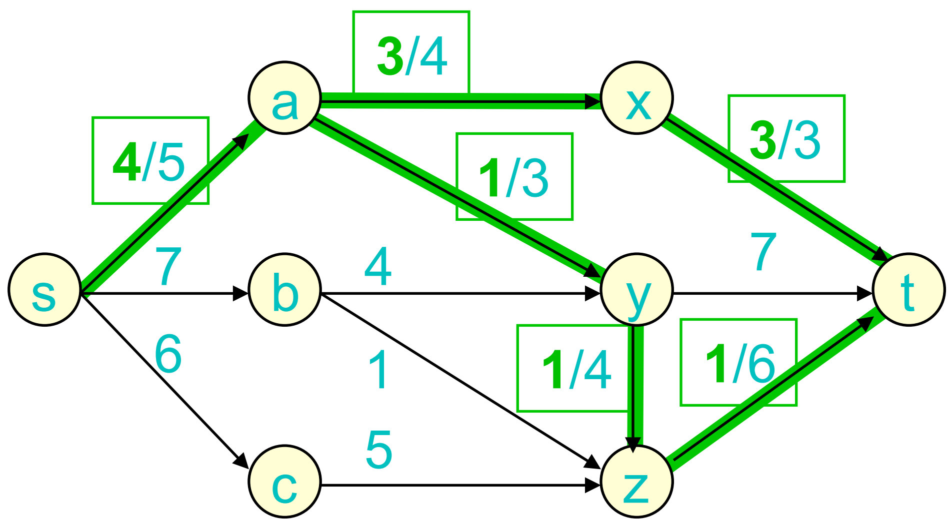
The *residual network* (w.r.t. f) is the graph $G_f = (V, E_f)$, where

$$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$$

An *augmenting path* (w.r.t. f) is a simple $s \rightarrow t$ path in G_f

i.e., acyclic

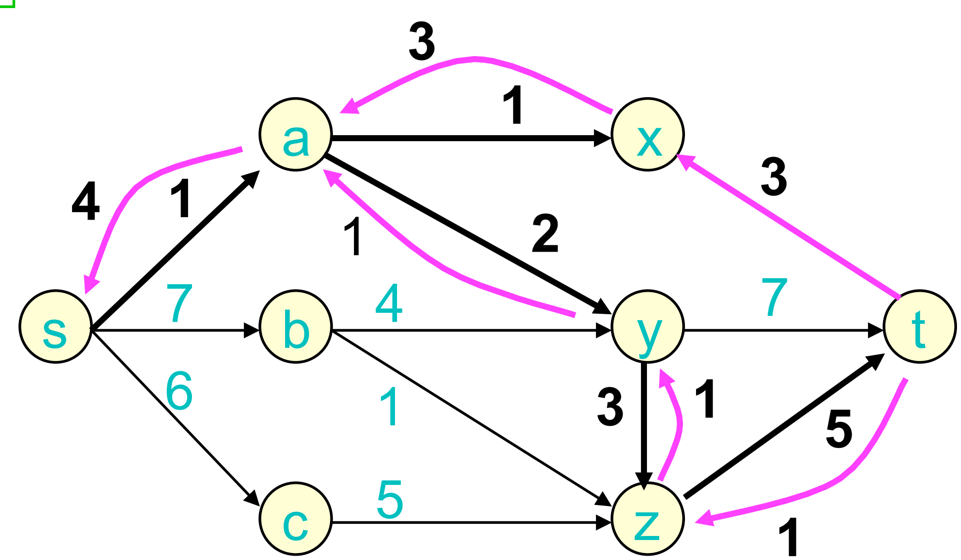
A Residual Network



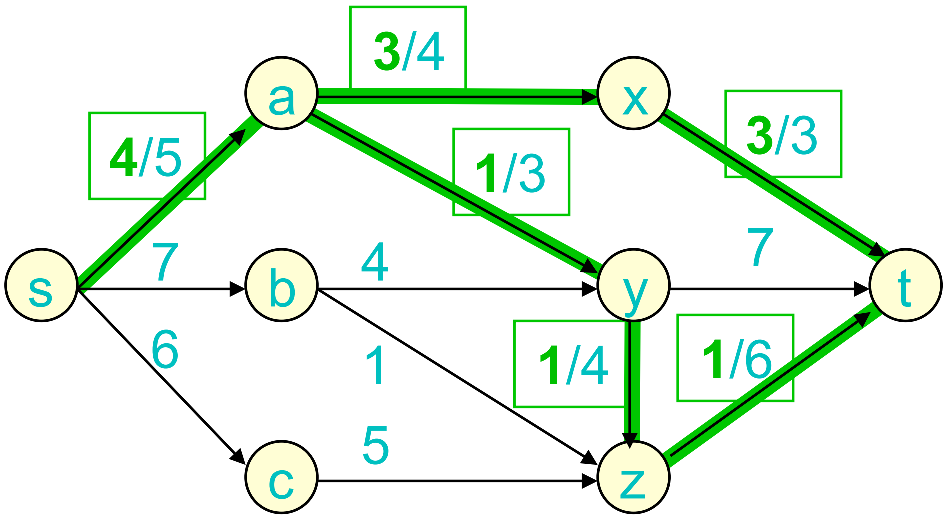
residual network: the graph

$G_f = (V, E_f)$, where

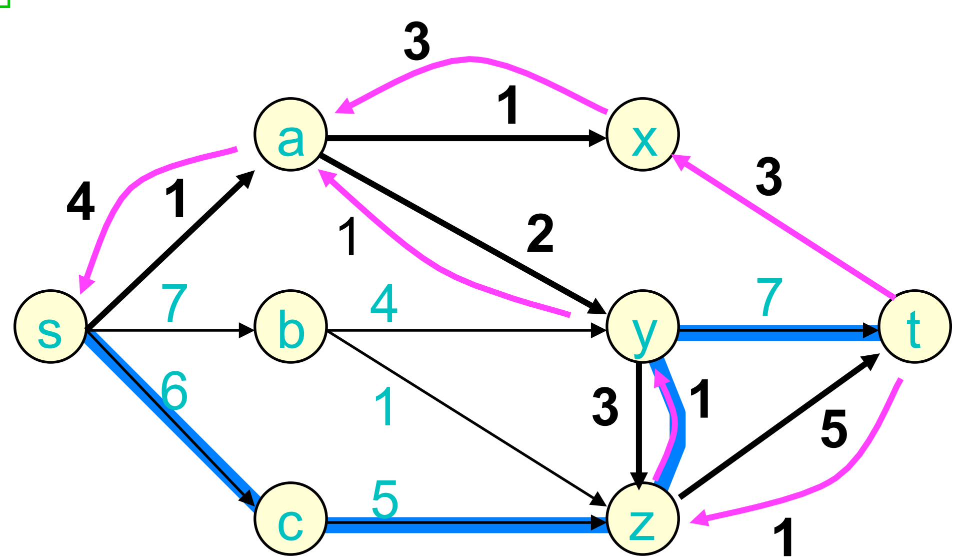
$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$



An Augmenting Path



augmenting path:
a simple $s \rightarrow t$ path in G_f

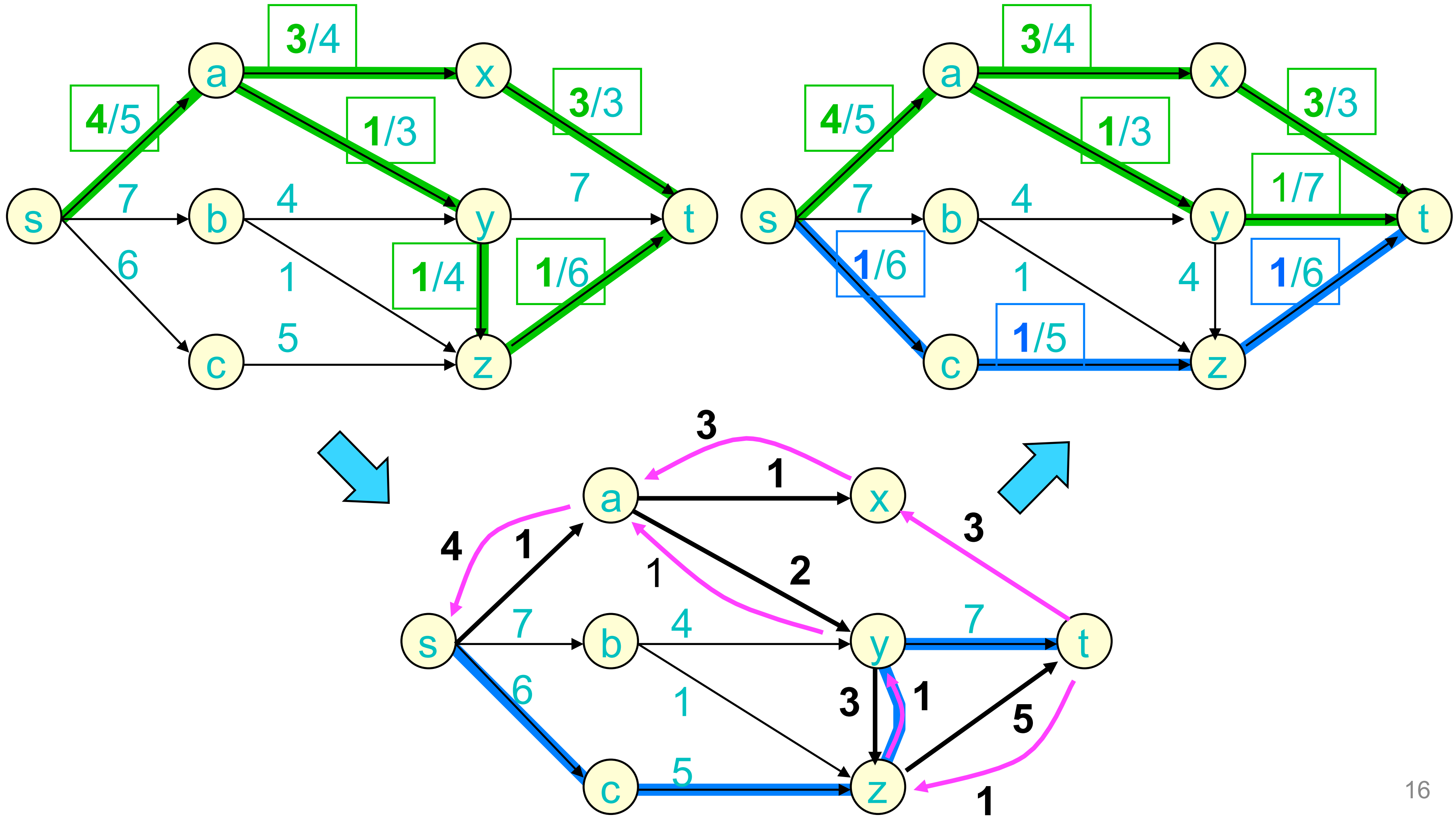


Lemma 1

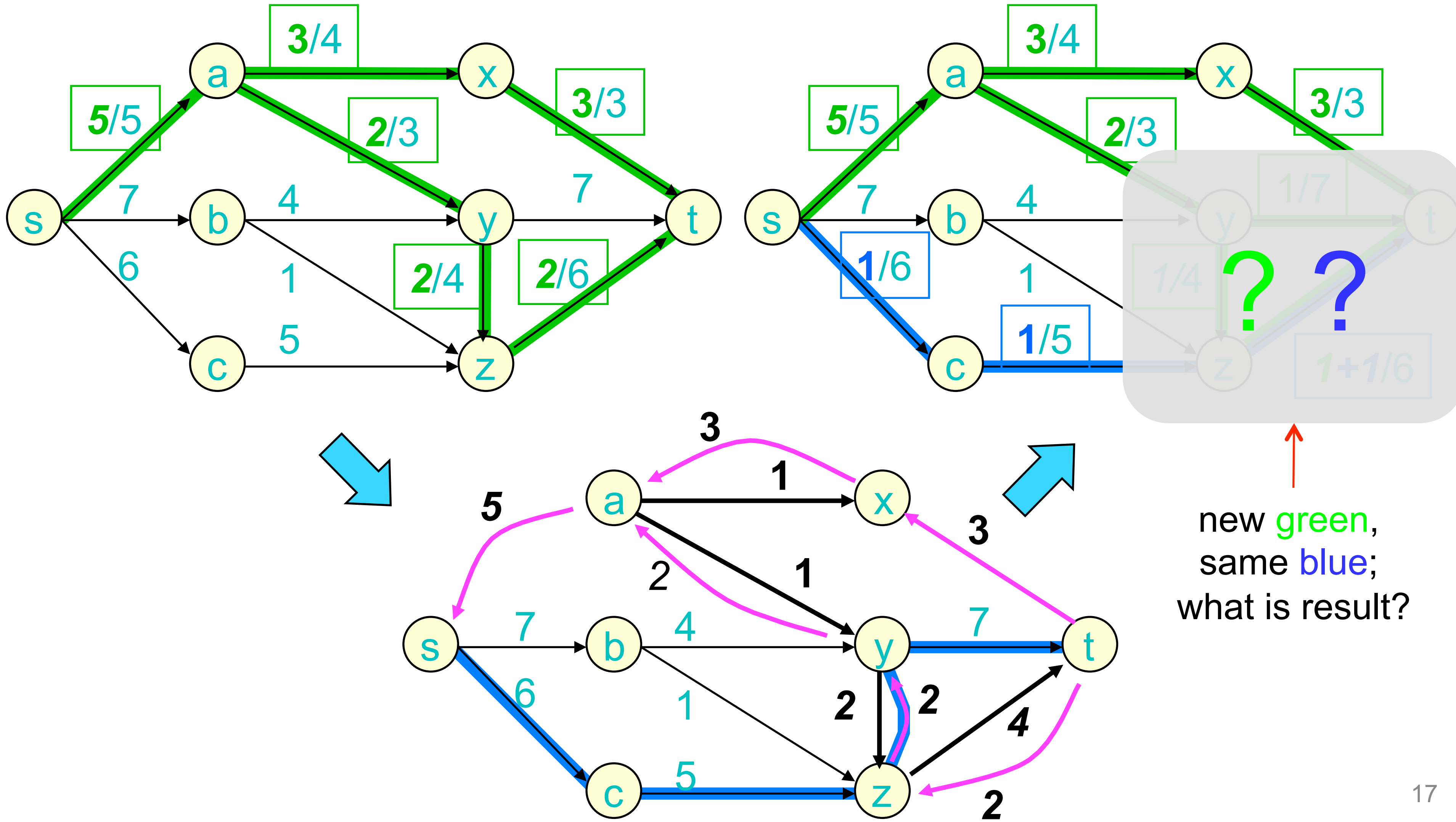
If f admits an augmenting path p, then f is not maximal.

