

# CSE 421: Algorithms

Graphs and Graph Algorithms

Larry Ruzzo

# Goals

Graphs: defns, examples, utility, terminology

Representation: input, internal

Traversal: Breadth- & Depth-first search

Five Graph Algorithms:

Connected components

Shortest Paths

Topological sort

Bipartiteness

Articulation points

} Review

} Review ?

# Graphs

An extremely important formalism for representing (binary) relationships

Objects: "vertices," aka "nodes"

Relationships between pairs:

"edges," aka "arcs"

Formally, a graph  $G = (V, E)$  is a pair of sets,  $V$  the vertices and  $E$  the edges

# Objects & Relationships

## The Kevin Bacon Game:

Obj: Actors

Rel: Two are related if they've been in a movie together

## Exam Scheduling:

Obj: Classes

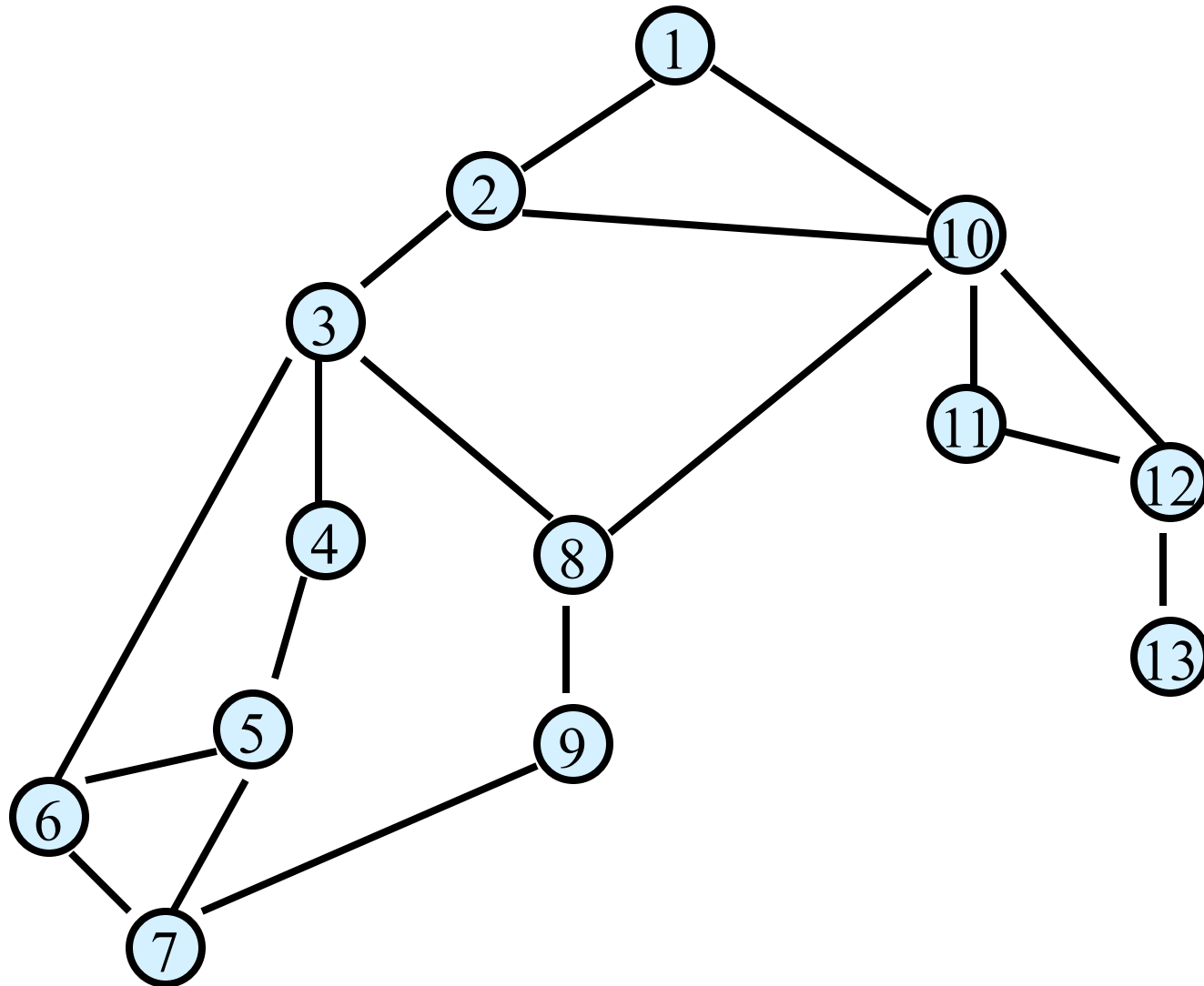
Rel: Two are related if they have students in common

## Traveling Salesperson Problem:

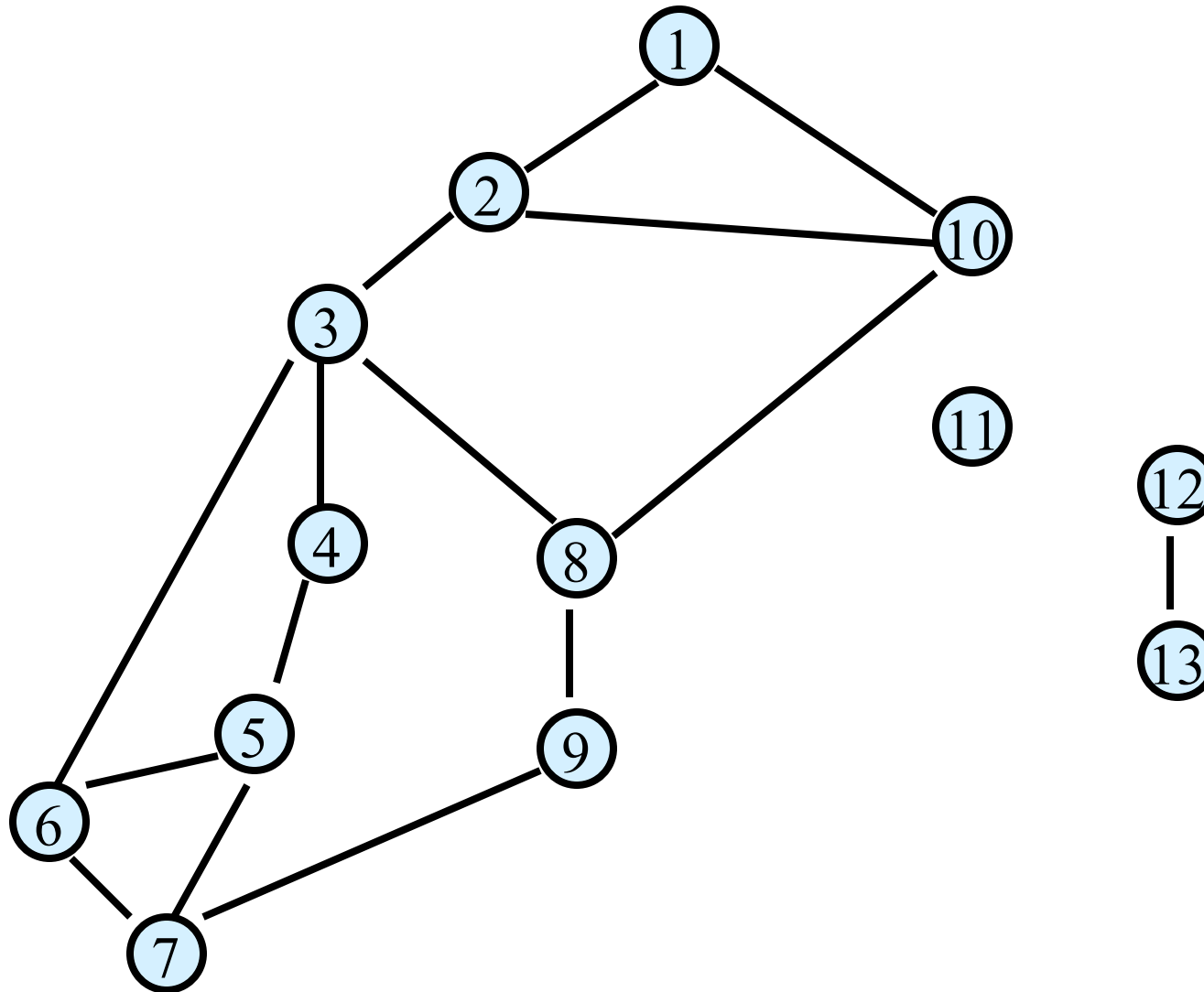
Obj: Cities

Rel: Two are related if can travel *directly* between them

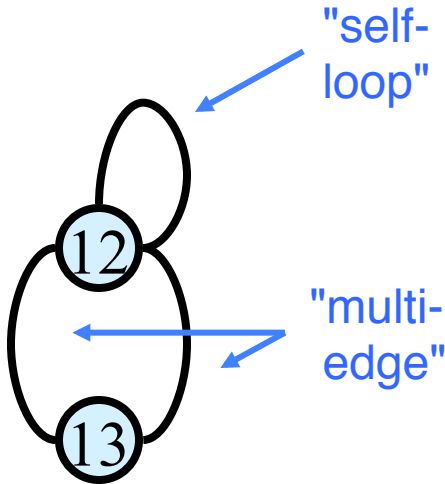
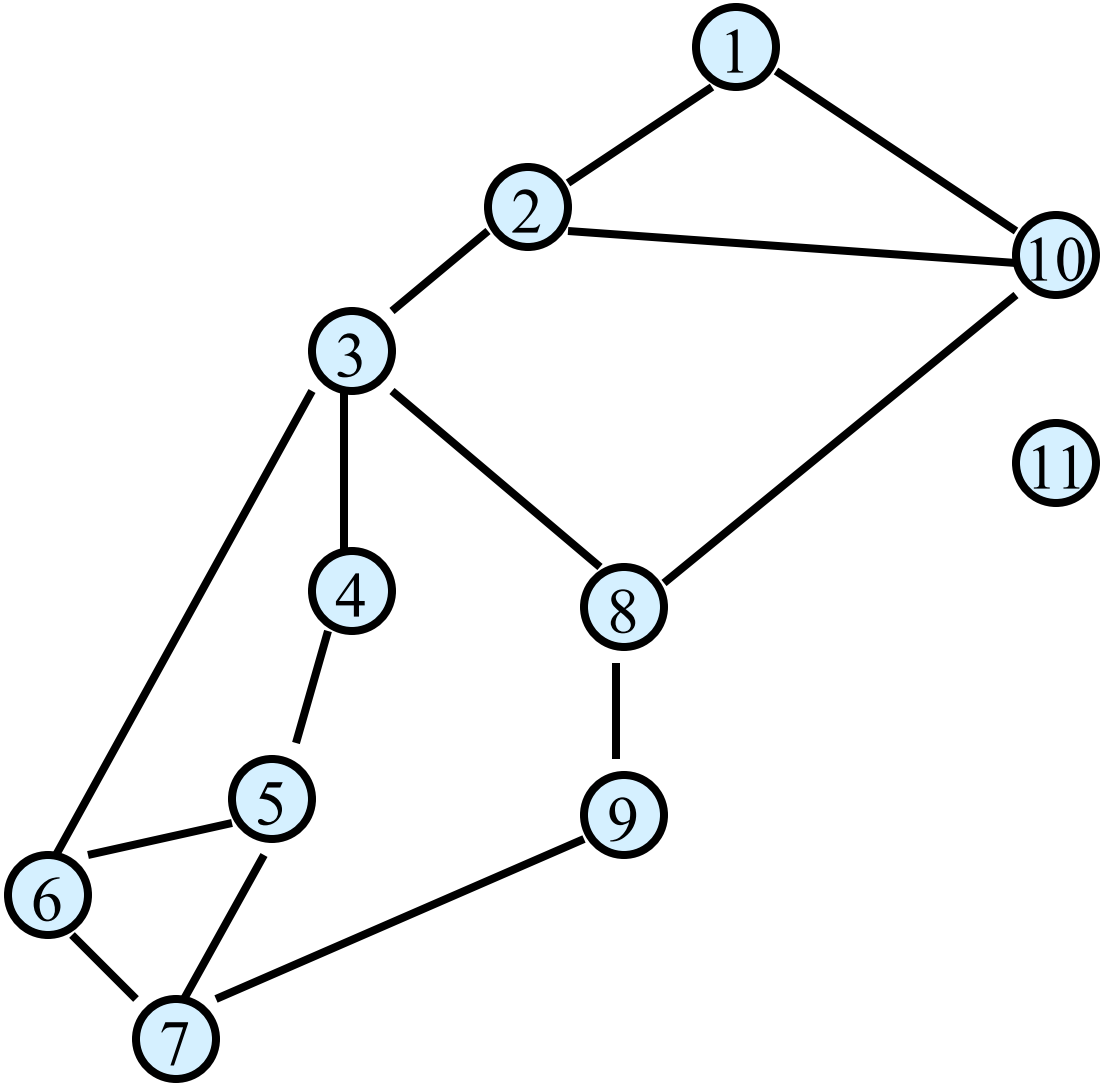
# Undirected Graph $G = (V, E)$