Proof: “obvious” -- augment along p by c_p, the min residual capacity of p’s edges.

Augmenting A Flow



Augmenting A Flow



Lemma 1’: Augmented Flows are Flows

If f is a flow & p an augmenting path of capacity c_p, then f' is also a valid flow, where

$$f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}$$

Proof:

a) Flow conservation – easy

b) Skew symmetry – easy

c) Capacity constraints – pretty easy; next slides

Lma 1': Augmented Flows are Flows

$$f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}$$

f a flow & p an aug path of cap c_p, then f' also a valid flow.

Proof (Capacity constraints):

(u,v), (v,u) not on path: no change

(u,v) on path:

$$\begin{aligned} f'(u,v) &= f(u,v) + c_p \\ &\le f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v) \end{aligned}$$

$$\begin{aligned} f'(v,u) &= f(v,u) - c_p \\ &< f(v,u) \\ &\le c(v,u) \end{aligned}$$

QED

Residual Capacity:

$0 < c_p \le c_f(u,v) = c(u,v) - f(u,v)$

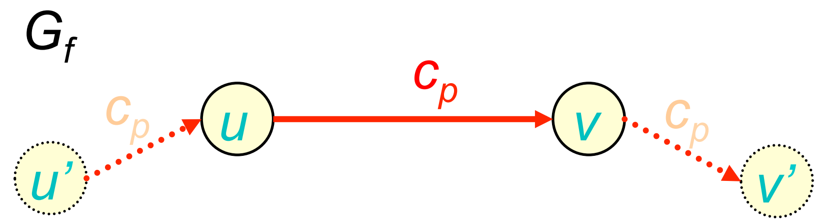
Cap Constraints:

$-c(v,u) \le f(u,v) \le c(u,v)$

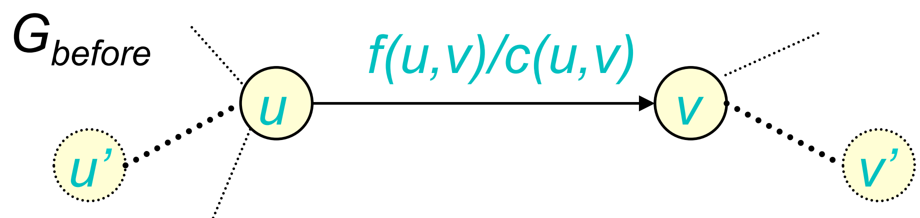
Lemma 1' Example – Case 1

Let (u,v) be any edge in augmenting path. Note

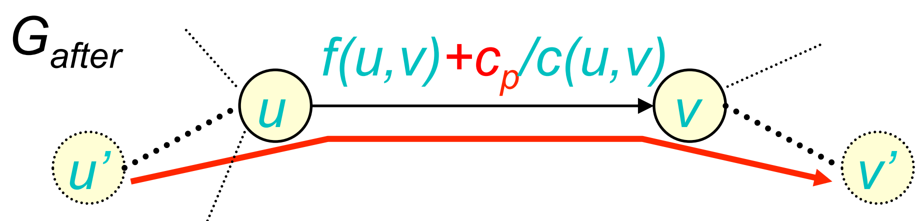
$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$



Case 1: $f(u,v) \geq 0$:



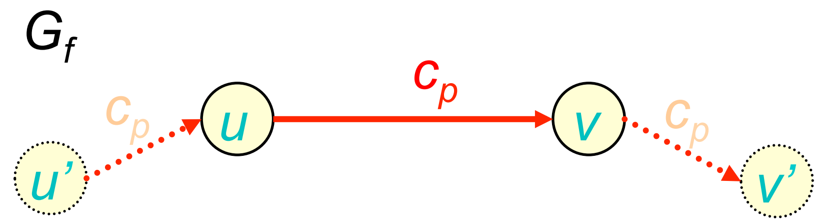
Add forward flow



Lemma 1' Example – Case 2

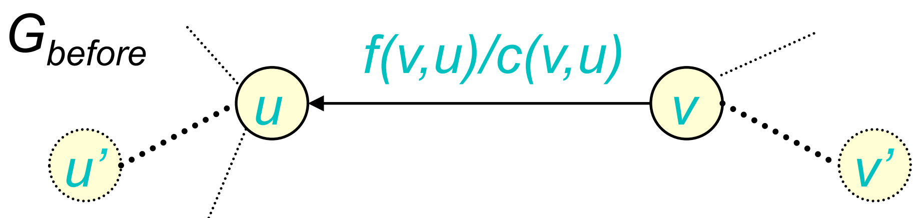
Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) – f(u,v) \geq c_p > 0$

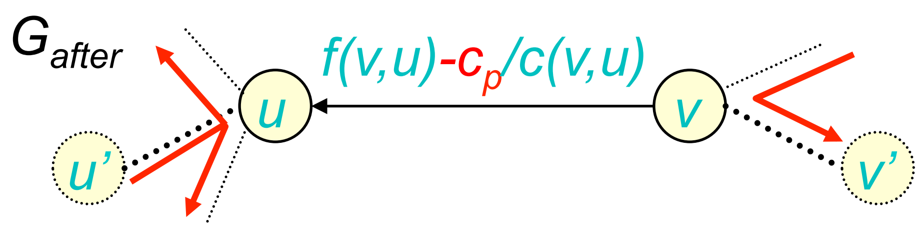


Case 2: $f(u,v) \leq -c_p$:

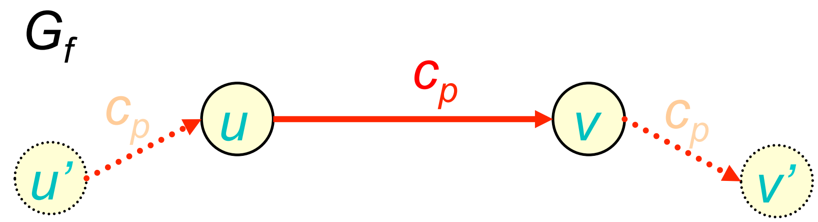
$f(v,u) = -f(u,v) \geq c_p$



Cancel/redirect reverse flow



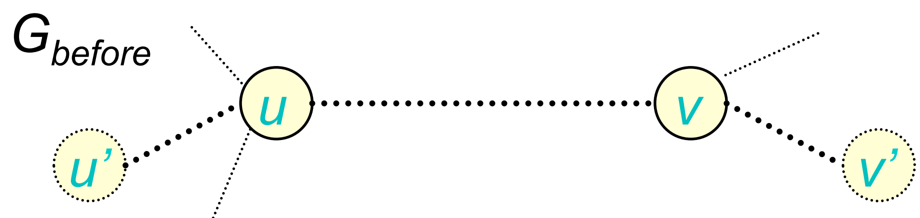
Lemma 1' Example – Case 3



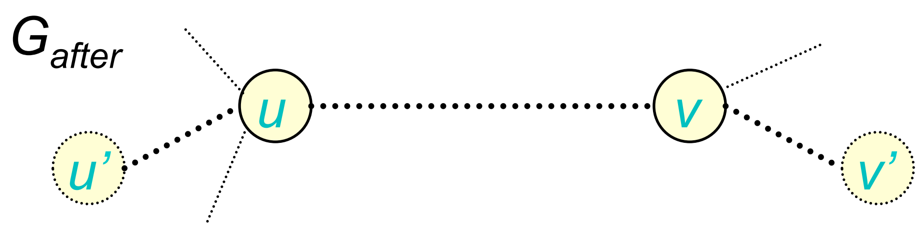
Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) – f(u,v) \geq c_p > 0$

Case 3: $-c_p < f(u,v) < 0$:

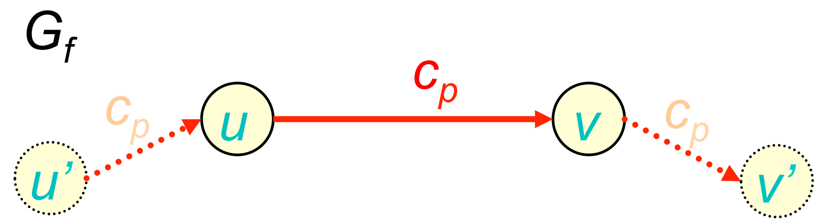


???



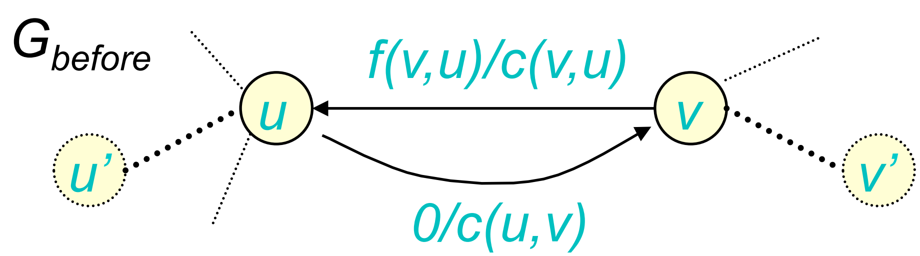
[E.g., $c_p = 8$, $f(u,v) = -5$]

Lemma 1' Example – Case 3



Let *(u,v)* be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$



Case 3: $-c_p < f(u,v) < 0$

$c_p > f(v,u) > 0$:

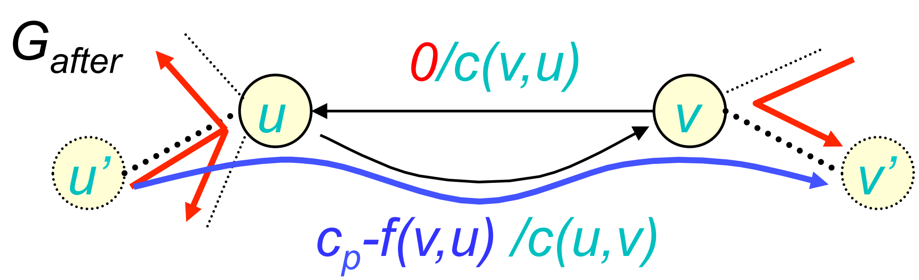
Both:

cancel/redirect

reverse flow

and

add forward flow



Ford-Fulkerson Method

While G_f has an augmenting path,
augment

Questions:

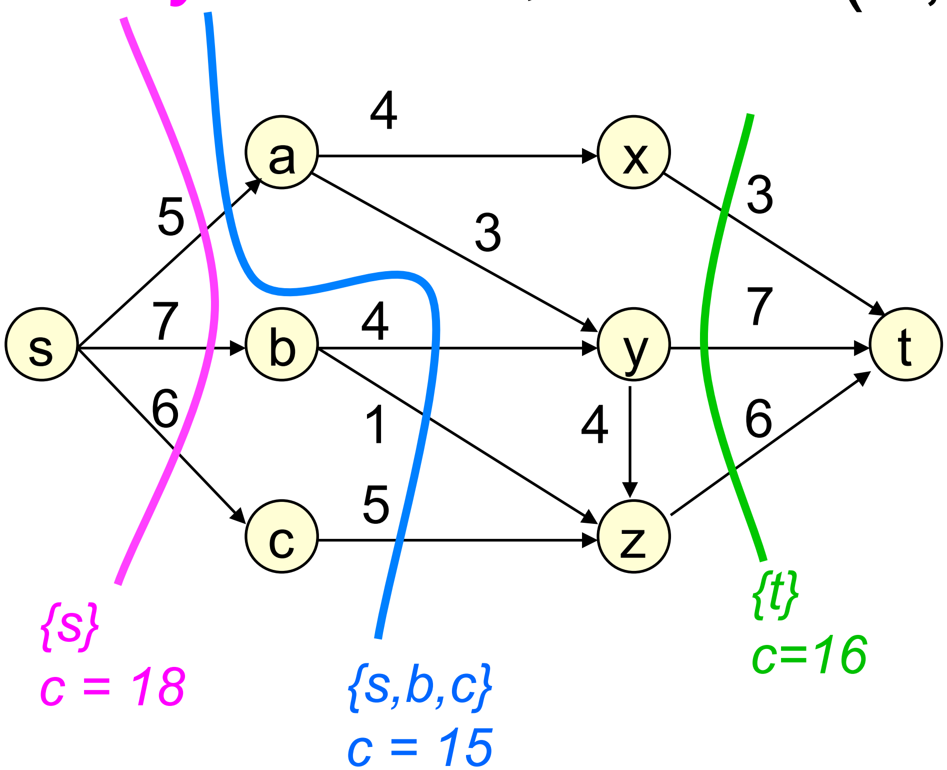
- » Does it halt?
- » Does it find a maximum flow?
- » How fast?

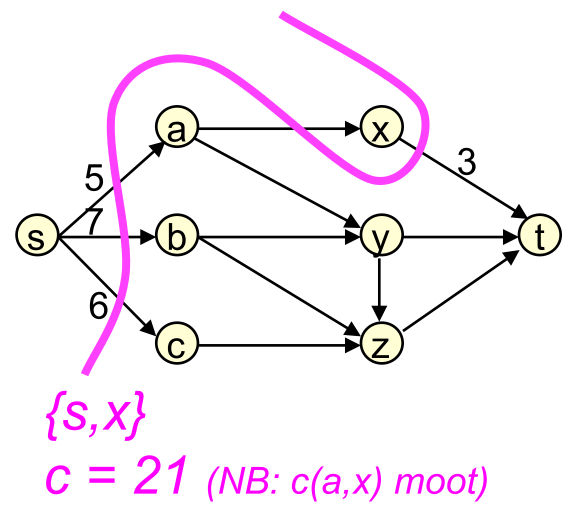
Cuts

A partition S, T of V is a *cut* if $s \in S$, $t \in T$.