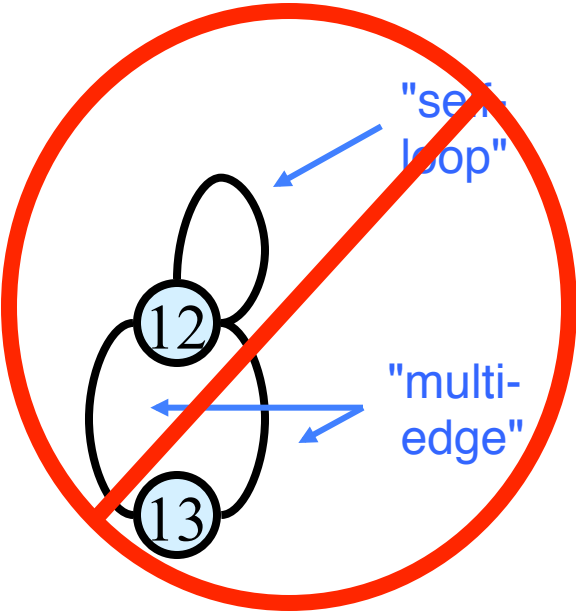
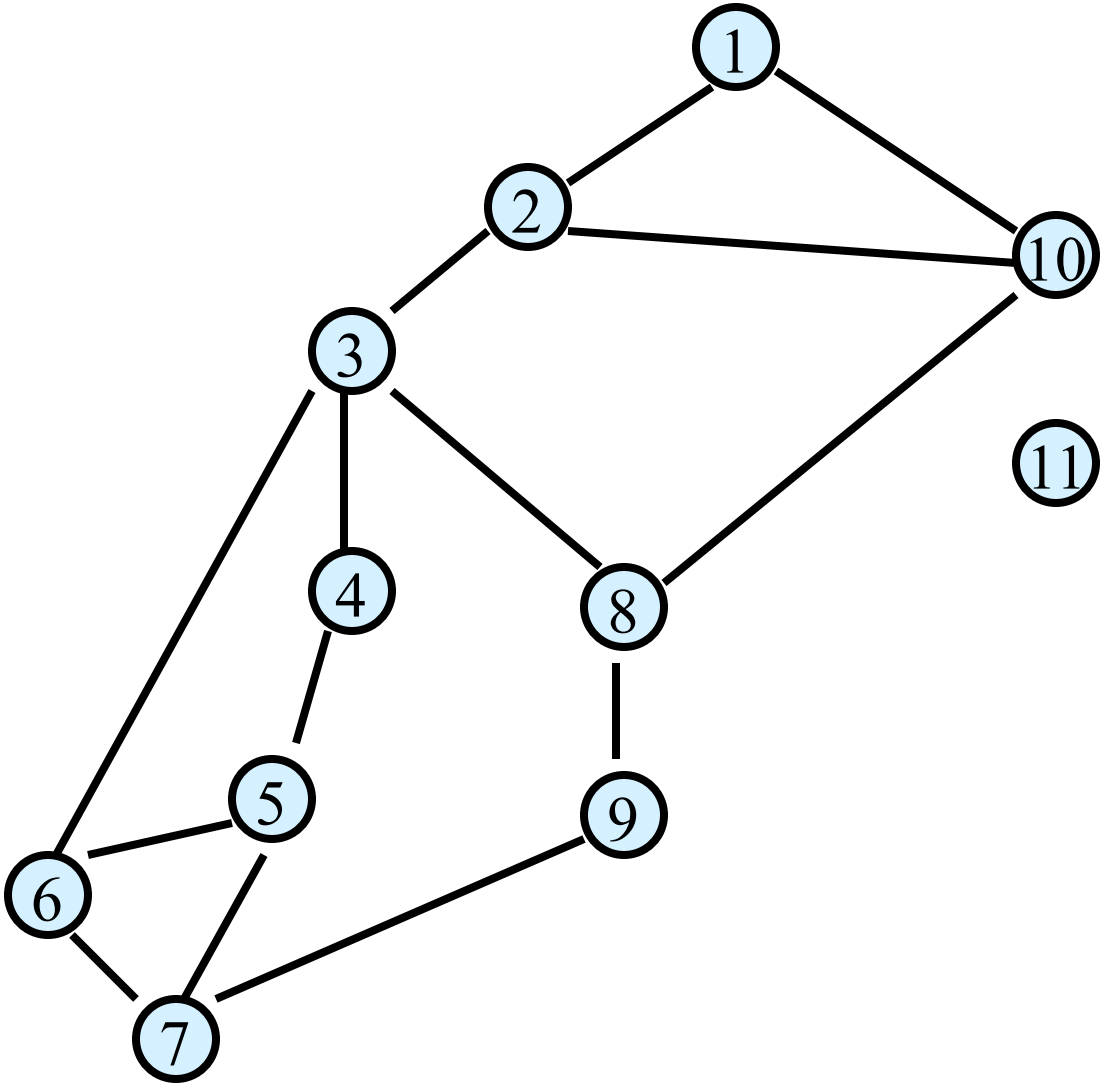
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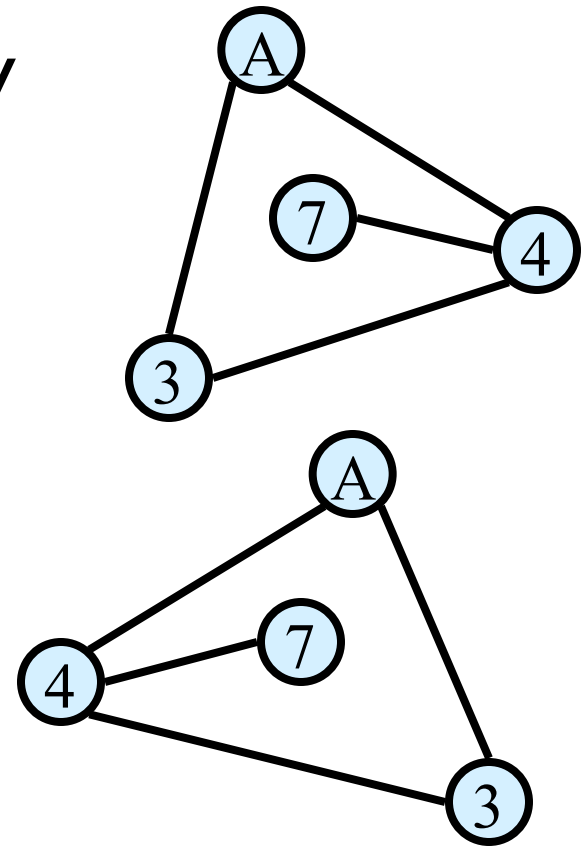
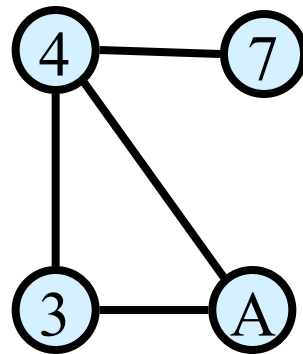
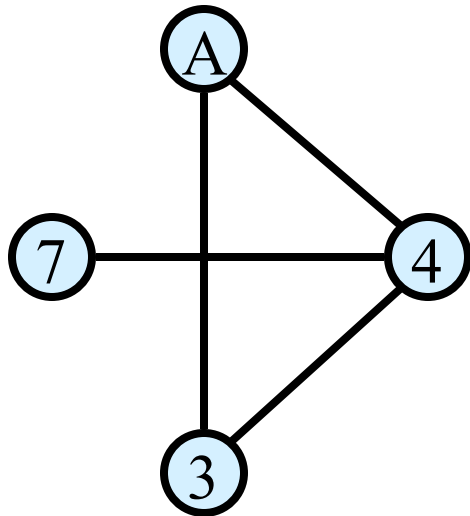
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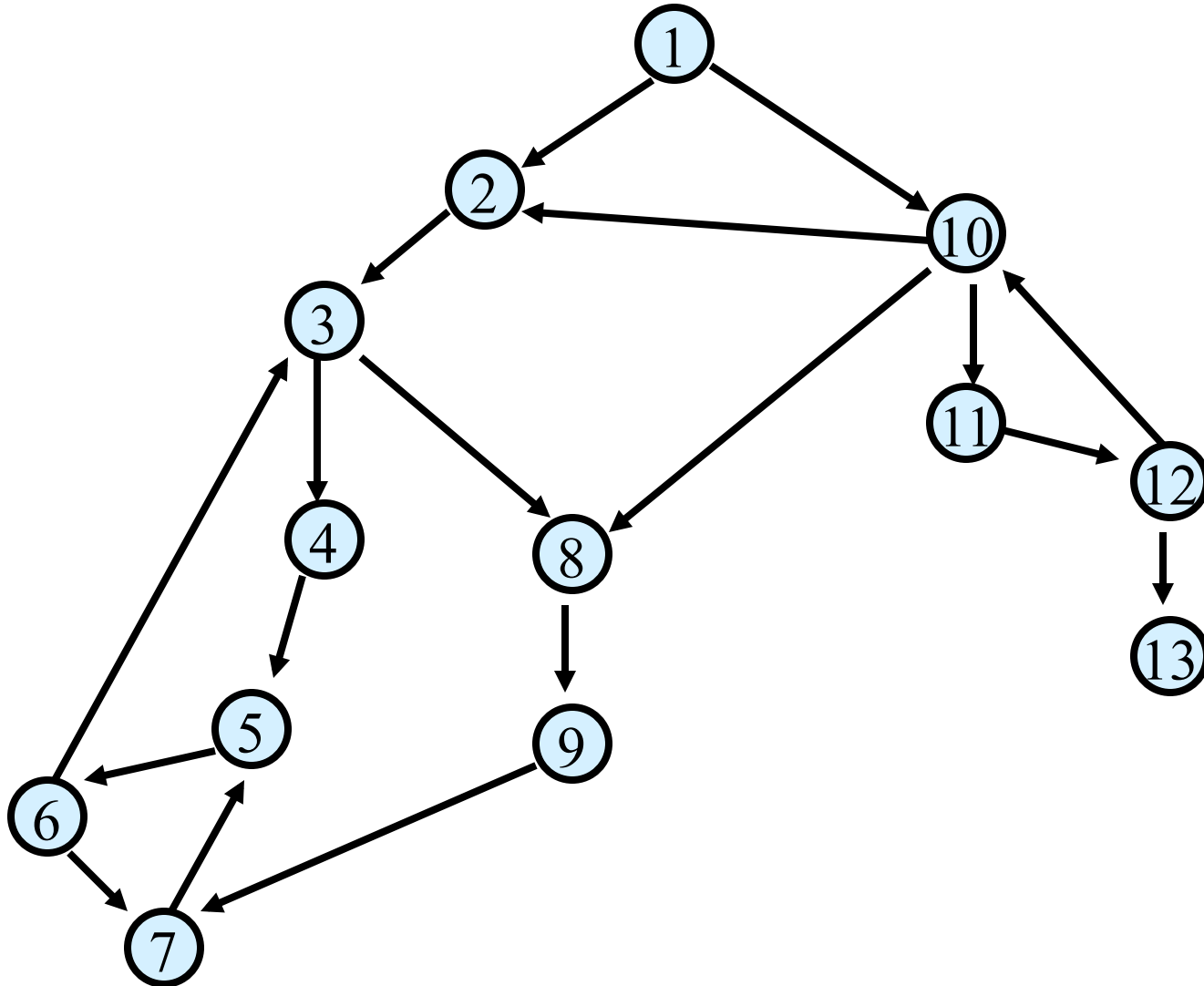