Capacity of cut S, T is $$c(S,T) = \sum_{\substack{u \in S \\ v \in T}} c(u,v)$$

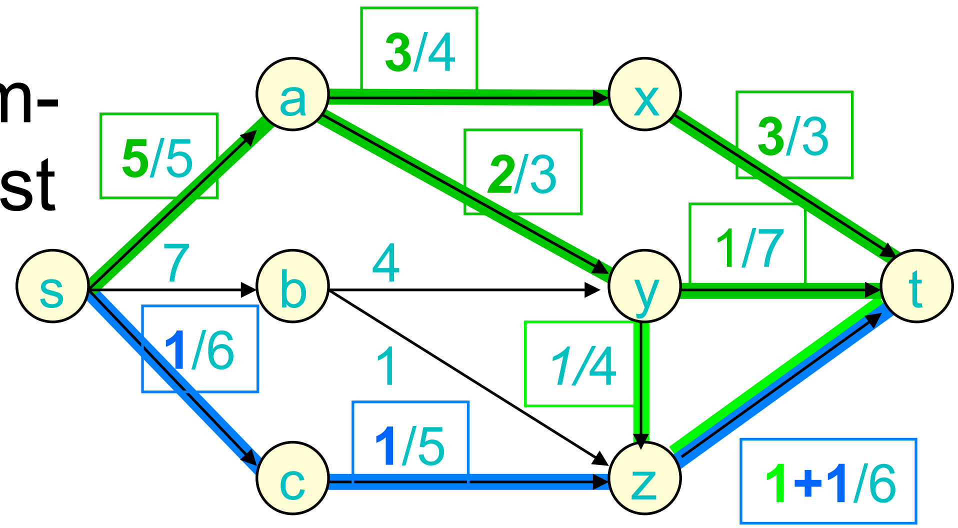
sum of caps of edges *from S to T*





Every Flow is a Sum of Paths

Every flow can be decomposed into sum of at most *m* simple augmenting paths, each using only "real" edges in G.



saxt	3
sayt	1
sayzt	1
sczt	1

Pf: Delete all edges with 0 flow. Find a simple *s*-*t* path *p*. Let *d* = min flow on any edge of *p*. Reduce flow by *d* along *p*. Repeat until flow = 0

(Note: helps next proof, but not algs, since don't know how to find these paths without flow…)

Lemma 2

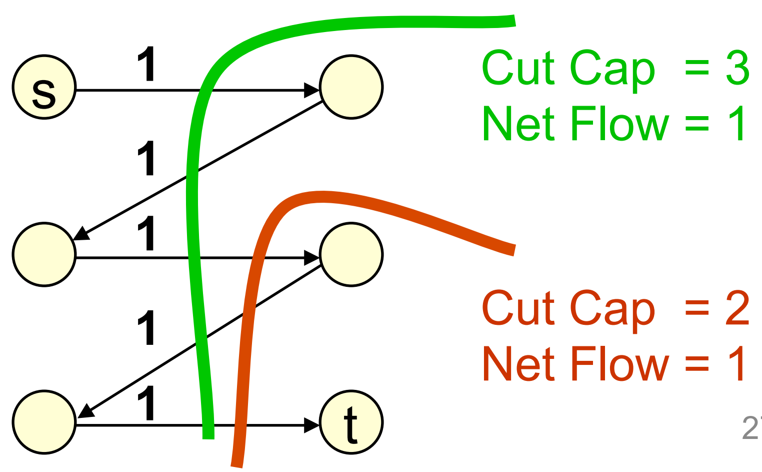
For any flow f and any cut S, T, net flow across cut $\underset{a}{=}$ total flow $\underset{b}{\leq}$ cut capacity

Proof:

a: Decompose. Track d_i flow units along p_i from s to t. Crosses cut an odd # of times; net = d_i. $\sum d_i = |f|$

b: Assign d_i's to last crossing; it's a forward edge totaled in $C(S,T)$, $\therefore \sum d_i \leq C(S,T)$

Cor: Max flow ≤ Min cut



Max Flow / Min Cut Theorem

For any flow f, the following are equivalent

(1) $|f| = c(S,T)$ for some cut S,T (a min cut)

(2) f is a maximum flow

(3) f admits no augmenting path

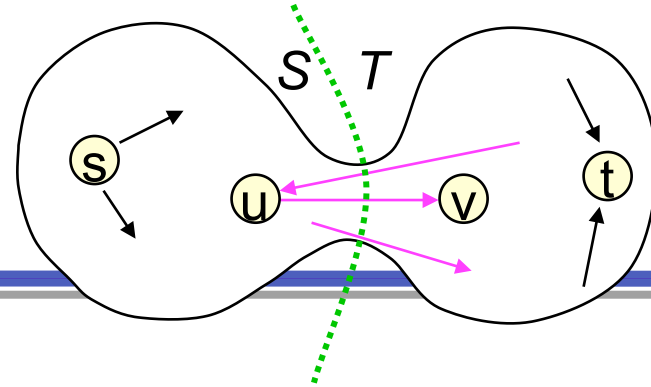
Proof:

(1) $\Rightarrow$ (2): corollary to lemma 2

(2) $\Rightarrow$ (3): contrapositive of lemma 1

$(3) \Rightarrow (1)$

(no aug) $\Rightarrow$ (cut)



Idea: where's the bottleneck

S = { u | $\exists$ an augmenting path wrt f from s to u }

$T = V - S;\ \ s \in S,\ t \in T$

For any (u,v) in $S \times T$, $\exists$ an augmenting path from s to u, but not to v.

$\therefore$ (u,v) has 0 residual capacity:

$(u,v) \in E \Rightarrow$ saturated $\qquad f(u,v) = c(u,v)$

$(v,u) \in E \Rightarrow$ no flow $\qquad f(u,v) = -f(v,u) = 0$

This is true for every edge crossing the cut, i.e.

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) =$$

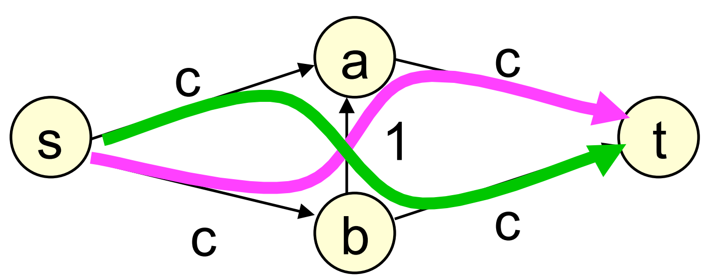
$$\sum_{u\in S, v\in T, (u,v)\in E} f(u,v) = \sum_{u\in S, v\in T, (u,v)\in E} c(u,v) = c(S,T)$$

Corollaries & Facts

If Ford-Fulkerson terminates, then it's found a max flow.

It will terminate if *c(e)* integer or rational (but may not if they're irrational).

However, may take exponential time, even with integer capacities:



$c = 10^{99}$, say

How to Make it Faster

Many possibilities. Three important ones:

- Edmonds-Karp '70; Dinitz '70 (below)
  - 1st "strongly" poly time alg. (next) $T = O(nm^2)$
- "Scaling" [Edmonds-Karp, '72; Dinitz '72]
  - do *largest* edges first; see text 7.3.
  - if C = max capacity, $T = O(m^2 \log C)$
- Preflow-Push [Goldberg, Tarjan '86]
  - see text 7.4 (optional) $T = O(n^3)$

Edmonds-Karp-Dinitz '70 Algorithm

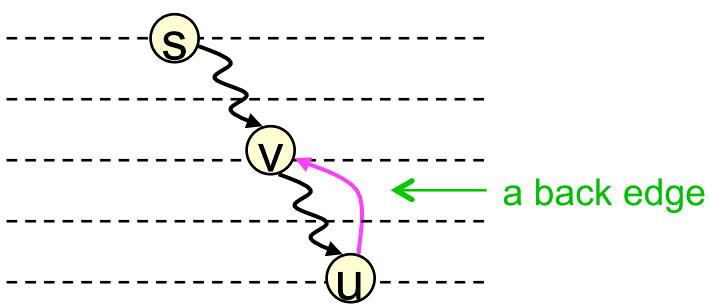
Use a shortest augmenting path
(via Breadth First Search in residual graph)

Time: $O(n\ m^2)$

BFS/Shortest Path Lemmas

Distance from s is never reduced by:

- Deleting an edge
  proof: no new (hence no shorter) path created
- Adding a back-edge (i.e., an edge (*u*,*v*), provided *v* is nearer than *u*)
  proof: BFS is unchanged, since *v* visited before (*u*,*v*) examined

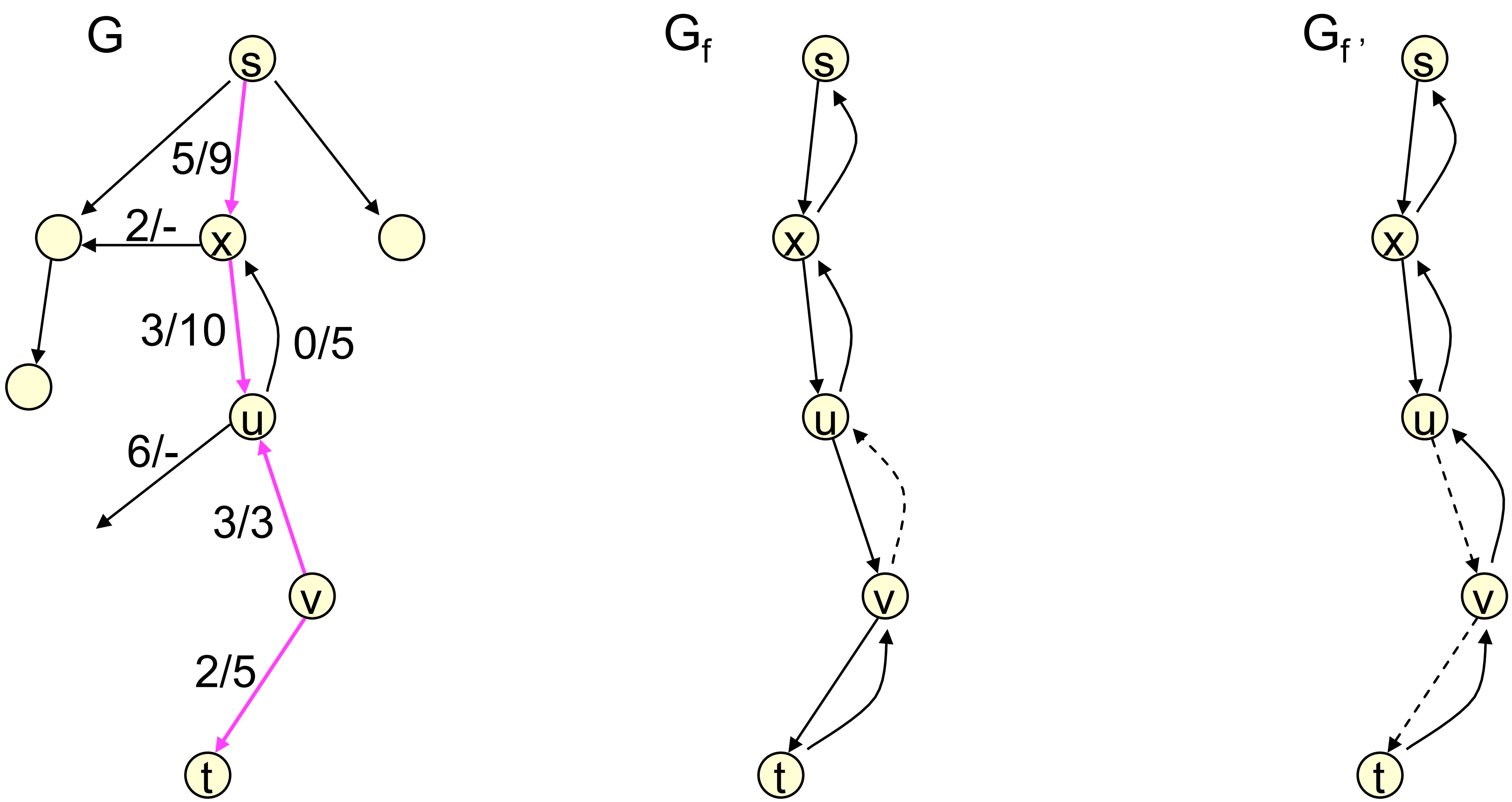


Lemma 3

Let f be a flow, G_f the residual graph, and p a shortest augmenting path. Then no vertex is closer to s in the new residual graph G_{f+p} after augmentation along p.