# Graphs don't live in Flatland

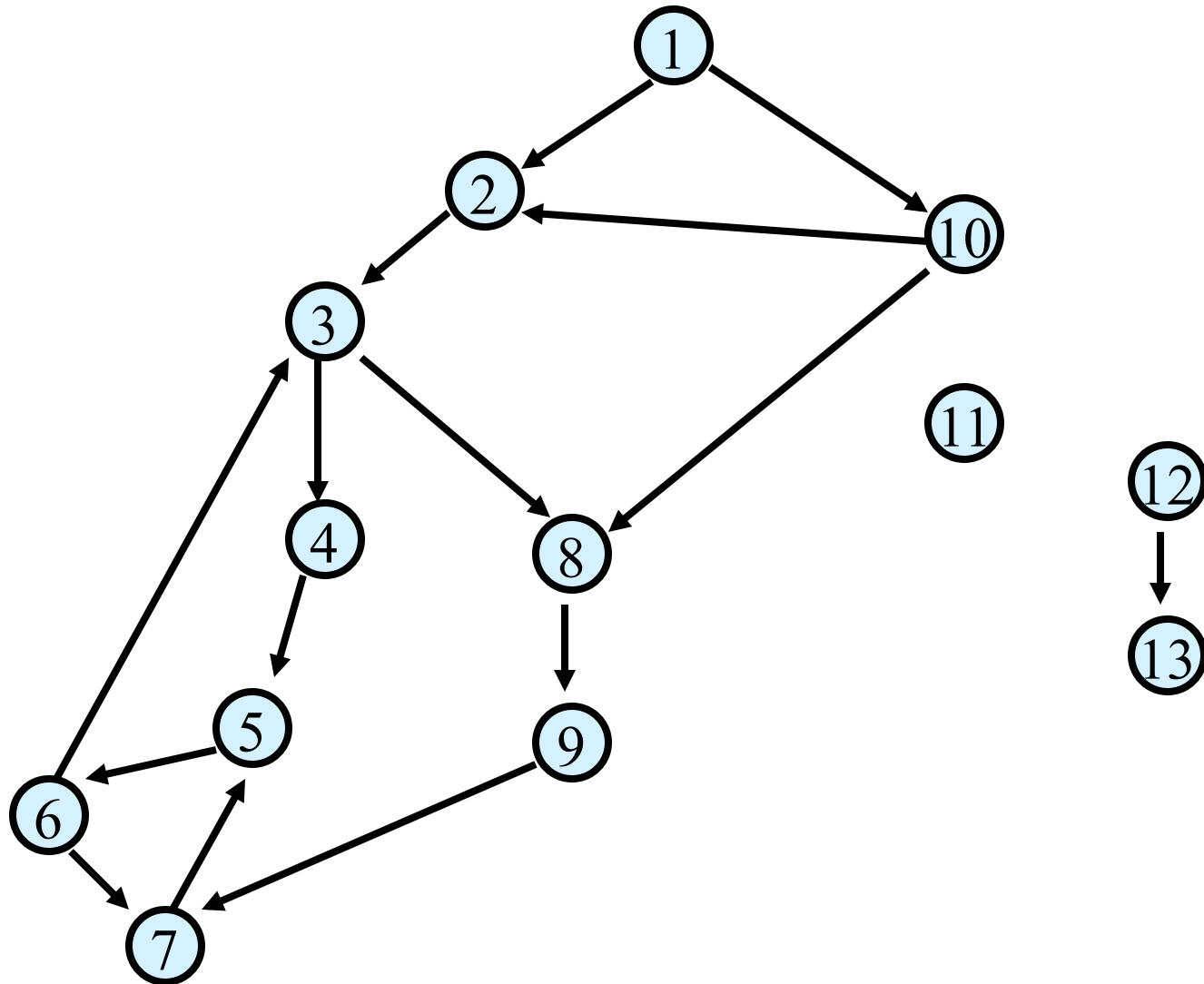
Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.



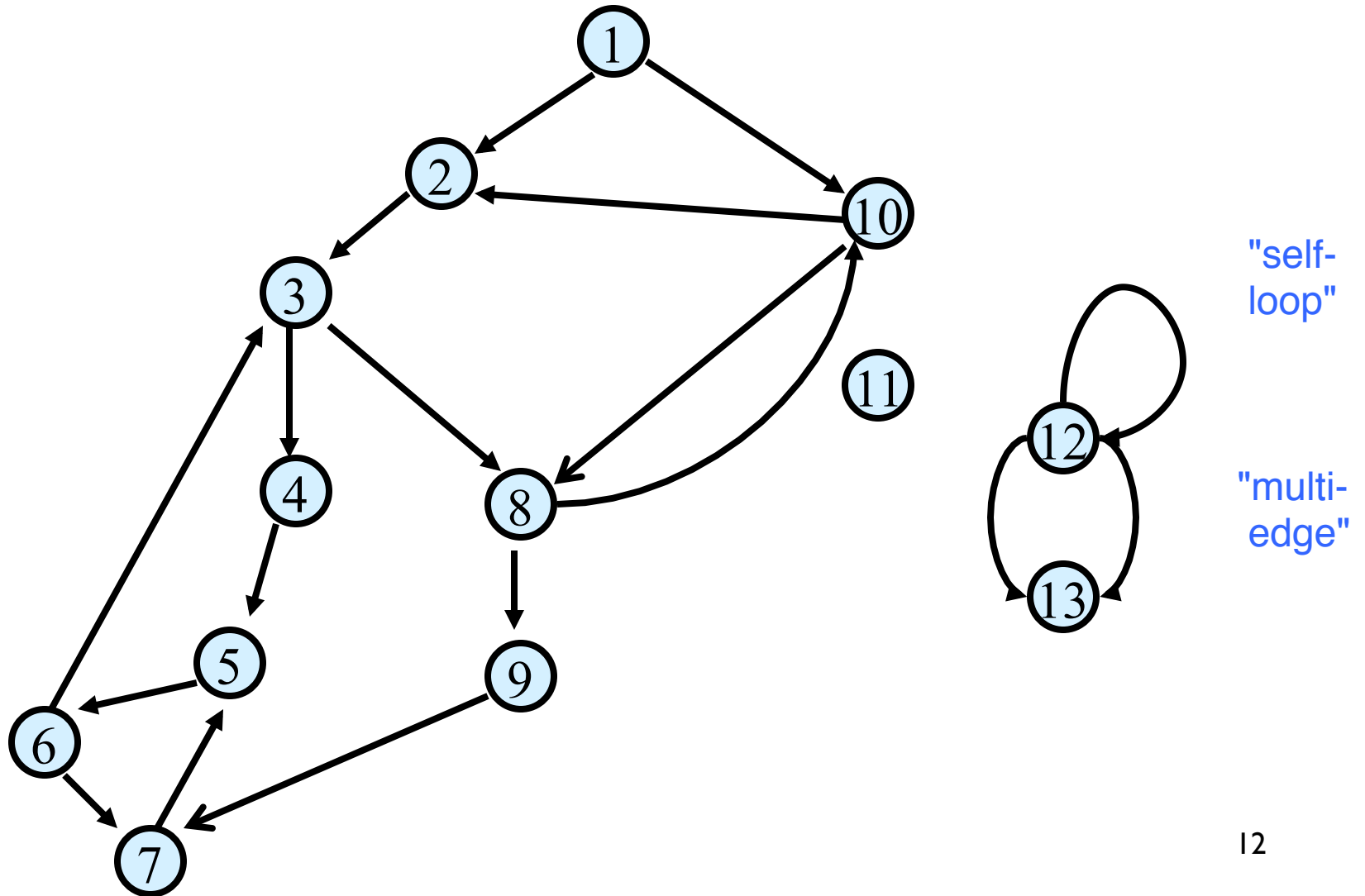
# Directed Graph $G = (V, E)$



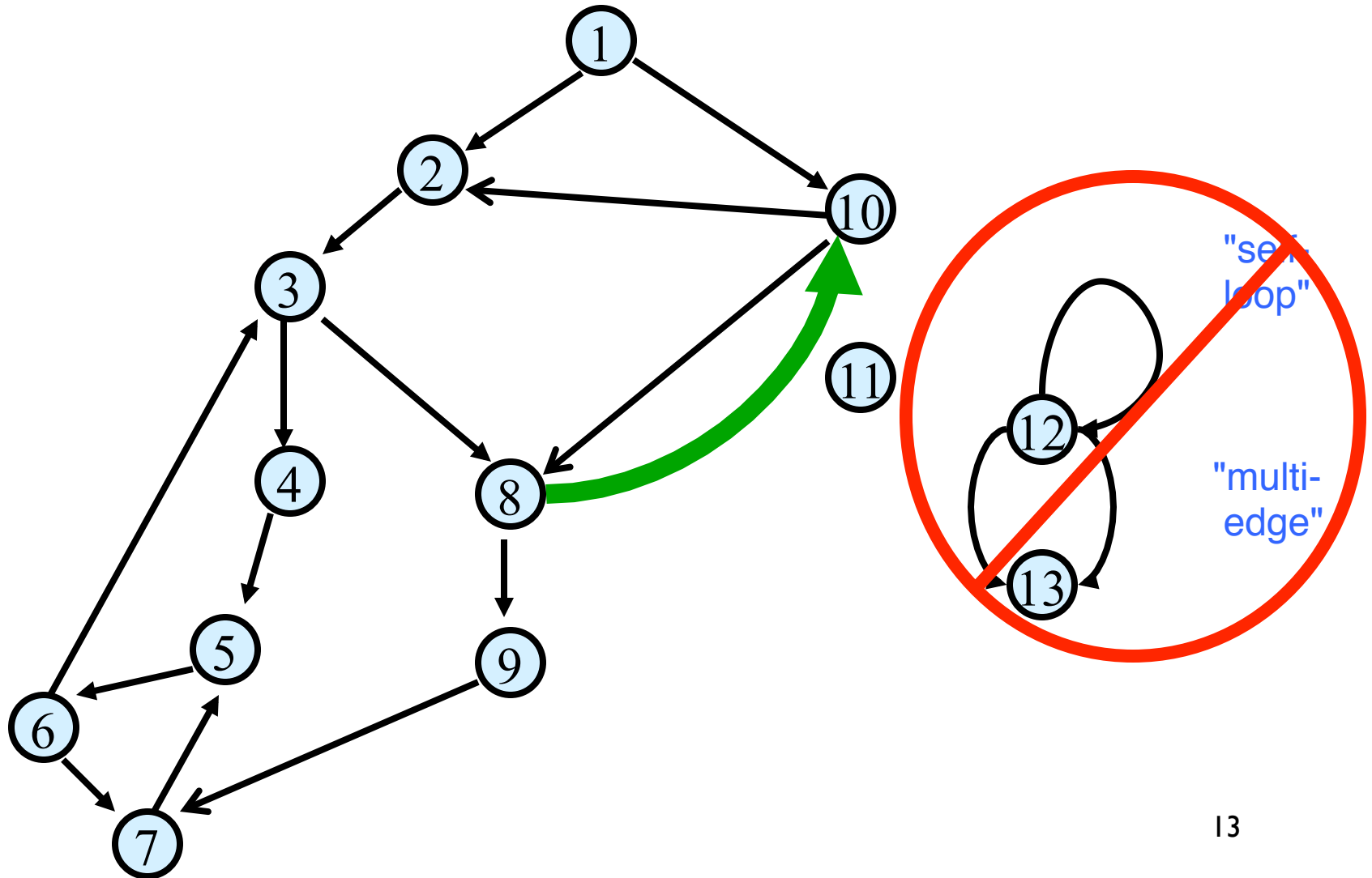
# Directed Graph $G = (V, E)$



# Directed Graph $G = (V, E)$



# Directed Graph $G = (V, E)$



# Specifying undirected graphs as input

What are the vertices?

Explicitly list them:

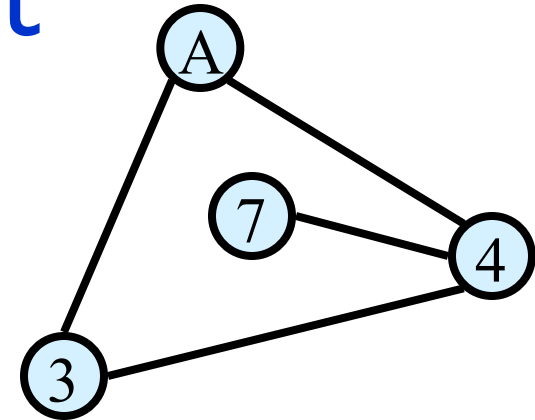
`{"A", "7", "3", "4"}`

What are the edges?

Either, set of edges

`{{A,3}, {7,4}, {4,3}, {4,A}}`

Or, (symmetric) adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

# Specifying directed graphs as input

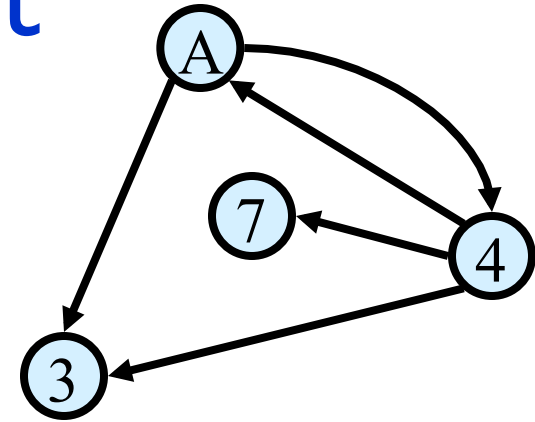
What are the vertices?

Explicitly list them:  
`{"A", "7", "3", "4"}`

What are the edges?

Either, set of directed edges:  
`{(A,4), (4,7), (4,3), (4,A), (A,3)}`

Or, (nonsymmetric)  
adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	0
3	0	0	0	0
4	1	1	1	0

# # Vertices vs # Edges

Let  $G$  be an undirected graph with  $n$  vertices and  $m$  edges. How are  $n$  and  $m$  related?

Since

every edge connects two different vertices (no loops),  
and no two edges connect the same two vertices (no  
multi-edges),

it must be true that:

$$0 \leq m \leq n(n-1)/2 = O(n^2)$$



# More Cool Graph Lingo

A graph is called *sparse* if  $m \ll n^2$ , otherwise it is *dense*

Boundary is somewhat fuzzy;  $O(n)$  edges is certainly sparse,  $\Omega(n^2)$  edges is dense.

Sparse graphs are common in practice

E.g., all planar graphs are sparse ( $m \leq 3n-6$ , for  $n \geq 3$ )

Q: which is a better run time,  $O(n+m)$  or  $O(n^2)$ ?

A:  $O(n+m) = O(n^2)$ , but  $n+m$  usually way better!

# Representing Graph $G = (V, E)$

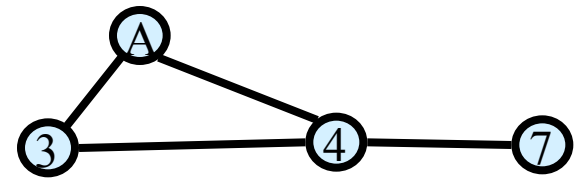
internally, indep of input format

Vertex set  $V = \{v_1, \dots, v_n\}$

Adjacency Matrix  $A$

$A[i,j] = 1$  iff  $(v_i, v_j) \in E$

Space is  $n^2$  bits



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Advantages:

$O(1)$  test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

$m \ll n^2$

# Representing Graph $G=(V,E)$

$n$  vertices,  $m$  edges

Adjacency List:

$O(n+m)$  words

Advantages:

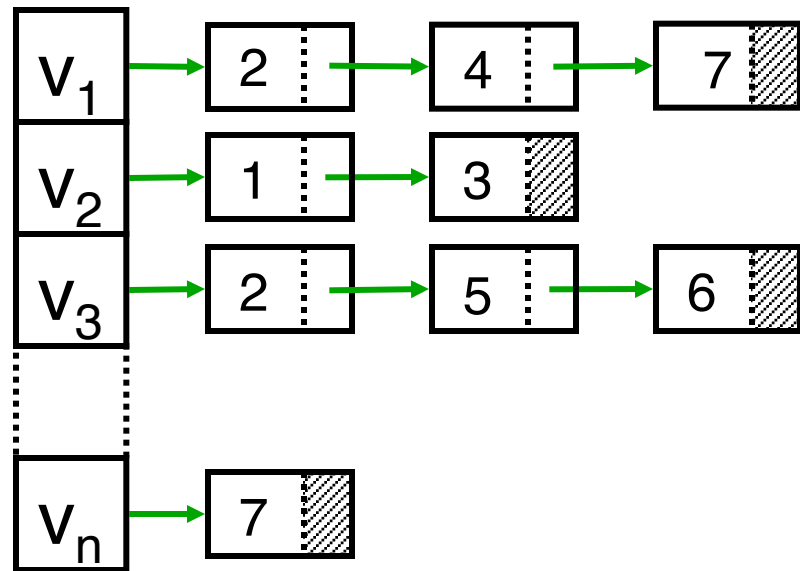
Compact for  
sparse graphs

Easily see all edges

Disadvantages

More complex data structure

no  $O(1)$  edge test



# Representing Graph $G=(V,E)$

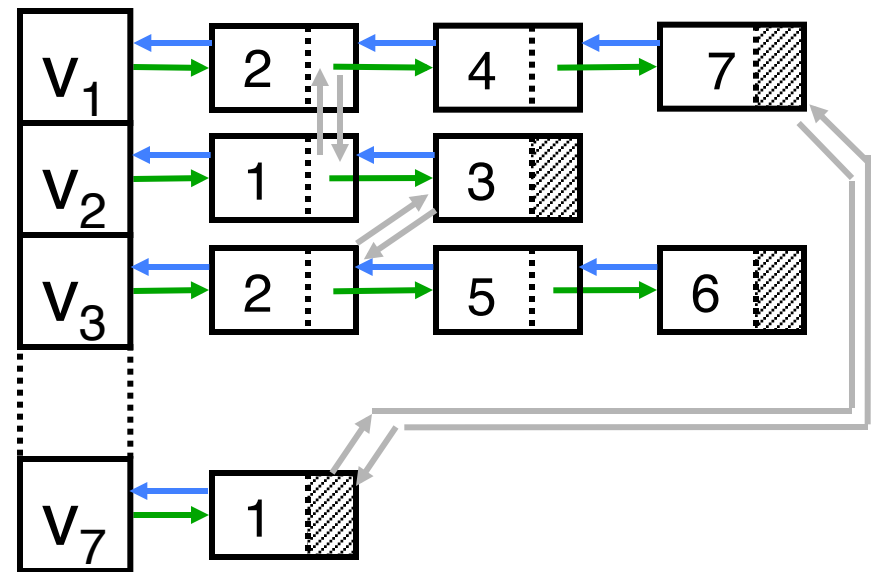
n vertices, m edges

Adjacency List:

$O(n+m)$  words

Back- and cross pointers allow easier traversal and deletion of edges, *if needed*, but don't bother if not:

- more work to build,
- more storage overhead ( $\sim 3m$  pointers)



# Graph Traversal

Learn the basic structure of a graph

"Walk," via edges, from a fixed starting vertex  $s$  to all vertices reachable from  $s$

Being *orderly* helps. Two common ways:

Breadth-First Search

Depth-First Search

# Breadth-First Search

Completely explore the vertices in order of their distance from  $s$

Naturally implemented using a queue

# Breadth-First Search

Idea: Explore from  $s$  in all possible directions, layer by layer.

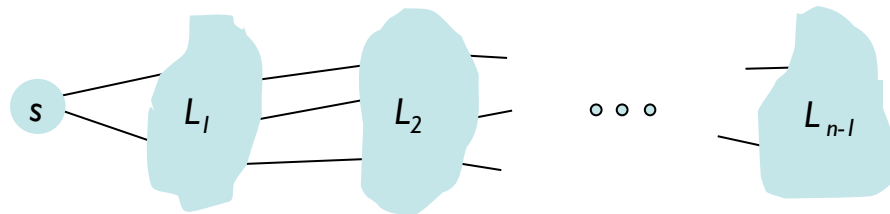
BFS algorithm.

$$L_0 = \{ s \}.$$

$L_1$  = all neighbors of  $L_0$ .

$L_2$  = all nodes not in  $L_0$  or  $L_1$ , and having an edge to a node in  $L_1$ .

$L_{i+1}$  = all nodes not in earlier layers, and having an edge to a node in  $L_i$ .



Theorem. For each  $i$ ,  $L_i$  consists of all nodes at distance (i.e., min path length) exactly  $i$  from  $s$ .

Cor: There is a path from  $s$  to  $t$  iff  $t$  appears in some layer.

# Graph Traversal: Implementation

Learn the basic structure of a graph

"Walk," via edges, from a fixed starting vertex  $s$  to all vertices reachable from  $s$

Three states of vertices

*undiscovered*

*discovered*

*fully-explored*



# BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

BFS(s)

mark s "discovered"

queue = { s }

while queue not empty

    u = remove\_first(queue)

    for each edge {u,x}

        if (x is undiscovered)