Proof: Augmentation only deletes edges, adds back edges

Augmentation vs BFS



Theorem 2

The Edmonds-Karp-Dinitiz Algorithm performs O(mn) flow augmentations

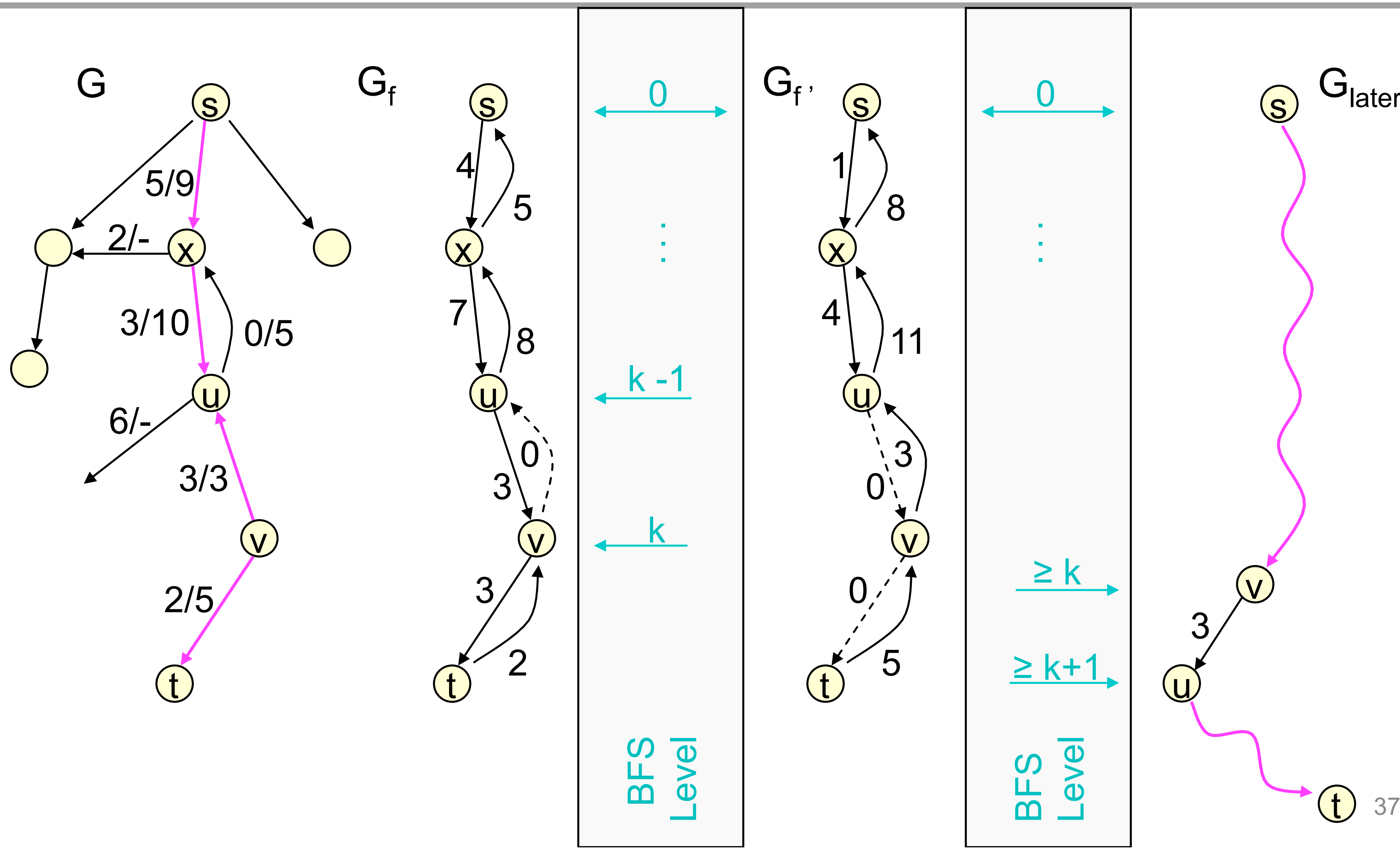
Proof:

$\{u,v\}$ is critical on augmenting path p if it's closest to s having min residual capacity.

Won't be critical again until farther from s.

So each edge critical at most n times.

Augmentation vs BFS Level

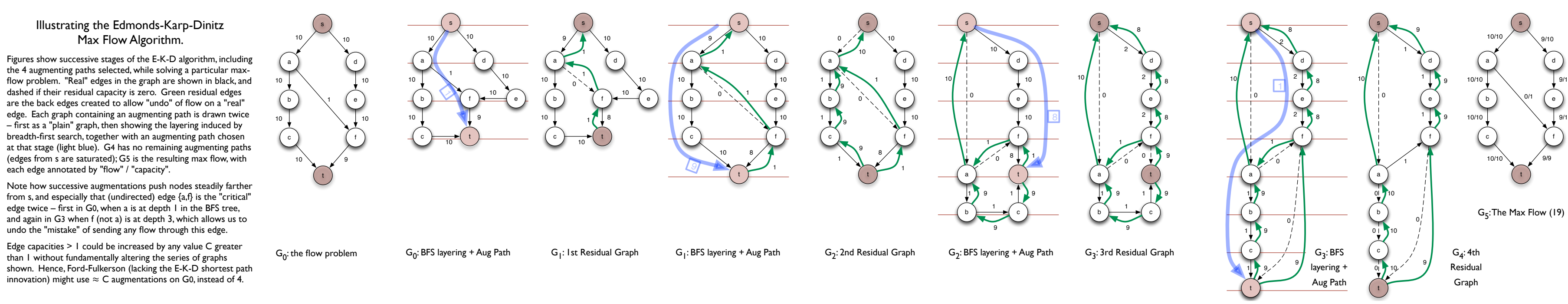


Corollary

Edmonds-Karp-Dinitz runs in $O(nm^2)$

Example

See “Edmonds-Karp-Dinitz Example” on course web page



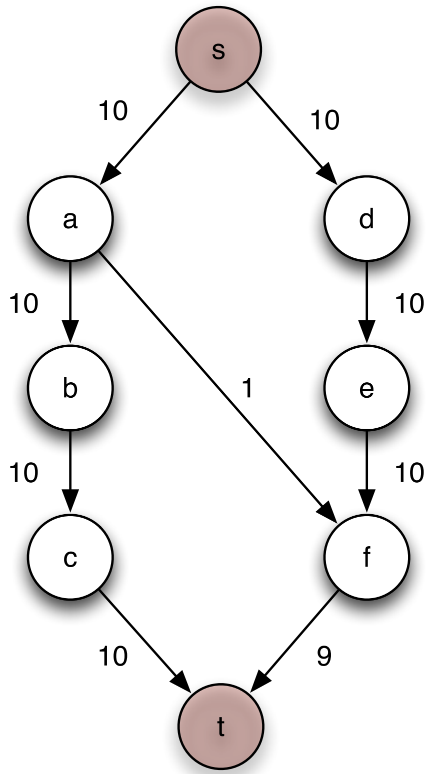
Illustrating the Edmonds-Karp-Dinitz Max Flow Algorithm.

Figures show successive stages of the E-K-D algorithm, including the 4 augmenting paths selected, while solving a particular max-flow problem. "Real" edges in the graph are shown in black, and dashed if their residual capacity is zero. Green residual edges are the back edges created to allow "undo" of flow on a "real" edge. Each graph containing an augmenting path is drawn twice – first as a "plain" graph, then showing the layering induced by breadth-first search, together with an augmenting path chosen at that stage (light blue). G4 has no remaining augmenting paths (edges from s are saturated); G5 is the resulting max flow, with each edge annotated by "flow" / "capacity".

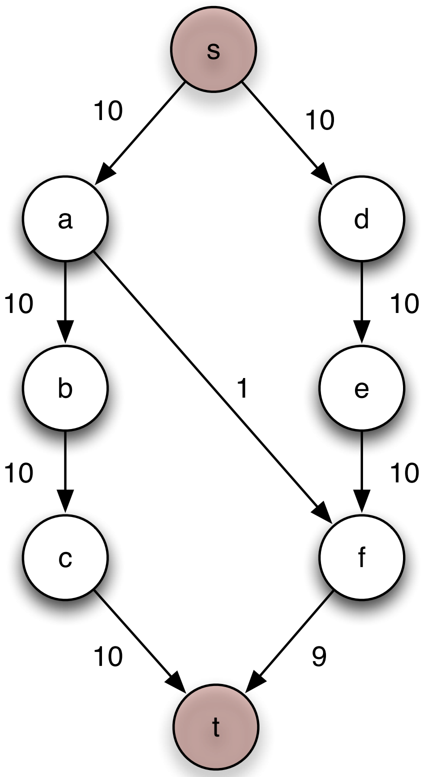
Note how successive augmentations push nodes steadily farther from s, and especially that (undirected) edge {a,f} is the "critical" edge twice – first in G0, when a is at depth 1 in the BFS tree, and again in G3 when f (not a) is at depth 3, which allows us to undo the "mistake" of sending any flow through this edge.

Edge capacities > 1 could be increased by any value C greater than 1 without fundamentally altering the series of graphs shown. Hence, Ford-Fulkerson (lacking the E-K-D shortest path innovation) might use ≈ C augmentations on G0, instead of 4.

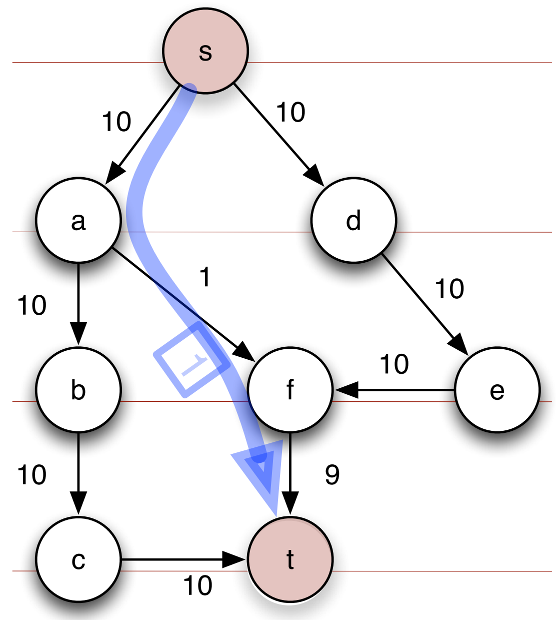
(file)