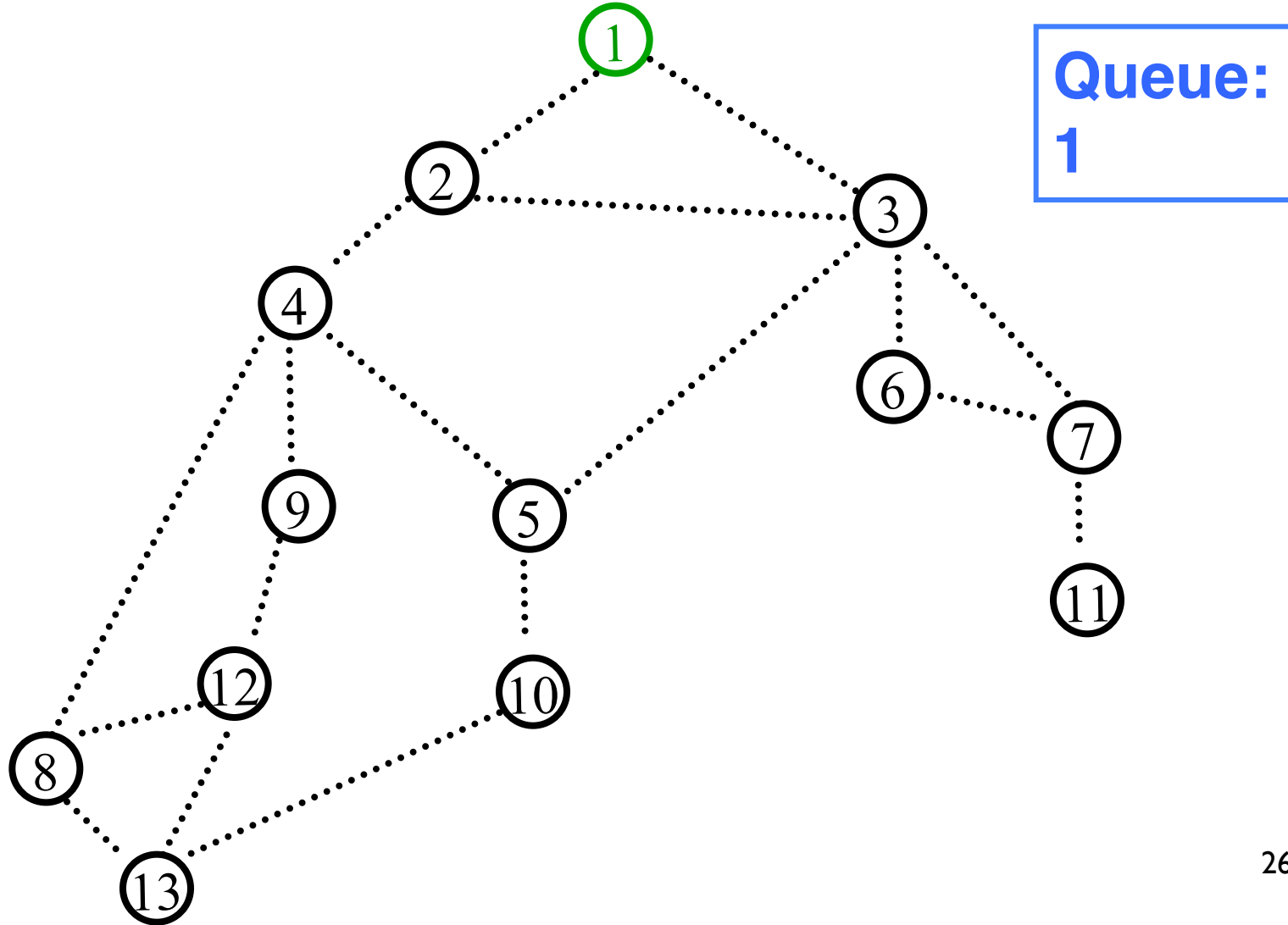
            mark x discovered

            append x on queue

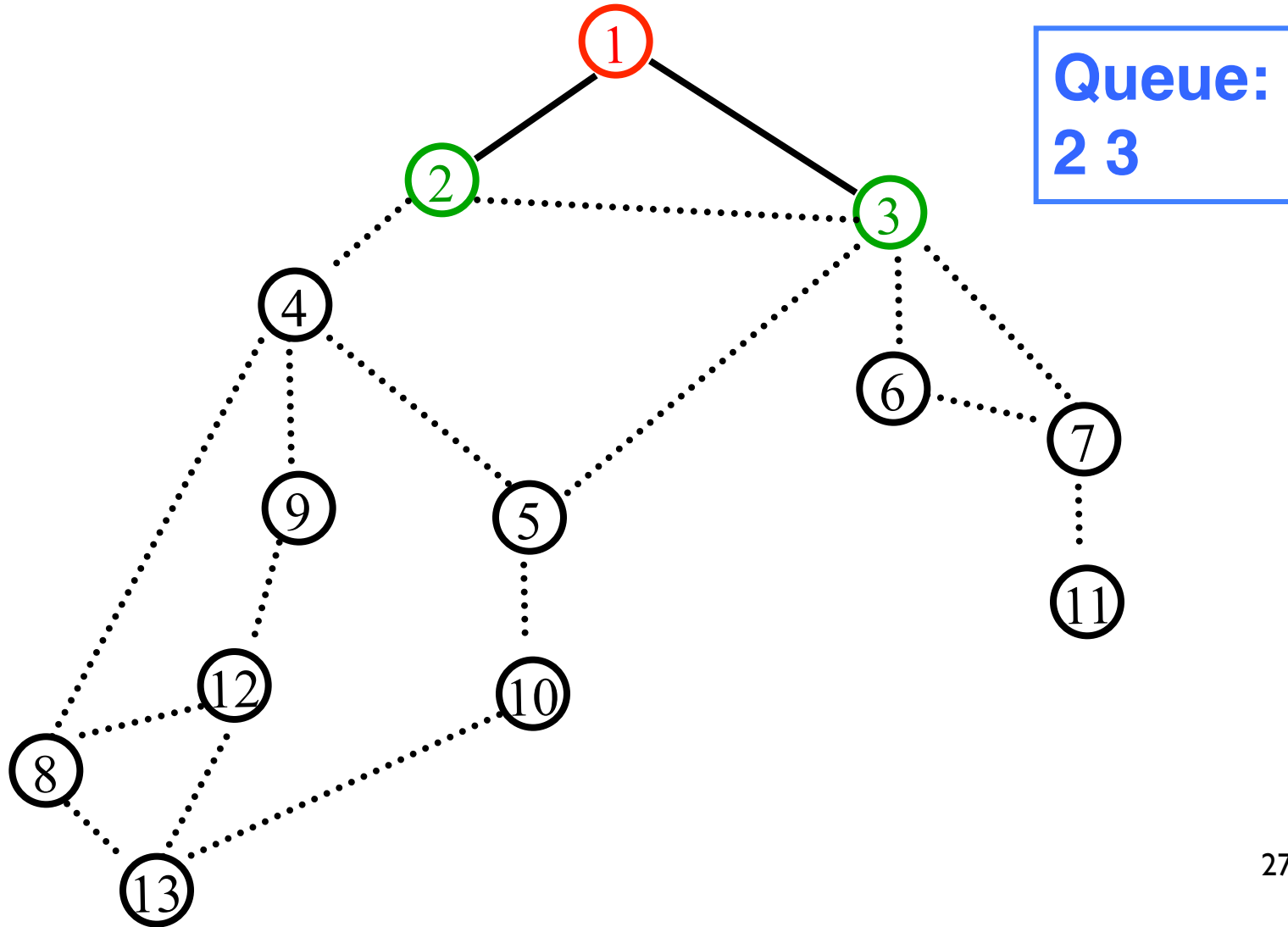
    mark u fully explored

Exercise: modify  
code to number  
vertices & compute  
level numbers

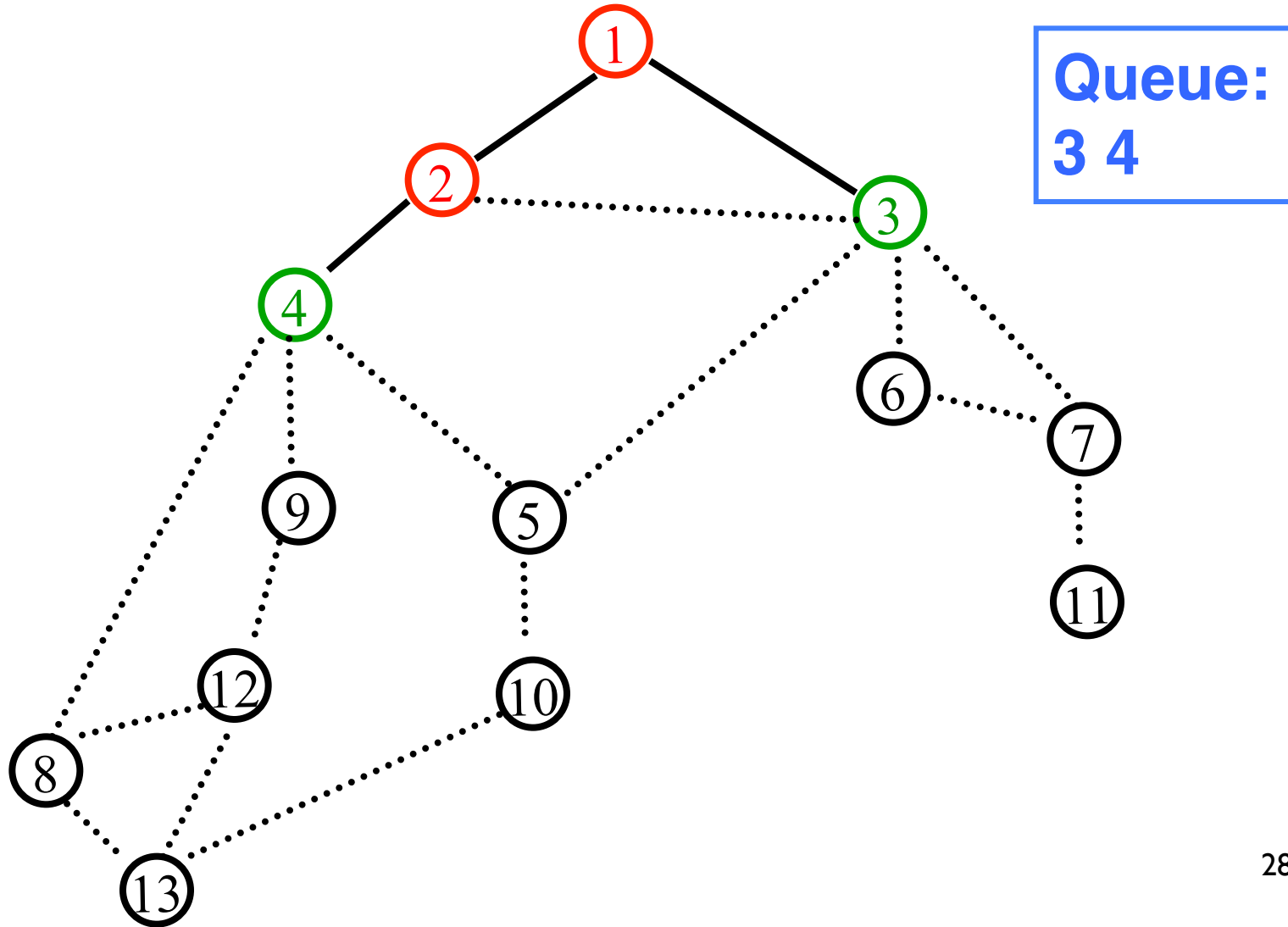
# BFS(v)



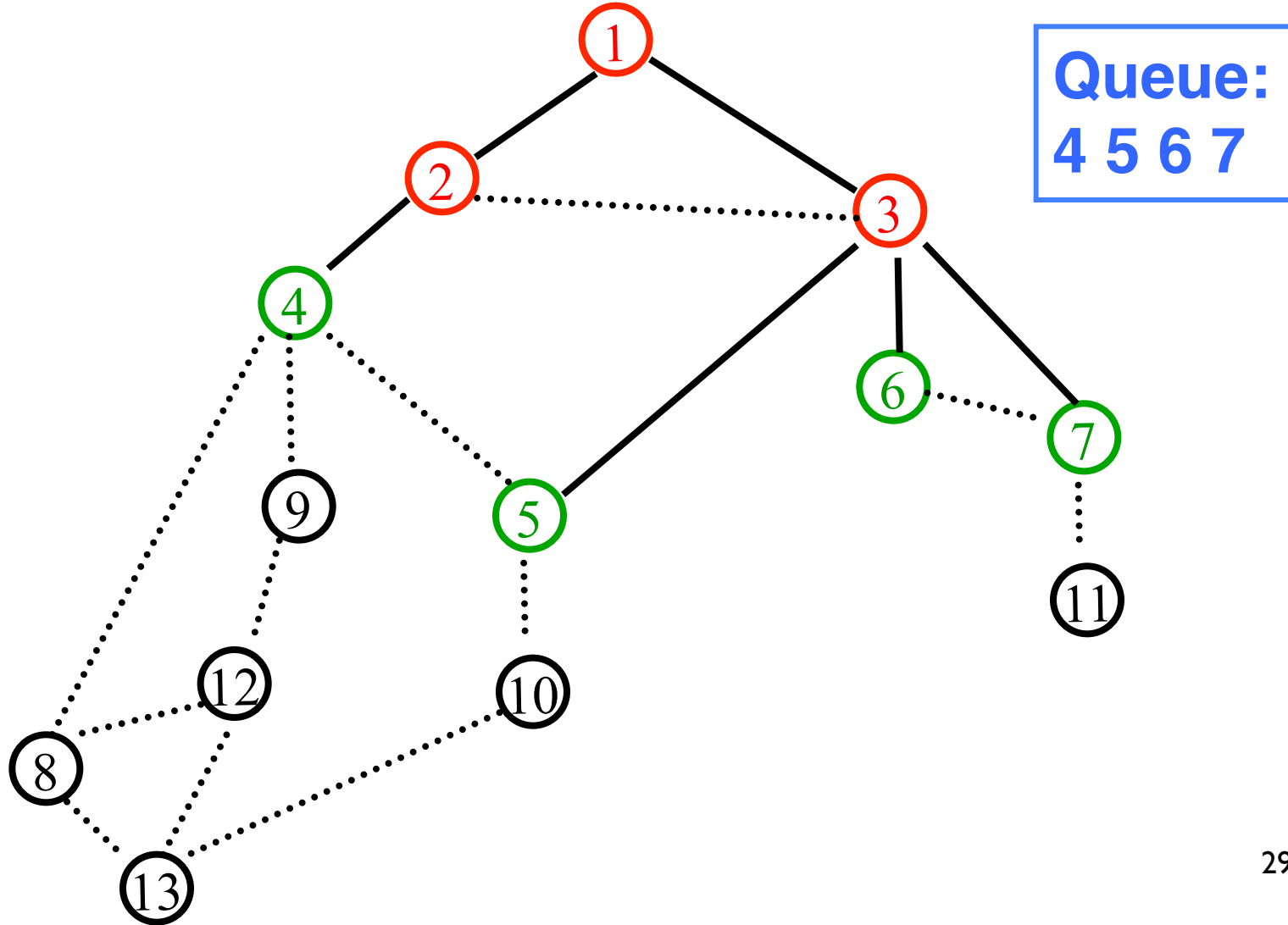
# BFS(v)



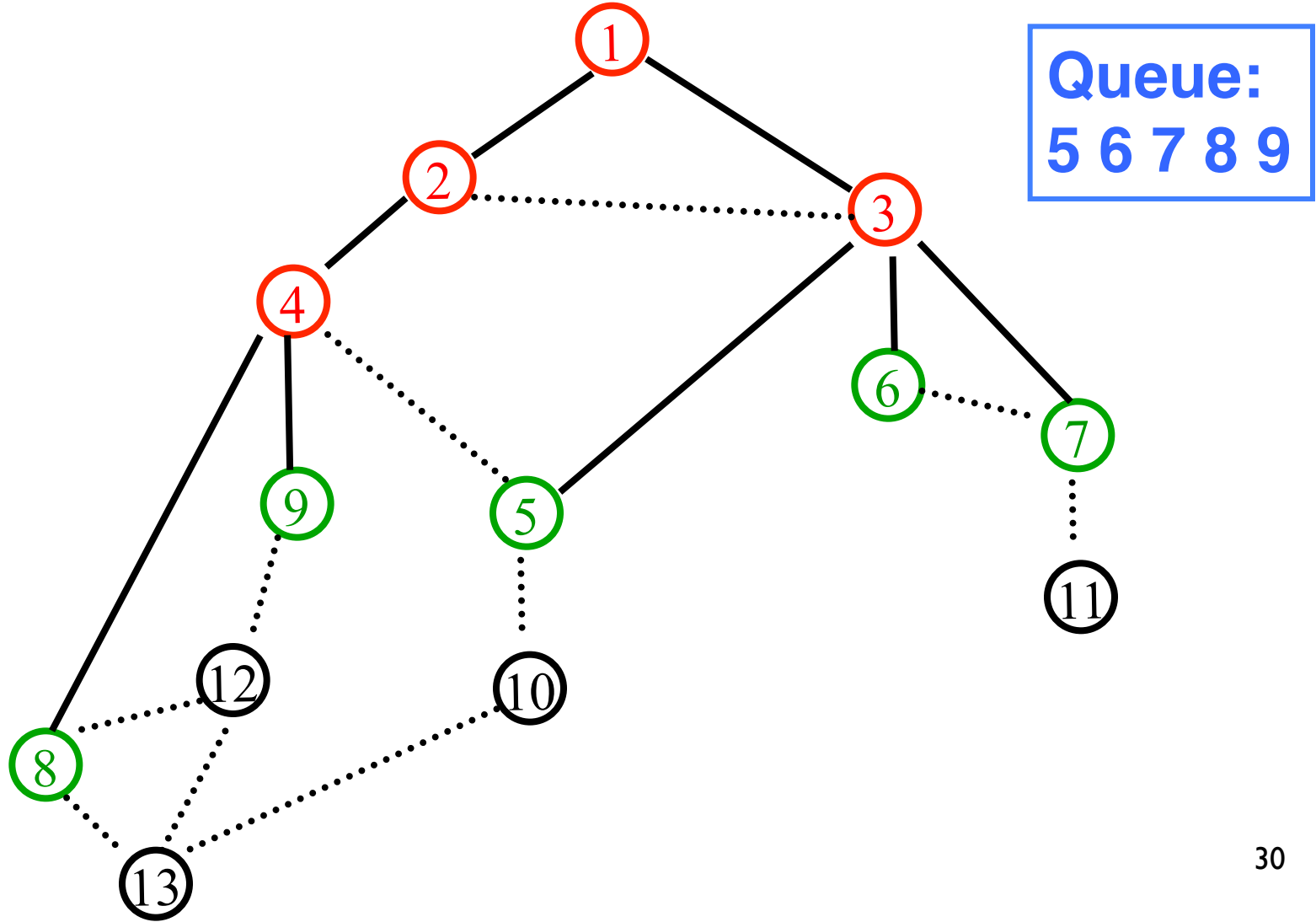
# BFS(v)



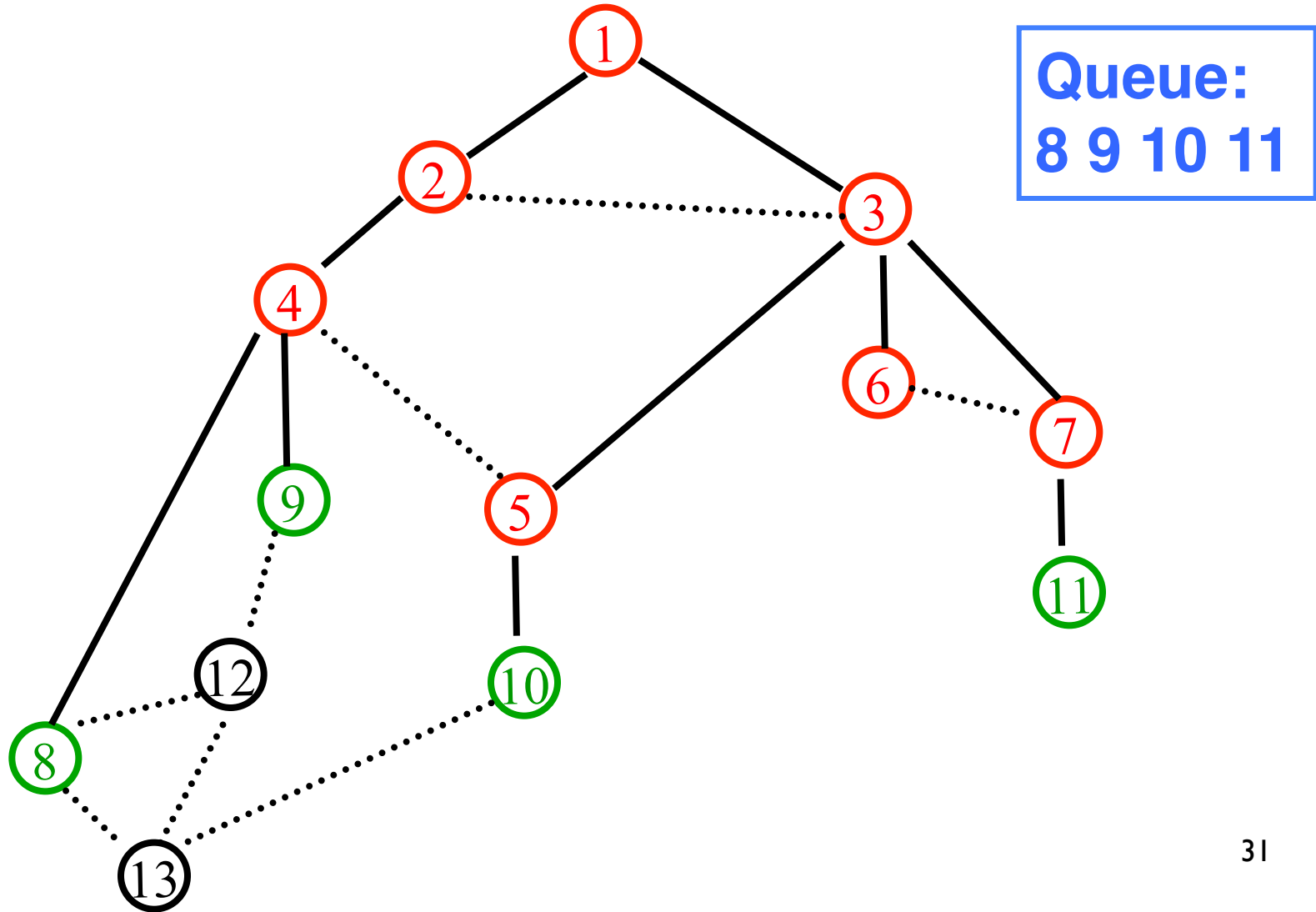
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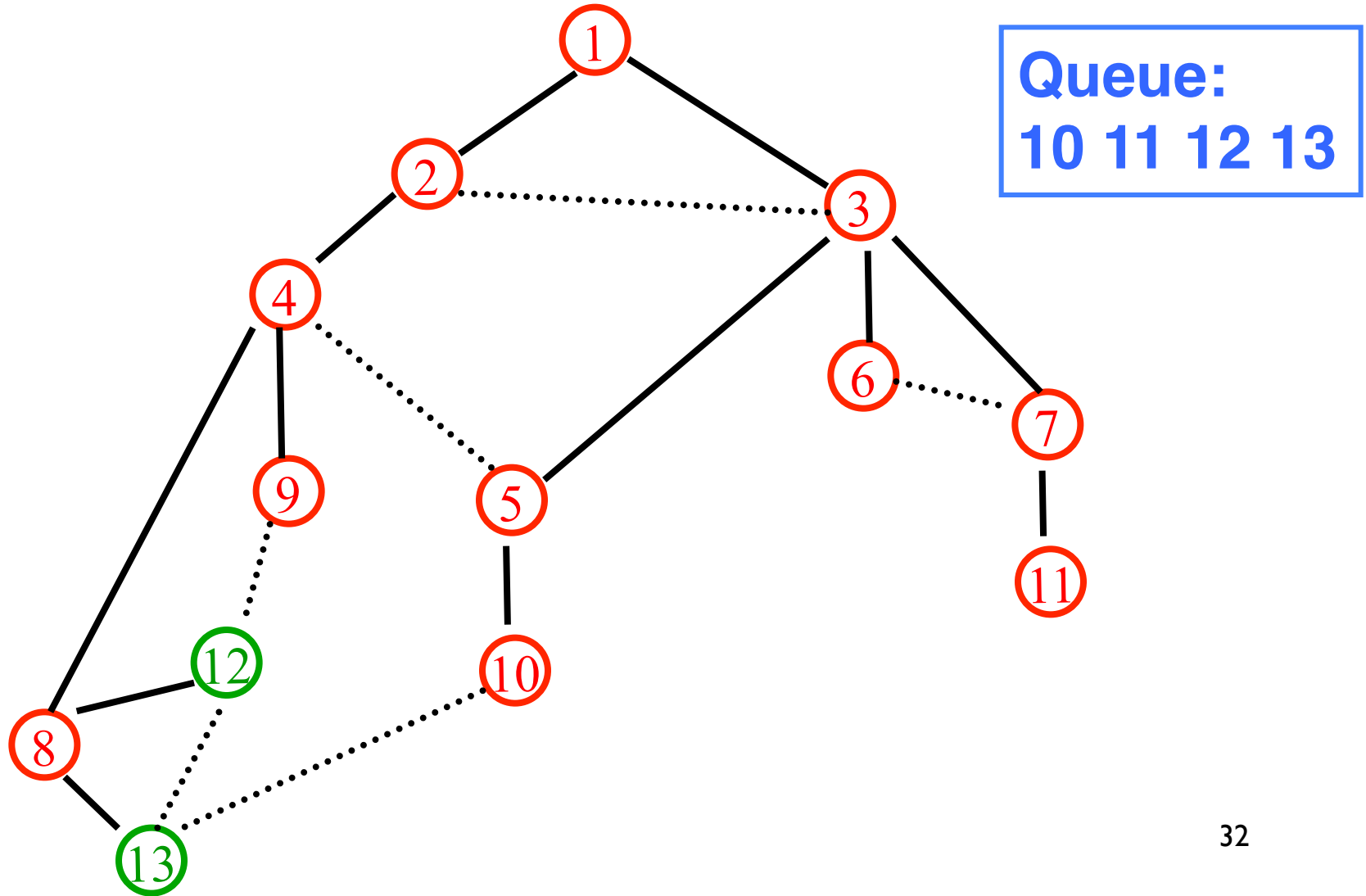
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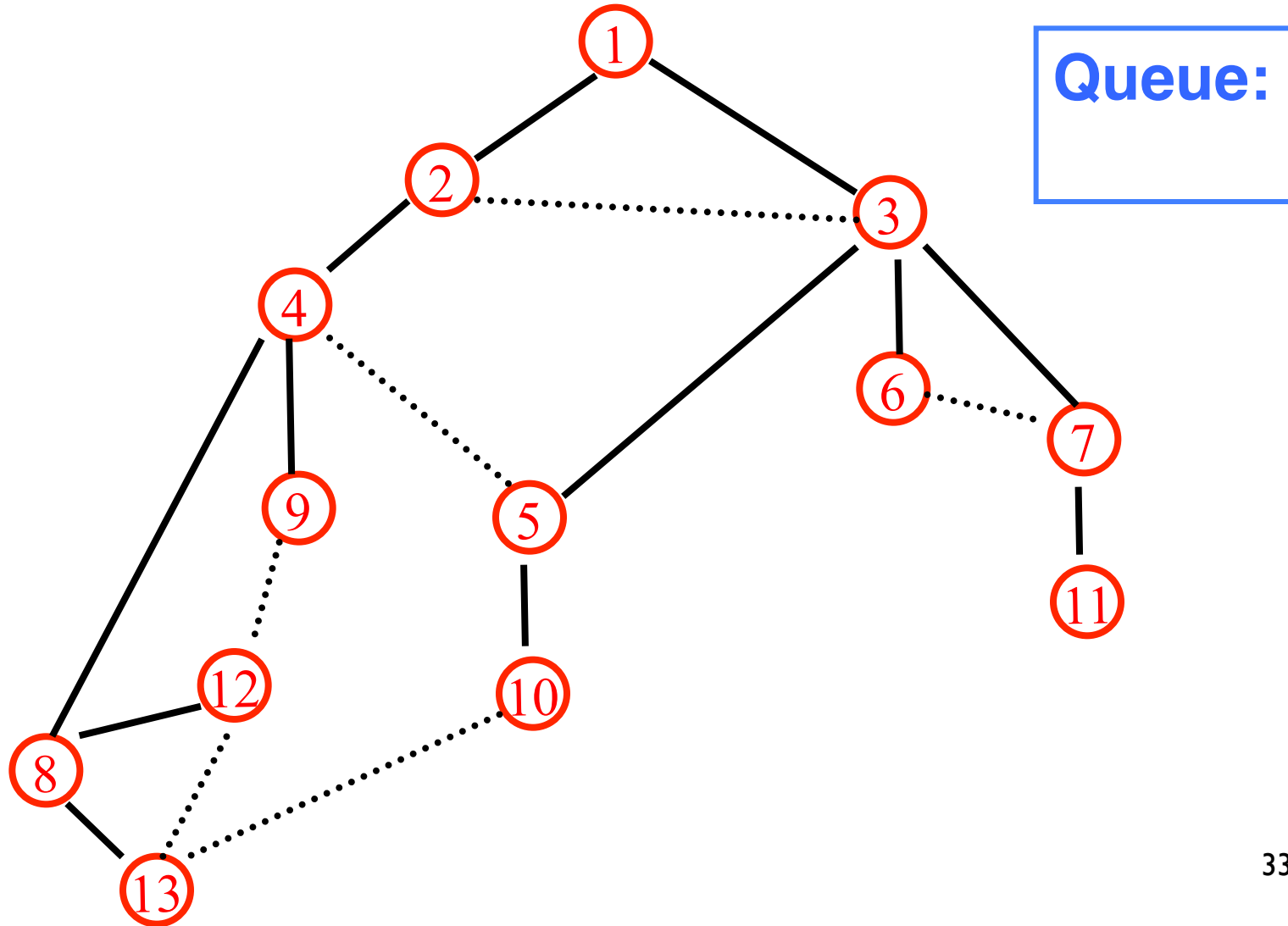