G_0: the flow problem

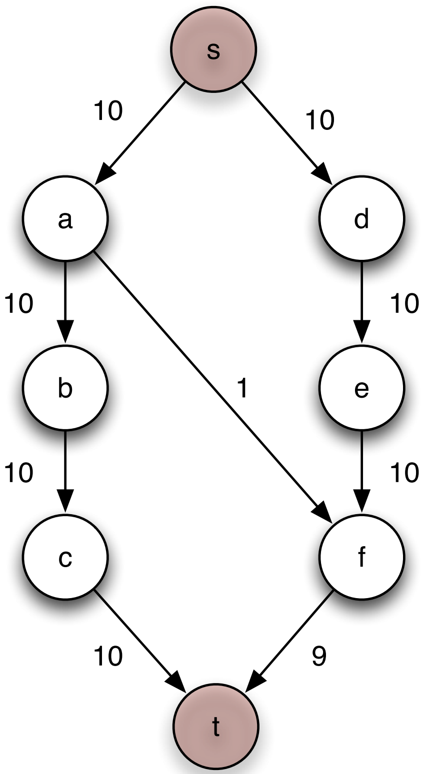


G_0: the flow problem

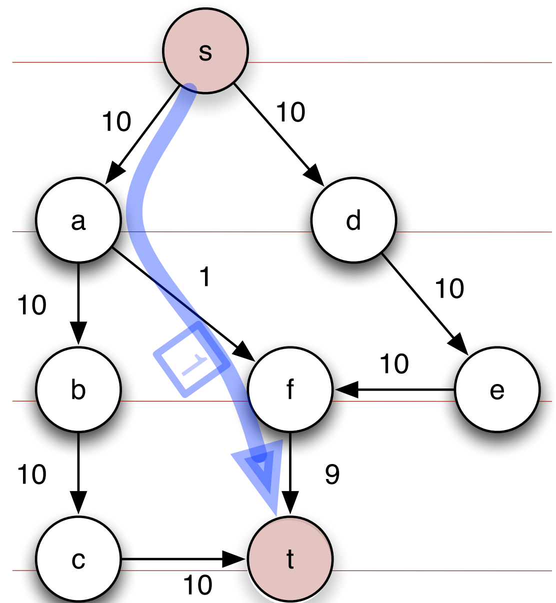


G_0: BFS layering + Aug Path

Critical edge: {a,f}

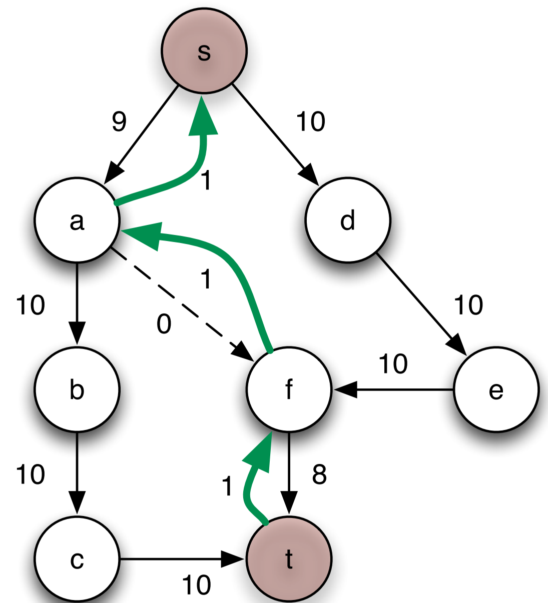


G_0: the flow problem

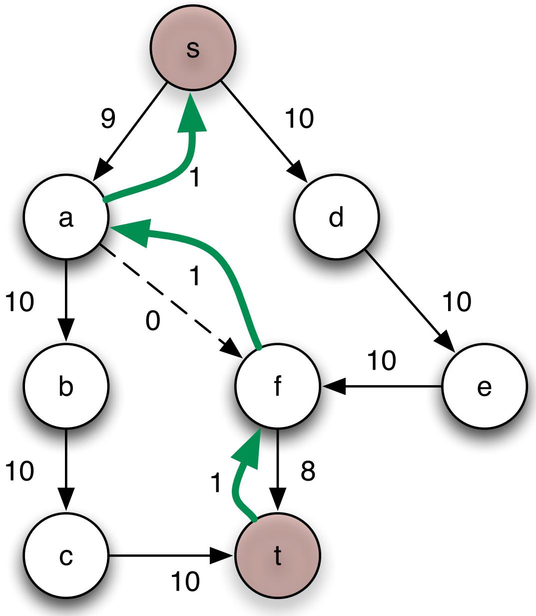


G_0: BFS layering + Aug Path

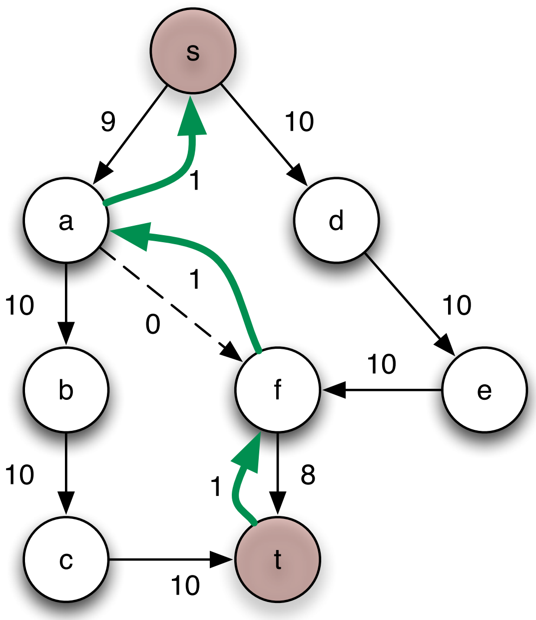
Critical edge: {a,f}

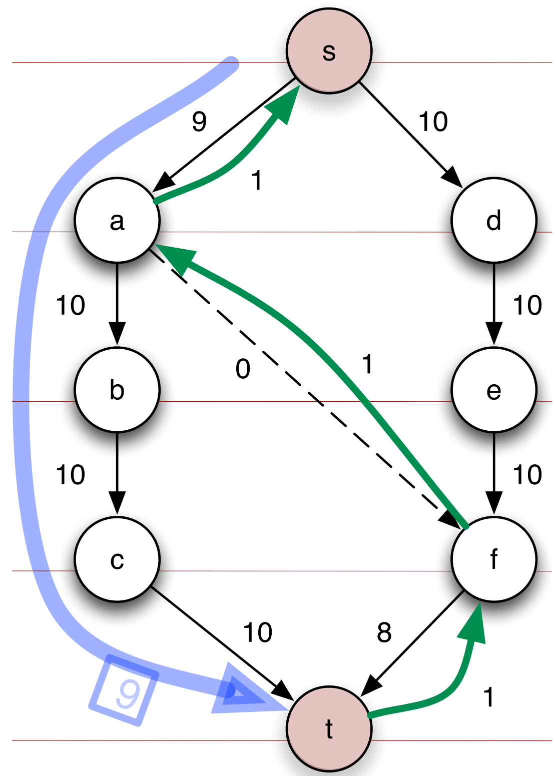


G_1: 1st Residual Graph



G_1: 1st Residual Graph

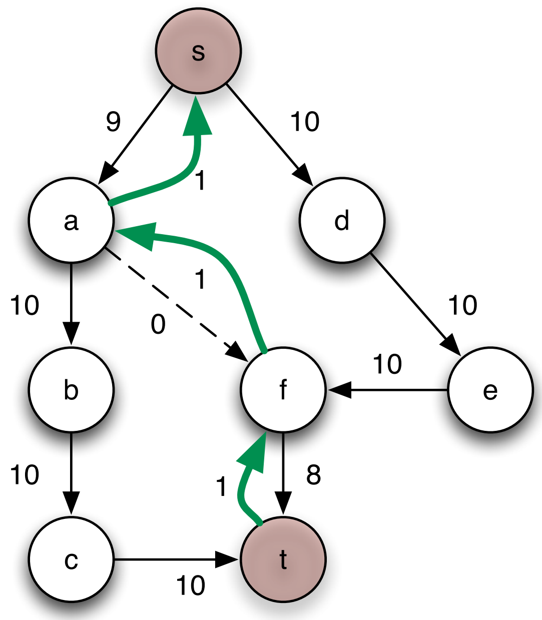




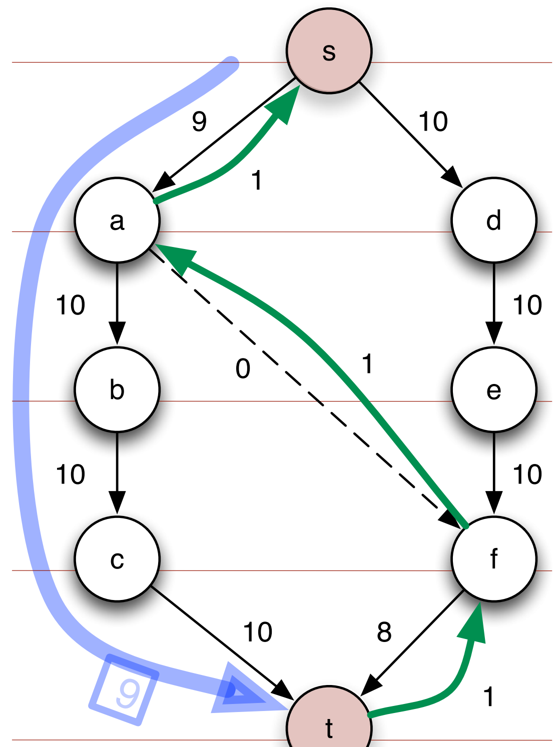
G_1: 1st Residual Graph

G_1: BFS layering + Aug Path

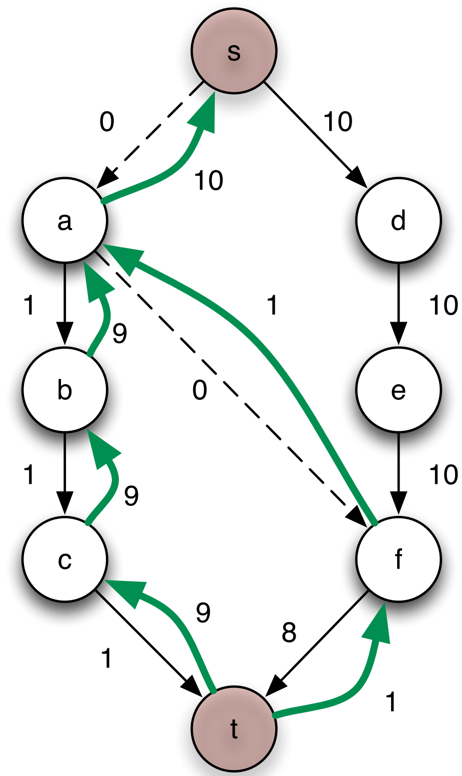
Critical edge: {s, a}



G_1: 1st Residual Graph

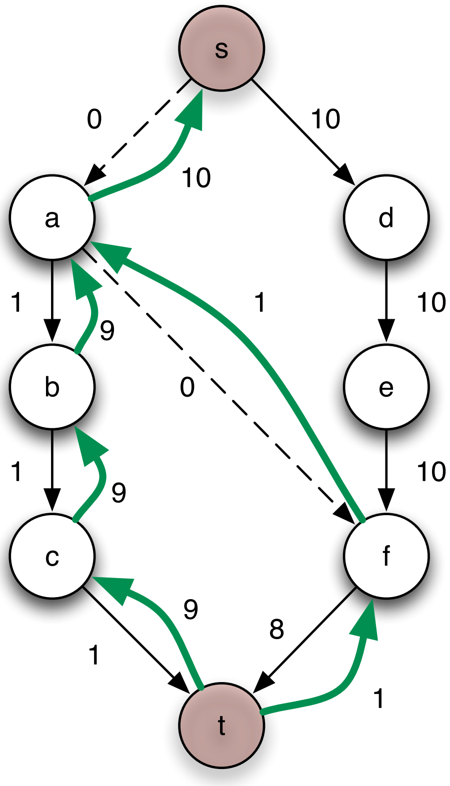


G_1: BFS layering + Aug Path

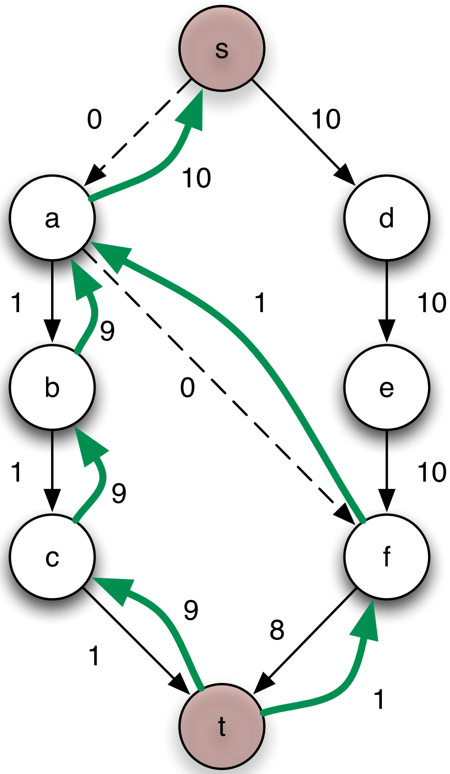