# BFS(v)





# BFS(v)



# BFS: Analysis, I

$O(n)$  Global initialization: mark all vertices "undiscovered"

+ BFS(s)

$O(1)$  mark s "discovered"

+ queue = { s }

$O(n)$  while queue not empty

x u = remove\_first(queue)

$O(n)$  for each edge {u,x}

if (x is undiscovered)

mark x discovered

append x on queue

mark u fully explored

=

$O(n^2)$

Simple analysis:  
2 nested loops.  
Get worst-case  
number of  
iterations of  
each; multiply.

# BFS: Analysis, II

Above analysis *correct*, but *pessimistic*, assuming  $G$  is sparse, edge list representation: can't have  $\Omega(n)$  edges incident to each of  $\Omega(n)$  distinct "u" vertices.  
Alt, more global analysis:

Each edge is explored once from each end-point, so *total* runtime of inner loop is  $O(m)$ , (assuming edge-lists)

Exercise: extend algorithm and analysis to non-connected graph

Total  $O(n+m)$ ,  $n = \#$  nodes,  $m = \#$  edges

# Properties of (Undirected) BFS(v)

BFS(v) visits  $x$  if and only if there is a path in  $G$  from  $v$  to  $x$ .

Edges into then-undiscovered vertices define a **tree** – the "breadth first spanning tree" of  $G$

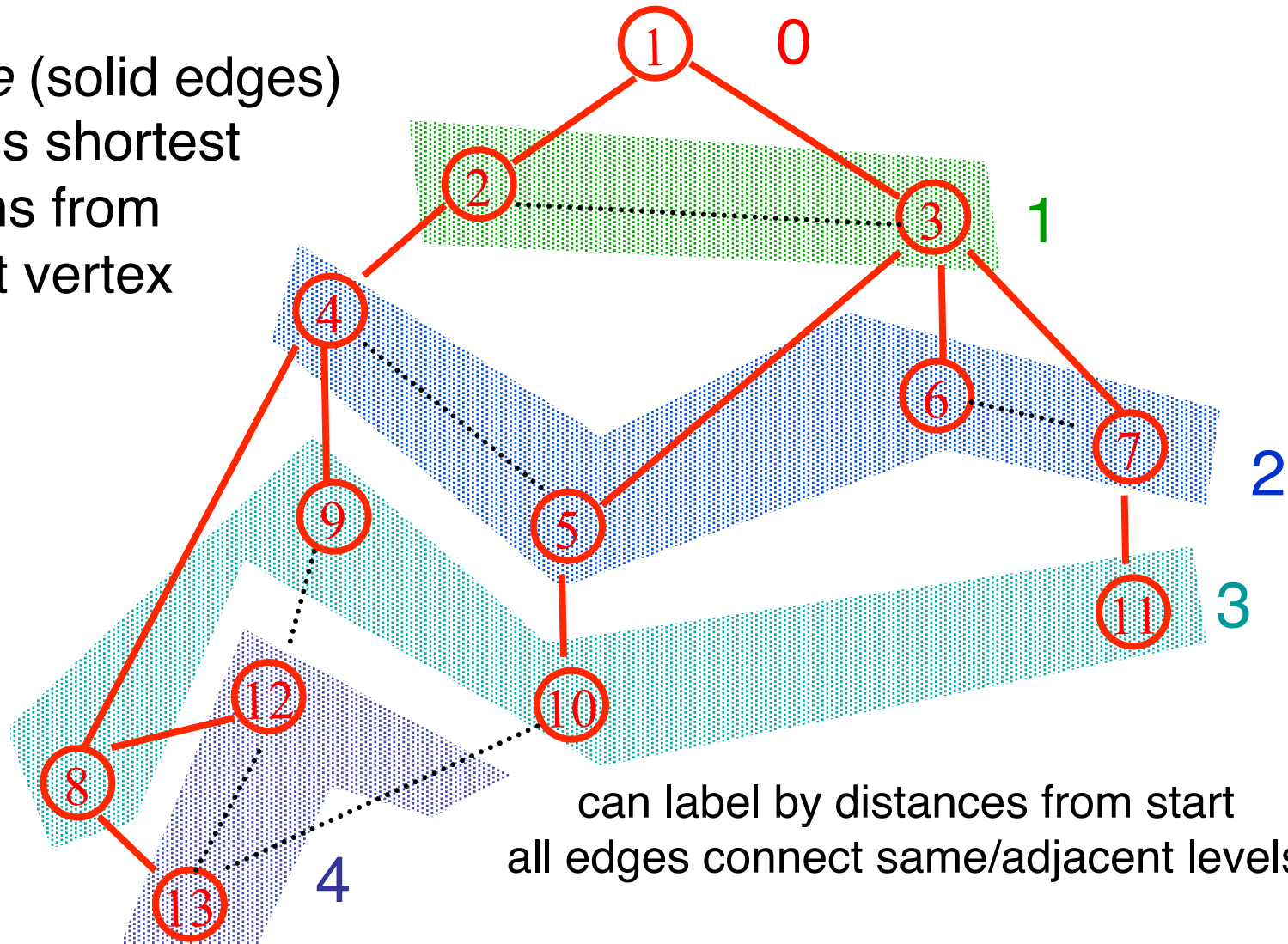
Level  $i$  in this tree are exactly those vertices  $u$  such that the shortest path (in  $G$ , not just the tree) from the root  $v$  is of length  $i$ .

**All** non-tree edges join vertices on the same or adjacent levels

not true  
of every  
spanning  
tree!