G_2: 2nd Residual Graph

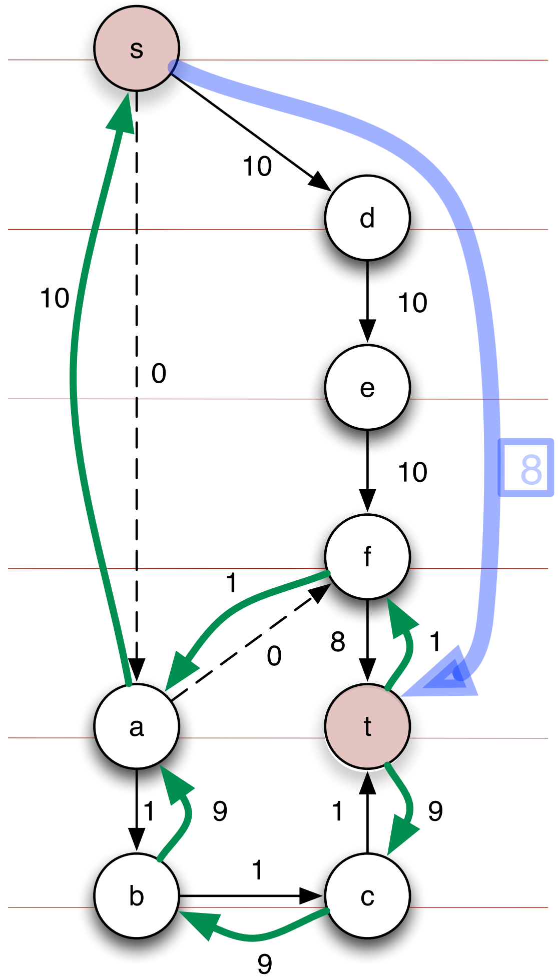
Critical edge: {s, a}



G_2: 2nd Residual Graph

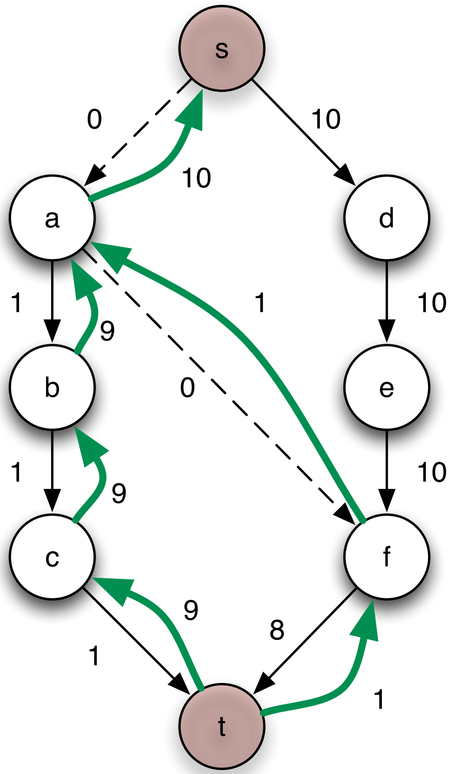


G_2: 2nd Residual Graph

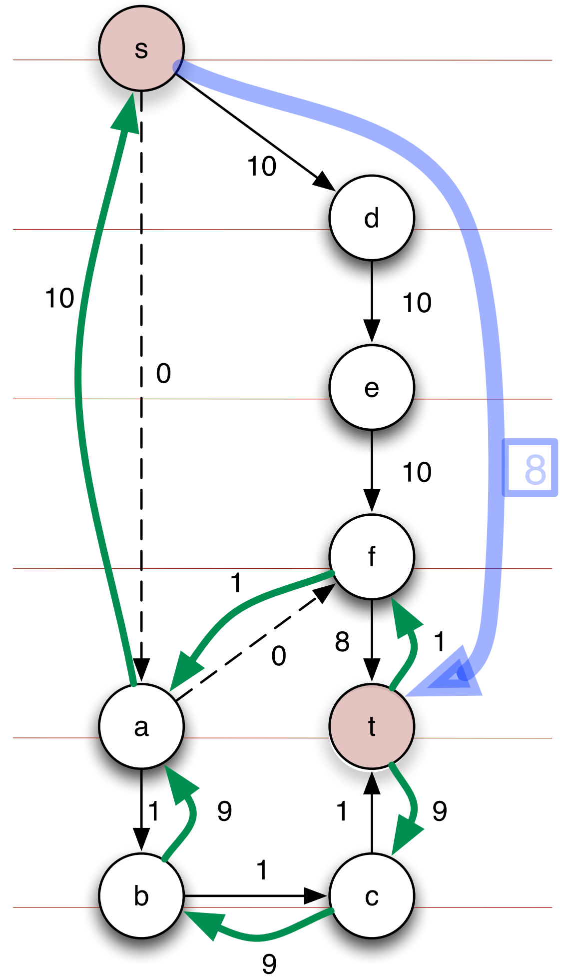


G_2: BFS layering + Aug Path

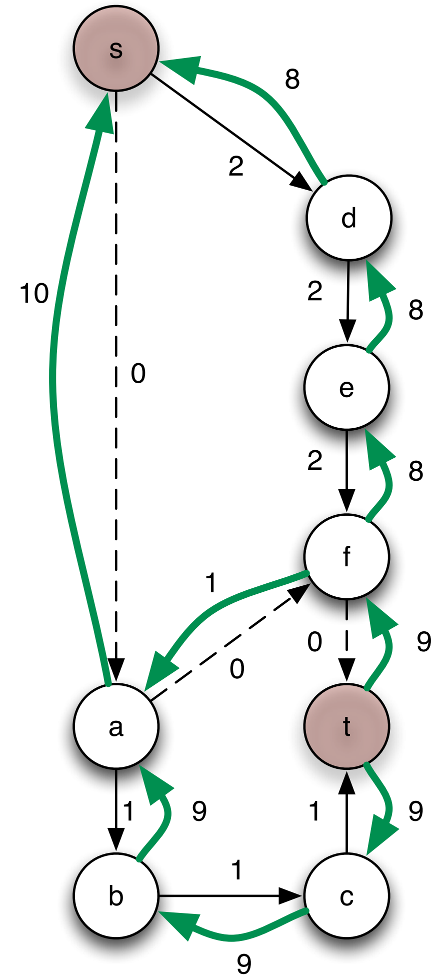
Critical edge: {f, t}



G_2: 2nd Residual Graph

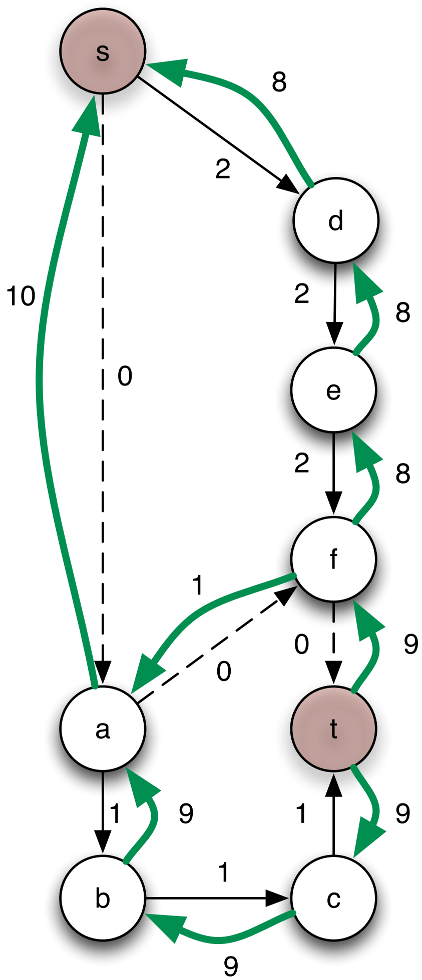


G_2: BFS layering + Aug Path

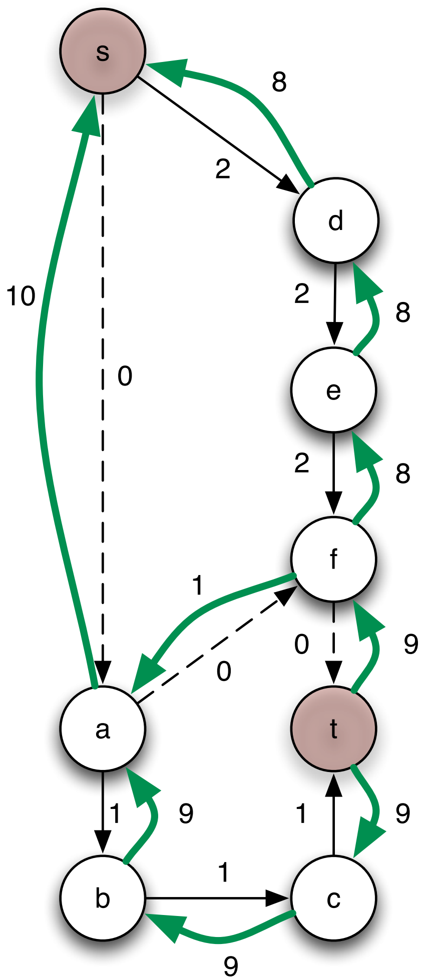


G_3: 3rd Residual Graph

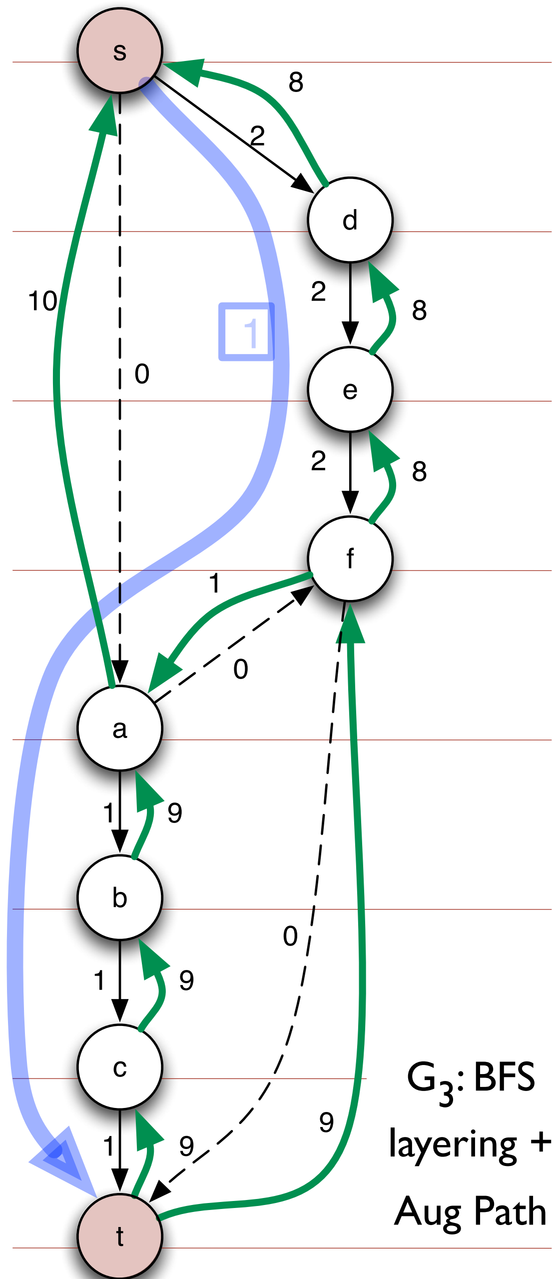
Critical edge: {f, t}



G_3: 3rd Residual Graph

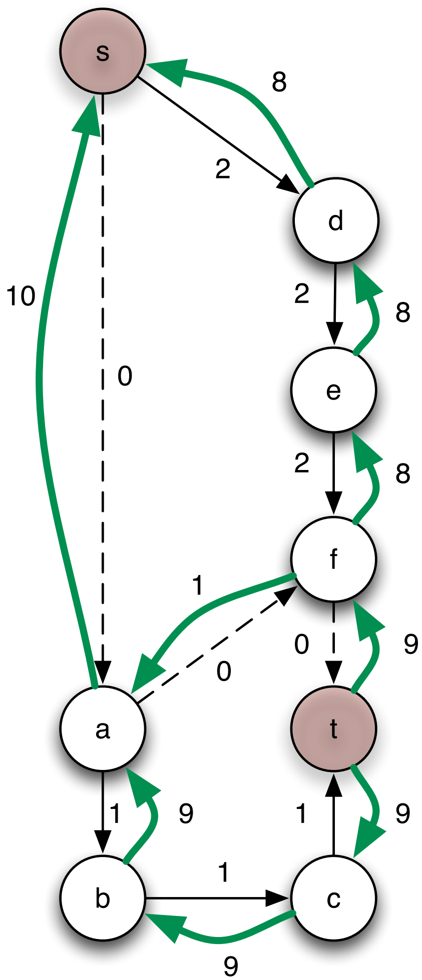


G$_3$: 3rd Residual Graph

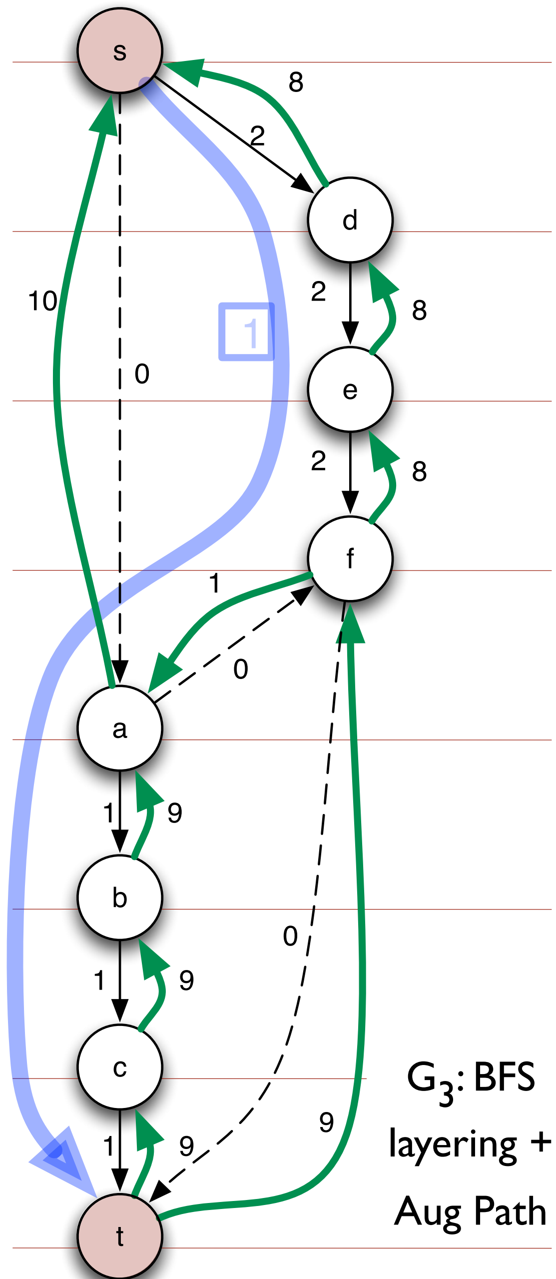


G$_3$: BFS layering + Aug Path

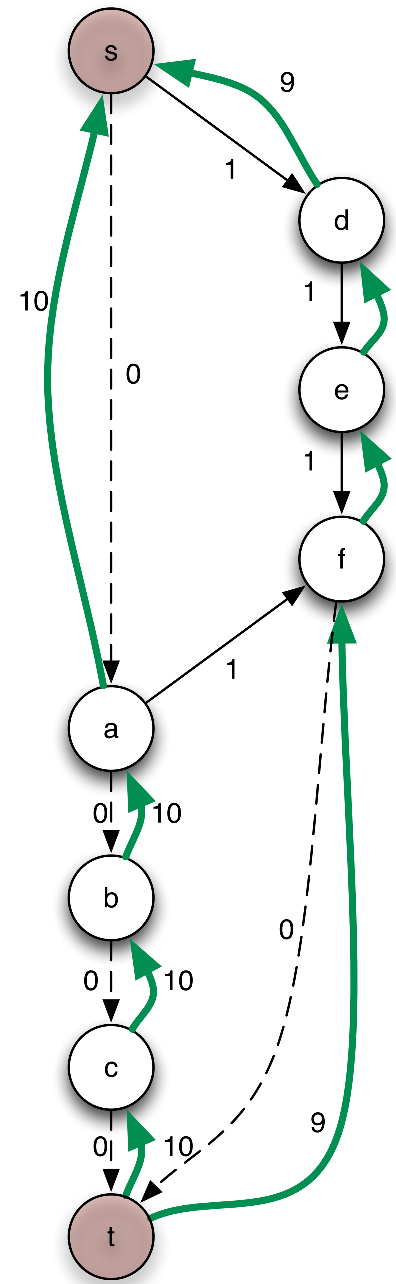
Critical edge: {a,f} (for the 2nd time)



G_3: 3rd Residual Graph

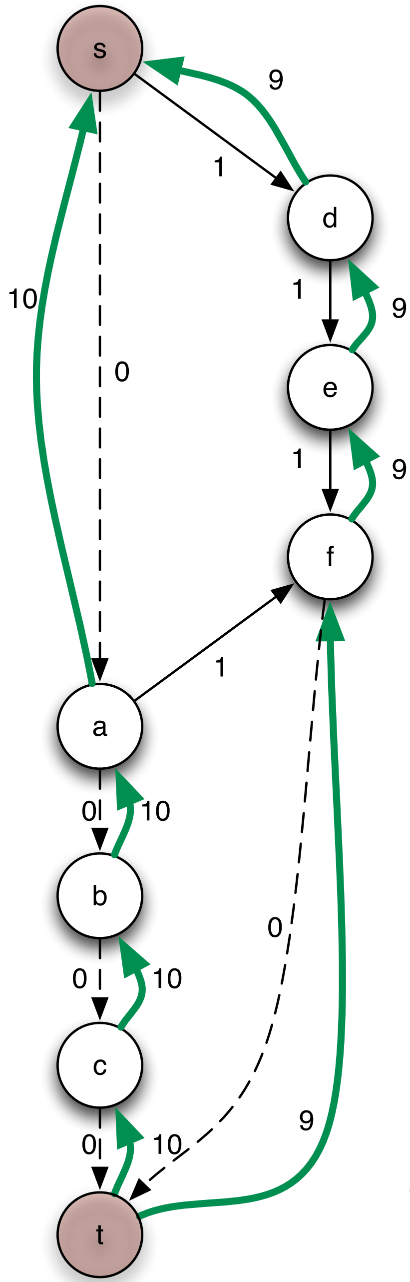


G_3: BFS layering + Aug Path

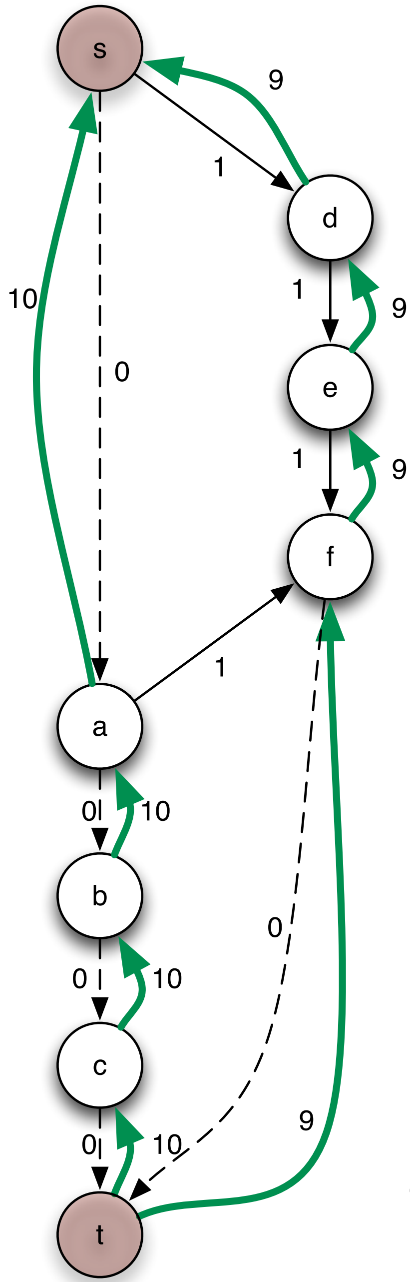


G_4: 4^{th} residual graph

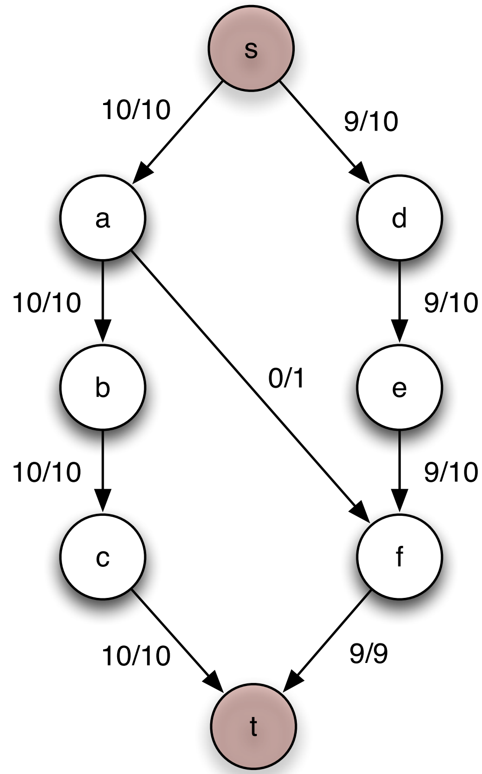
Critical edge: {a,f} (for the 2^{nd} time)



G_4: 4^{th} residual graph



G_4: 4^{th} residual graph



G_5:The Max Flow (19)

Flow Applications

Applications of Max Flow

Many!

Most look nothing like flow, at least superficially, but are deeply connected

Several interesting examples in 7.5-7.13

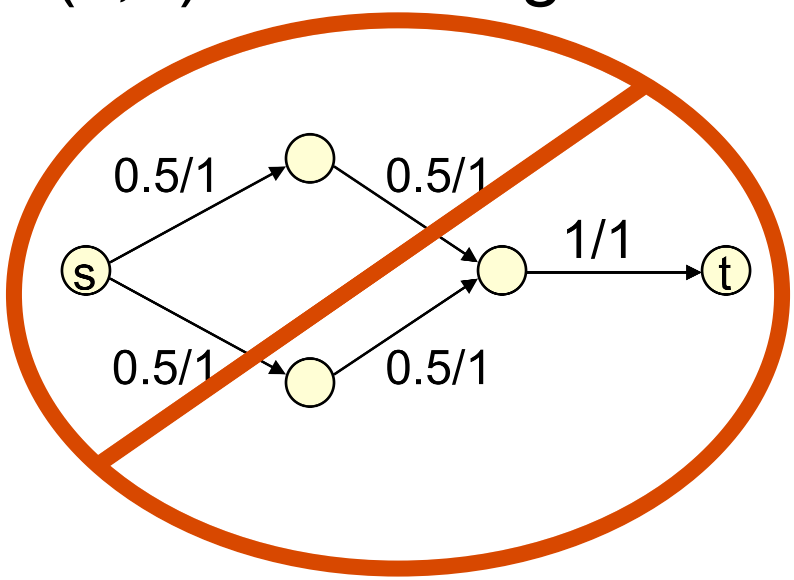
(7.8-7.11, 7.13 are optional, but interesting. Airline scheduling and image segmentation are especially recommended.)

A few more in following slides

Flow Integrality Theorem

Useful facts: If all capacities are integers

- Some max flow has an integer value
- Ford-Fulkerson method finds a max flow in which $f(u,v)$ is an integer for all edges (u,v)



A valid flow,
but unnecessary

7.6: Disjoint Paths

Given a digraph with designated nodes *s*,*t*, are there *k* edge-disjoint paths from *s* to *t*?

You might try depth-first search; you might fail…

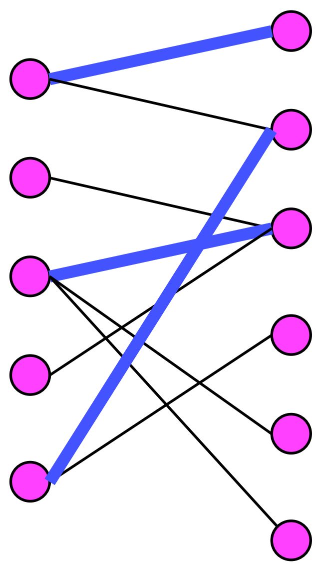
Instead:“edge caps=1, is max flow ≥ *k*?” Success!

Max-flow/min-cut also implies max number of edge disjoint paths = min number of edges whose removal separates *s* from *t*.

Many variants: node-disjoint, undirected, …

See 7.6

7.5: Bipartite Maximum Matching



Bipartite Graphs:

$G = (V,E)$

$V = L \cup R \ \ (L \cap R = \emptyset)$

$E \subseteq L \times R$

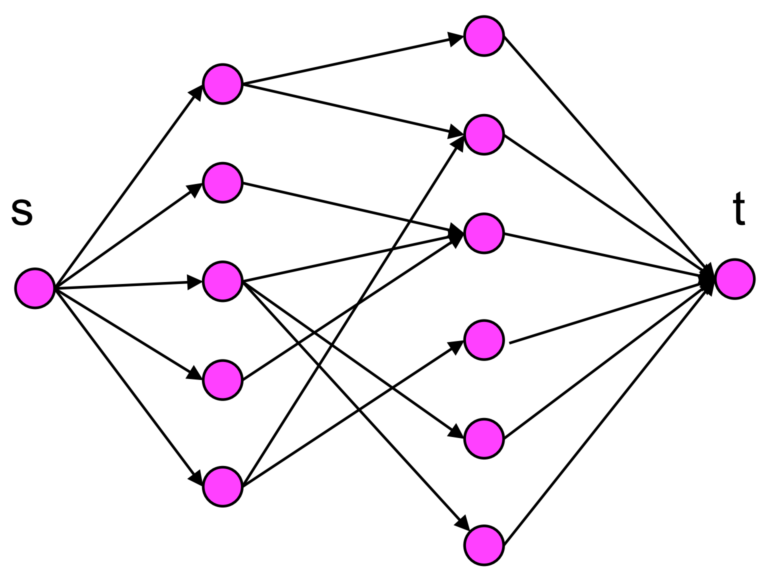
Matching:

A set of edges $M \subseteq E$ such that no two edges touch a common vertex

Problem:

Find a max size matching M

Reducing Matching to Flow



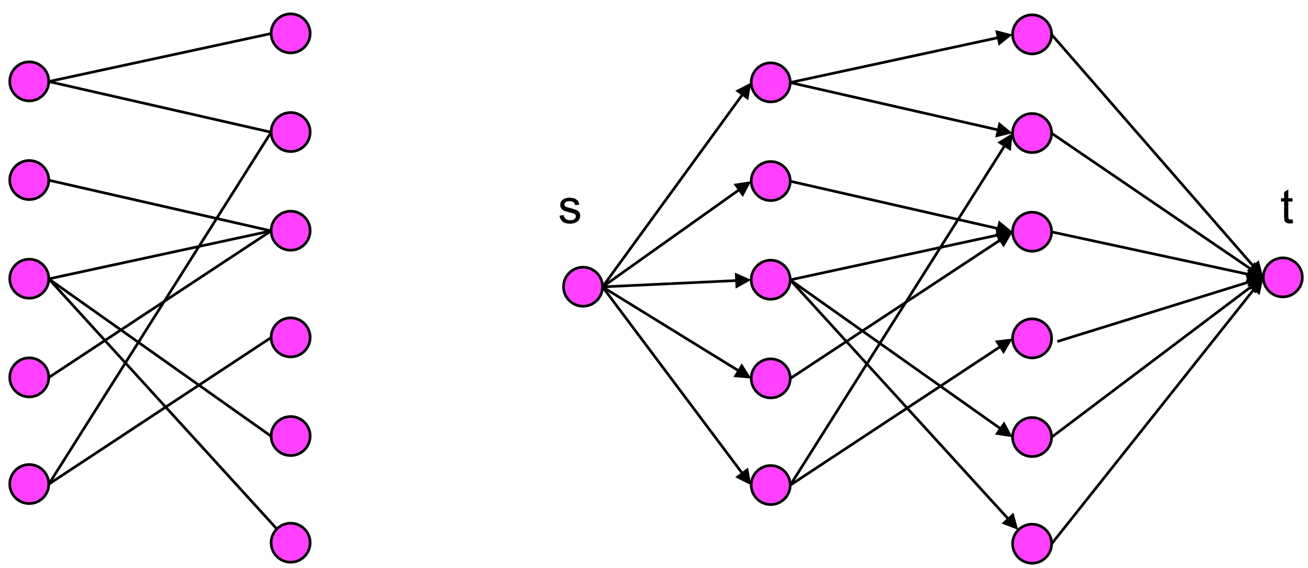
Given bipartite G, build flow network N as follows:

- Add source s, sink t
- Add edges $s \rightarrow L$
- Add edges $R \rightarrow t$
- All edge capacities 1

Theorem:
Max flow iff
max matching

Reducing Matching to Flow

Theorem: Max matching size = max flow value



$M \rightarrow f$? Easy – send flow only through M

$f \rightarrow M$? Flow Integrality Thm, + cap constraints

Notes on Matching

Max Flow Algorithm is probably overly general here

But most direct matching algorithms use "augmenting path"-type ideas similar to that in max flow – See text (& homework?)

Time $mn^{1/2}$ possible via Edmonds-Karp

7.12 Baseball Elimination

Some slides by Kevin Wayne

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

» Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.

» $w_i + g_i < w_j \Rightarrow$ team *i* eliminated.

» Sports writers rarely give a deeper analysis

» Sufficient, but not necessary!

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- » Philly can win 83, but still eliminated . . .
- » If Atlanta loses a game, then some other team wins one.

Remark. Depends on *both* how many games already won and left to play, *and* on which opponents.

Baseball Elimination

Baseball elimination problem.

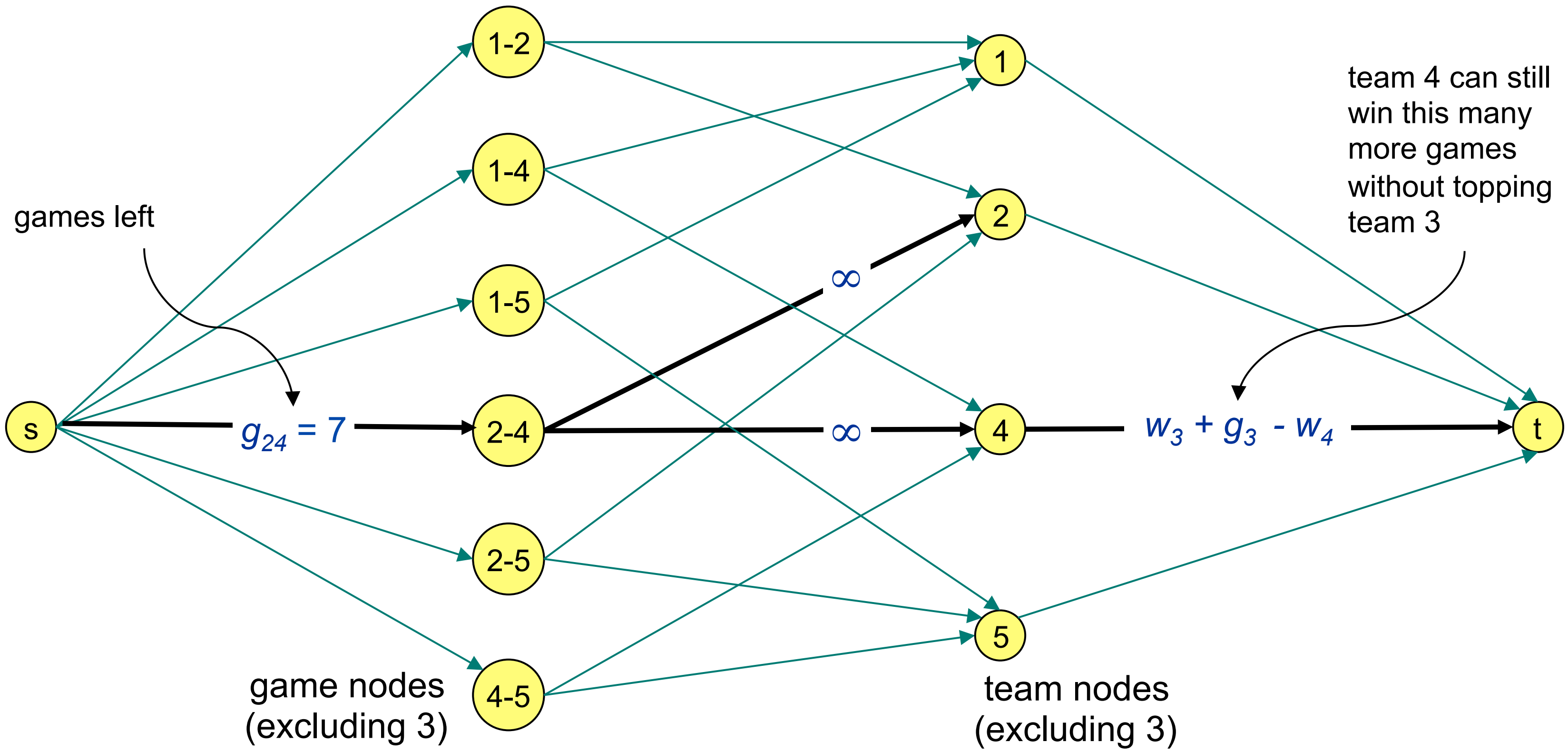
- Set of teams S.
- Distinguished team $s \in S$.
- Team x has won w_x games already.
- Teams x & y play each other g_{xy} more times.
- Can team s finish with (or tie for) most wins?

  I.e., is there a way to *allocate* wins of the remaining games so that s ends on top?

Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

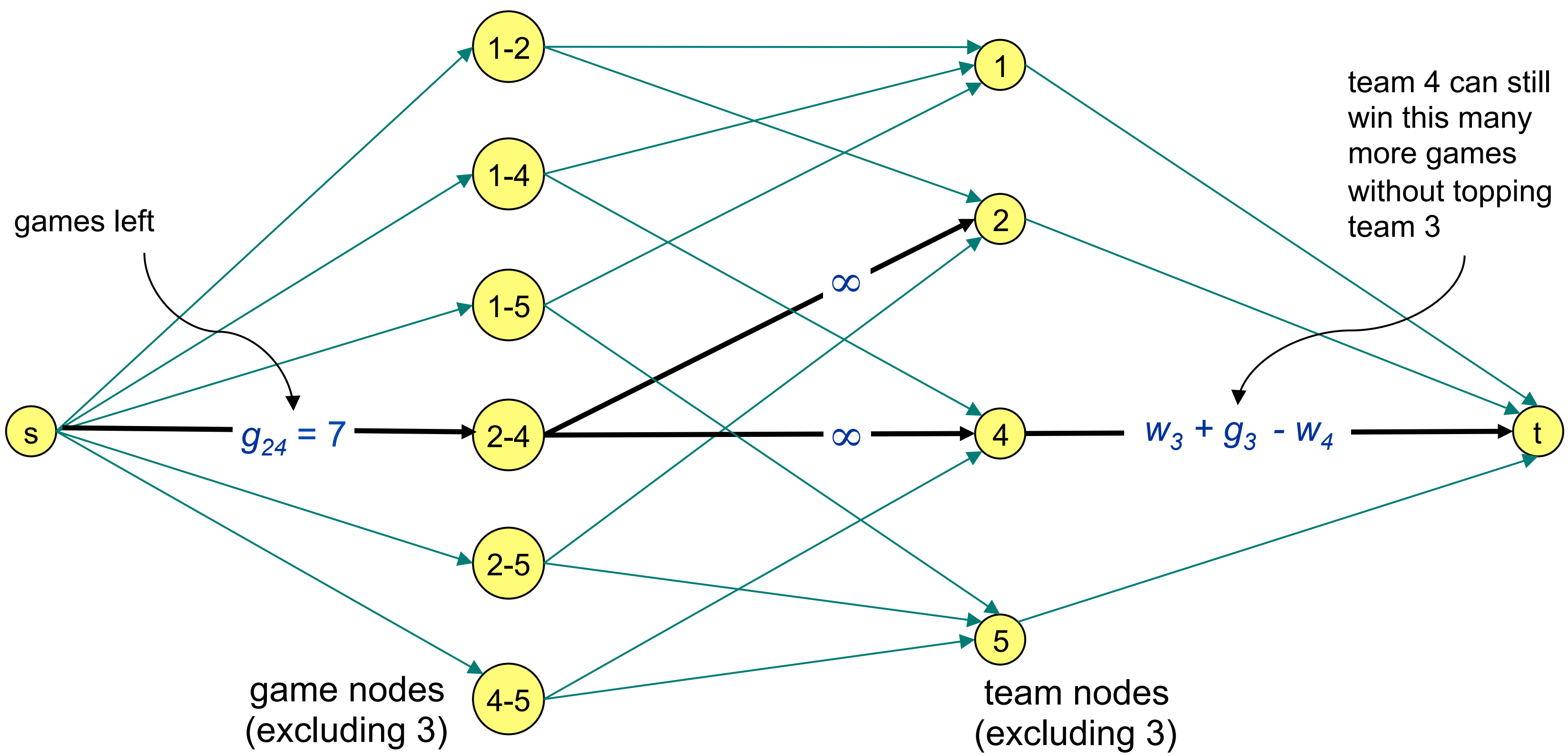
- One unit of flow = one win
- Assume team 3 wins all remaining games $\Rightarrow$ $w_3 + g_3$ wins.
- Divvy remaining games so that all teams have $\leq$ $w_3 + g_3$ wins.



Baseball Elimination: Max Flow Logic

Team 3 is eliminated iff max flow < games left.

- Integrality ⇒ each remaining *x : y* game added to # wins for *x* or *y*.
- Capacities on *(x, t)* edges ensure no team wins too many games.
- Capacities on *(s,x-y)* edges ensure no team plays too many games.
- In max flow, unsaturated source edge = unplayed game; if played, (either) winner would push ahead of team 3



Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams have a chance of finishing the season with most wins?

Detroit could finish season with 49 + 27 = 76 wins.

Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams could finish the season with most wins?

- Detroit could finish season with 49 + 27 = 76 wins.

Certificate of elimination. R = {NY, Bal, Bos, Tor}

- Have already won w(R) = 278 games.
- Must win at least r(R) = 27 more.
- Average team in R wins at least 305/4 > 76 games.

Baseball Elimination: Explanation for Sports Writers

Certificate of elimination

$$T \subseteq S, \quad w(T) := \overbrace{\sum_{i \in T} w_i}^{\#\text{ wins}}, \quad g(T) := \overbrace{\sum_{\{x,y\} \subseteq T} g_{x\,y}}^{\#\text{ remaining games}},$$

If $\overbrace{\dfrac{w(T) + g(T)}{|T|}}^{\text{LB on avg \# games won}} > w_z + g_z$ then *z* eliminated (by subset *T*).

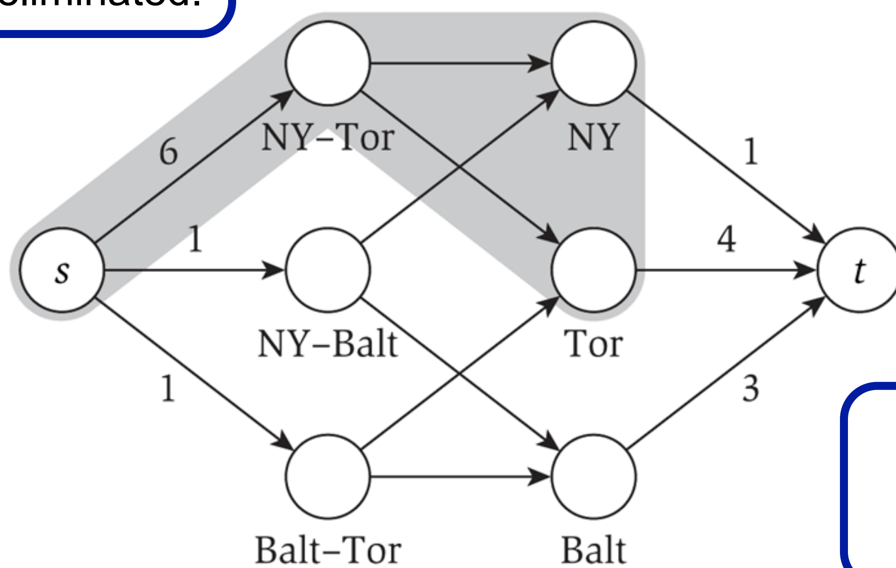
Theorem. [Hoffman-Rivlin 1967] Team *z* is eliminated iff there exists a subset *T** that eliminates *z*.

Proof idea. Let *T** = teams on source side of min cut.

	w	*l*	*g*	NY	Balt	Tor	Bos
NY	90		11	-	1	6	4
Baltimore	88		6	1	-	1	4
Toronto	87		10	6	1	-	4
Boston	79		12	4	4	4	-

$g^* = 1+6+1 = 8$

(90 + 87 + 6) / 2 > 91,
so the set T = {NY, Tor}
proves Boston is eliminated.



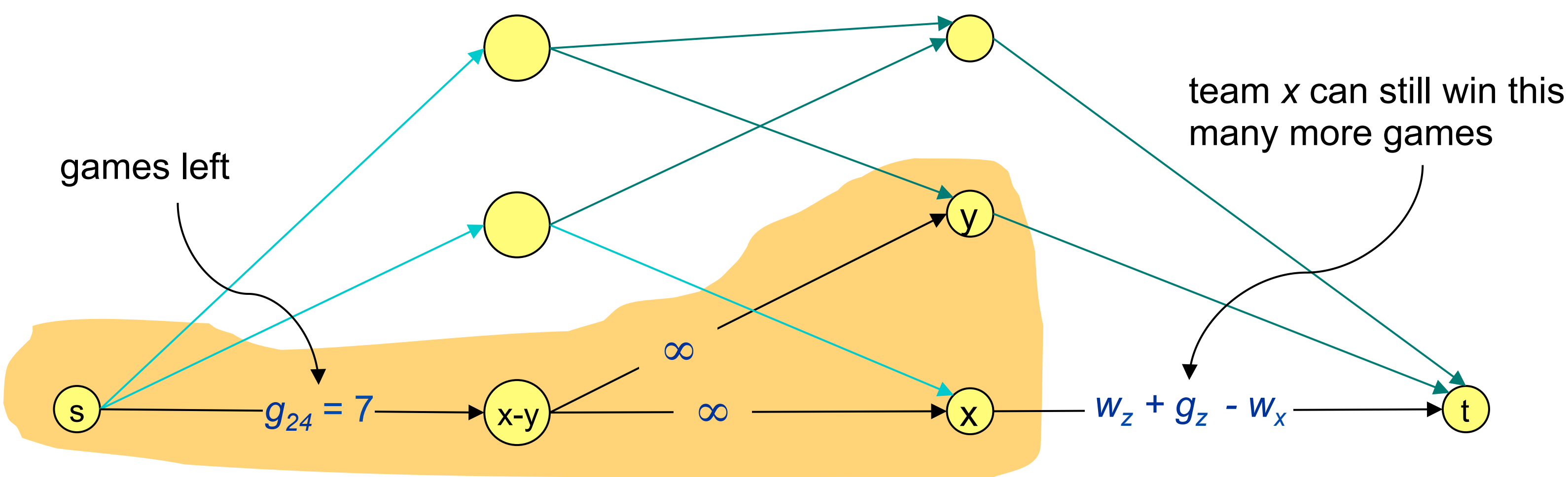
Note: T = {NY,Tor, Balt} is NOT a certificate, since (90+88+87+8)/3 = 91

Fig 7.21 Min cut ⇒ no flow of value g^*, so Boston eliminated.

Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

- Use max flow formulation, and consider min cut (A, B).
- Define T^* = team nodes on source side of min cut.
- Observe $x\text{-}y \in A$ iff both $x \in T^*$ and $y \in T^*$.
  - infinite capacity edges ensure if $x\text{-}y \in A$ then $x \in A$ and $y \in A$
  - if $x \in A$ and $y \in A$ but $x\text{-}y \notin T^*$, then adding $x\text{-}y$ to A decreases capacity of cut



Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

Use max flow formulation, and consider min cut (*A, B*).

Define *T** = team nodes on source side of min cut.

Observe *x-y* $\in$ *A* iff both *x* $\in$ *T** and *y* $\in$ *T**.

$$
\begin{aligned}
g(S-\{z\}) \quad &> \quad cap(A,\ B) \\
&= \quad \overbrace{g(S-\{z\})-g(T^*)}^{\text{capacity of game edges leaving A}} \ + \ \overbrace{\sum_{x\in T^*}(w_z+g_z-w_x)}^{\text{capacity of team edges leaving A}} \\
&= \quad g(S-\{z\})-g(T^*) \ - \ w(T^*) \ + \ |T^*|(w_z+g_z)
\end{aligned}
$$

Rearranging:

$$
w_z+g_z \ < \ \frac{w(T^*)+g(T^*)}{|T^*|}
$$

Matching & Baseball: Key Points

Can (sometimes) take problems that seemingly have *nothing* to do with flow & reduce them to a flow problem

How? Build a clever network; map allocation of *stuff* in original problem (match edges; wins) to allocation of *flow* in network. Clever edge capacities constrain solution to mimic original problem in some way. Integrality useful.

Matching & Baseball: Key Points

Furthermore, in the baseball example, min cut can be translated into a succinct *certificate* or *proof* of some property that is much more transparent than “see, I ran max-flow and it says flow must be less than g^*”.

These examples suggest why max flow is so important – *it’s a very general tool used in many other algorithms*.

Even more broadly – *reduction* is a powerful tool for algorithm design/analysis

Max Flow: Summary

- Important problem with a long history
- Properties and Tools:
  - » Duality: Max Flow – Min Cut Theorem
  - » Flow Integrality Theorem
  - » Residual graphs/augmenting paths
- Algorithms:
  - » Ford-Fulkerson ("method"); O(nmC) w/ rational caps
  - » Edmonds-Karp-Dinitz '70: shortest aug first, $O(nm^2)$
- Many applications:
  - » Disjoint paths, bipartite matching, "baseball," …
  - » "Reduction" as a key alg design technique