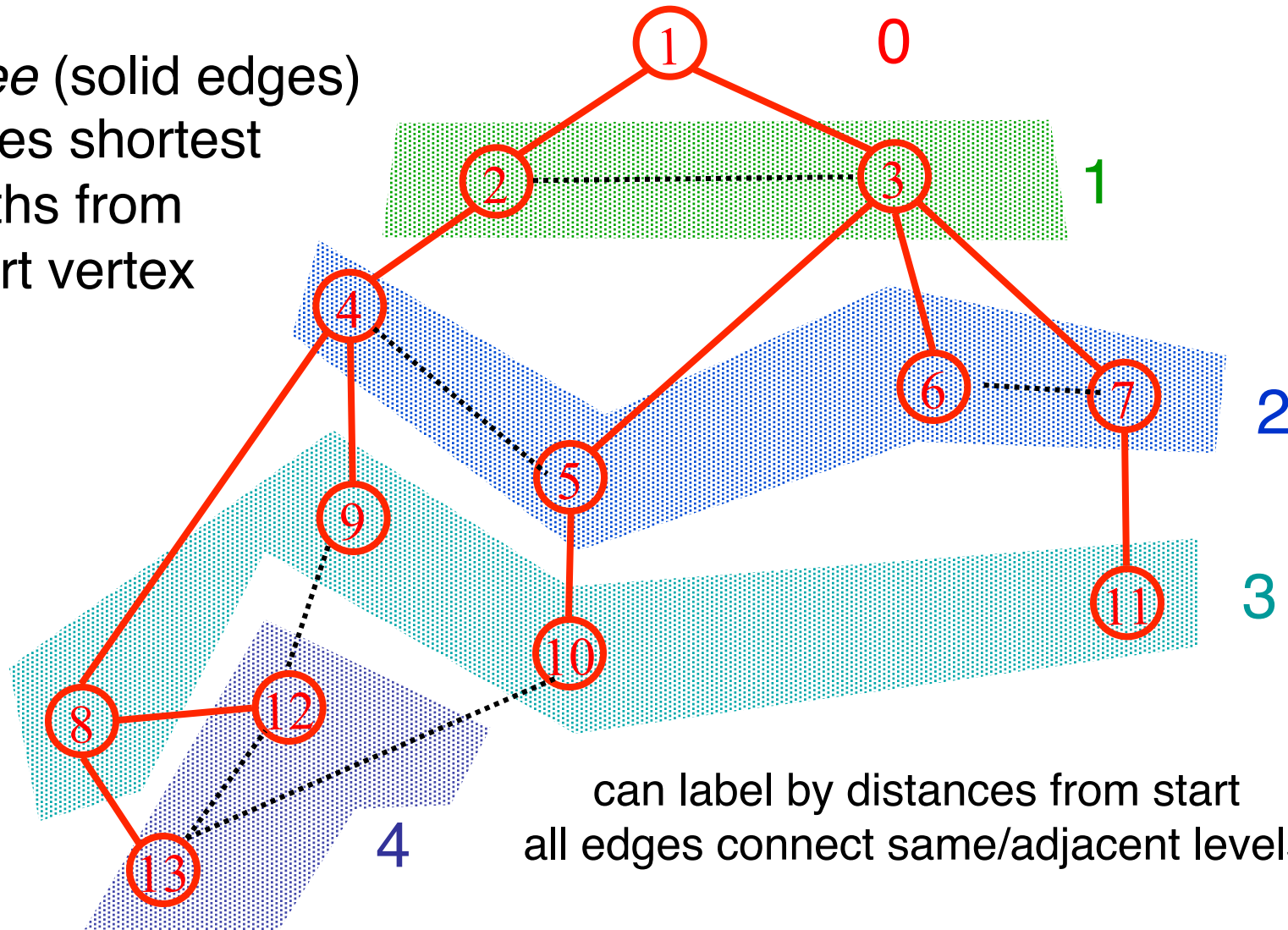
# BFS Application: Shortest Paths

*Tree* (solid edges)  
gives shortest  
paths from  
start vertex



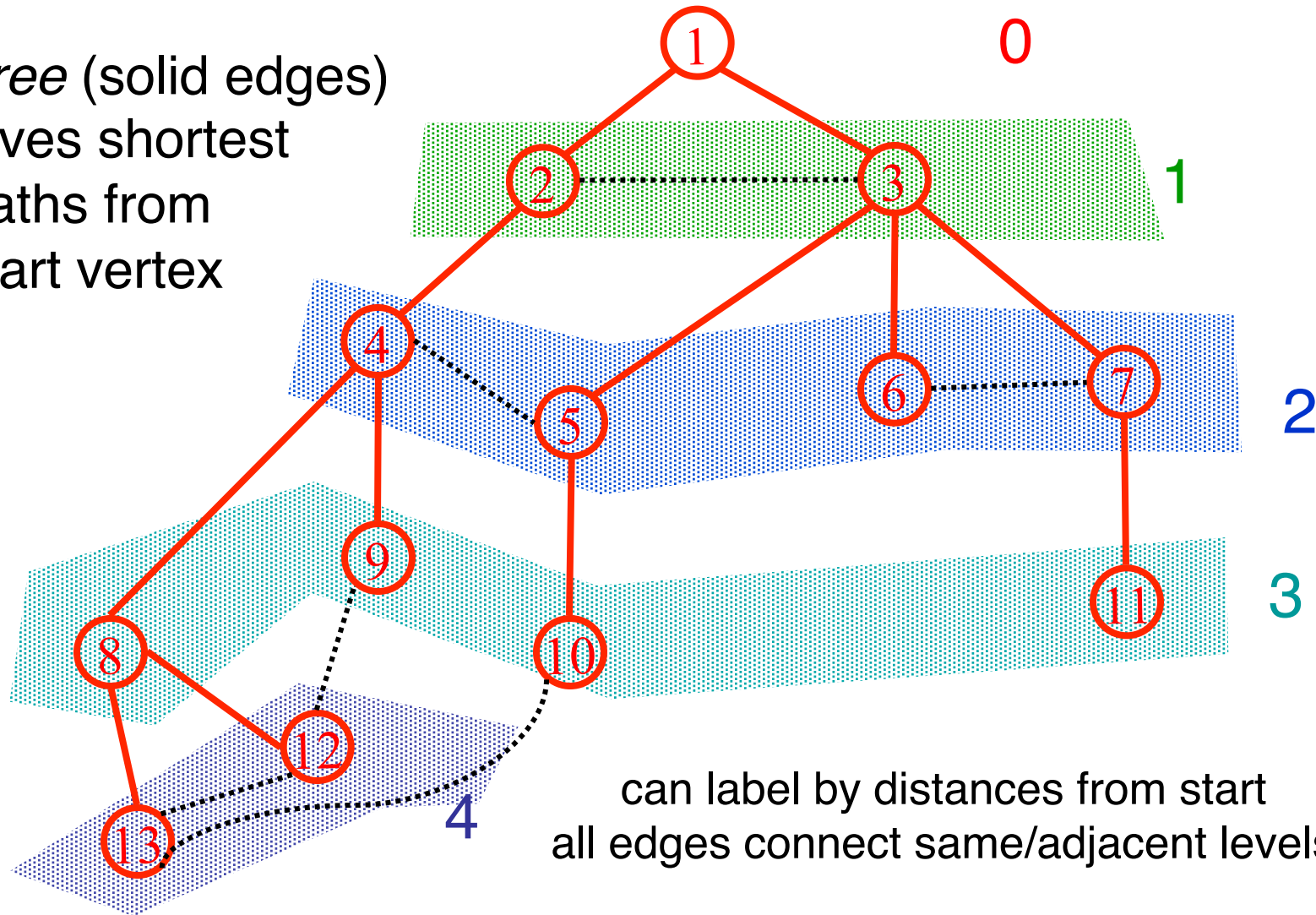
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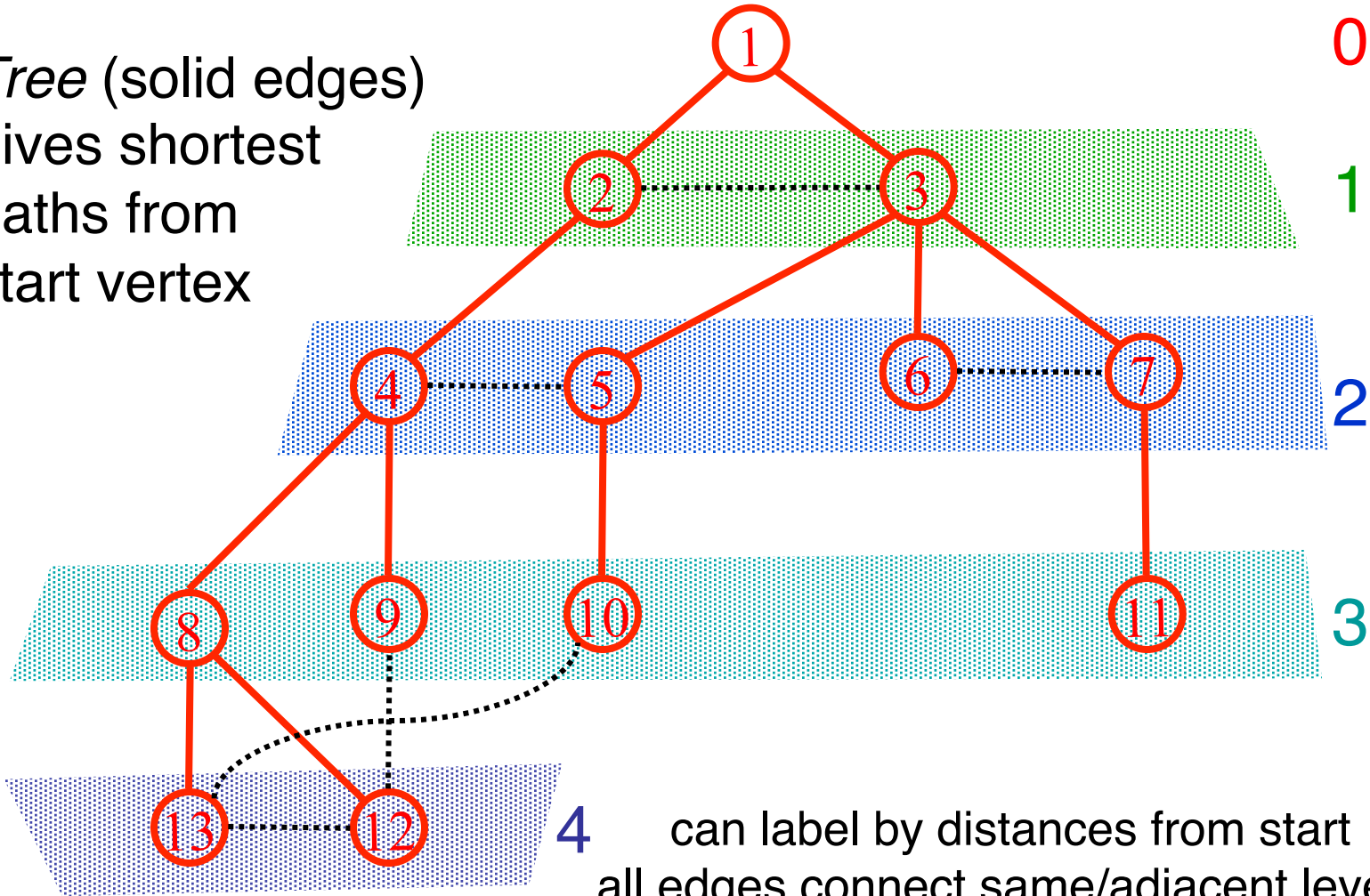
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# BFS Application: Shortest Paths

Tree (solid edges)  
gives shortest  
paths from  
start vertex



4 can label by distances from start  
all edges connect same/adjacent levels<sub>40</sub>



# Why fuss about trees?

Trees are simpler than graphs

Ditto for algorithms on trees vs algs on graphs

So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure

E.g., BFS finds a tree s.t. level-jumps are minimized

DFS (below) finds a different tree, but it also has interesting structure...

# Graph Search Application: Connected Components

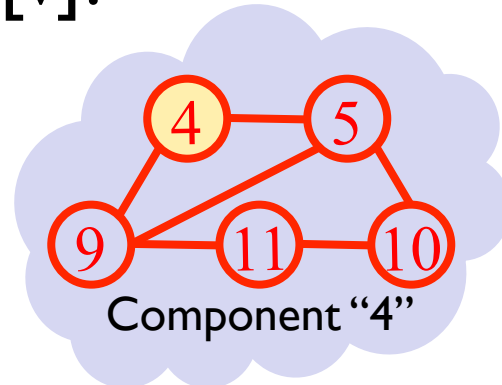
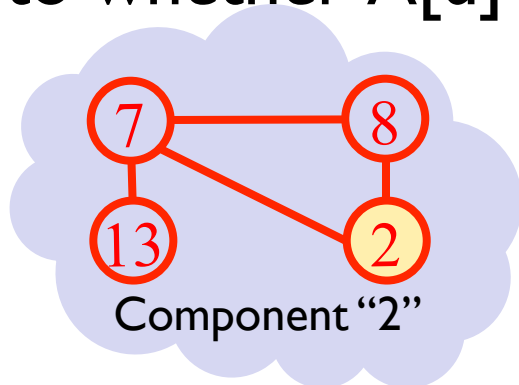
Want to answer questions of the form:

Given vertices  $u$  and  $v$ , is there a path from  $u$  to  $v$ ?

Idea: create array  $A$  such that

$A[u]$  = smallest numbered vertex that is connected to  $u$ . Question reduces to whether  $A[u]=A[v]$ ?

Q: Why not use 2-d array  $Path[u,v]$ ?



$A[8] = A[13]$ ? Y  
 $A[8] = A[9]$ ? N

# Graph Search Application: Connected Components

initial state: all  $v$  undiscovered

for  $v = 1$  to  $n$  do

  if  $\text{state}(v) \neq \text{fully-explored}$  then

    BFS( $v$ ): setting  $A[u] \leftarrow v$  for each  $u$  found  
    (and marking  $u$  discovered/fully-explored)

  endif

endfor

Total cost:  $O(n+m)$  Naively, *three* nested loops  $\Rightarrow O(n^3)$ , but careful look at BFS( $v$ ) shows  $O(n_i+m_i)$  if  $v$ 's component has  $n_i$  nodes &  $m_i$  edges;  $\sum n_i+m_i = n+m$ . Idea: each edge is touched twice, once from each end. (True for DFS, too)

## 3.4 Testing Bipartiteness

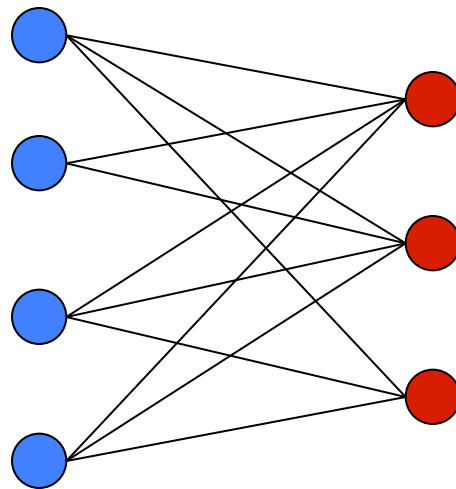
# Bipartite Graphs

Def. An undirected graph  $G = (V, E)$  is *bipartite (2-colorable)* if the nodes can be colored red or blue such that no edge has both ends the same color.

## Applications.

Stable marriage: men = red, women = blue

Scheduling: machines = red, jobs = blue



*a bipartite graph*

"bi-partite" means "two parts." An equivalent definition:  $G$  is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

# Testing Bipartiteness

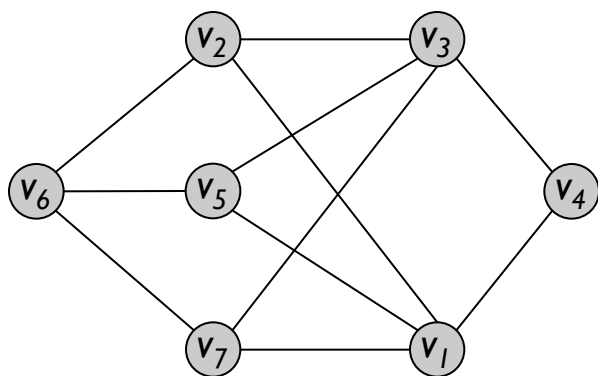
Testing bipartiteness. Given a graph  $G$ , is it bipartite?

Many graph problems become:

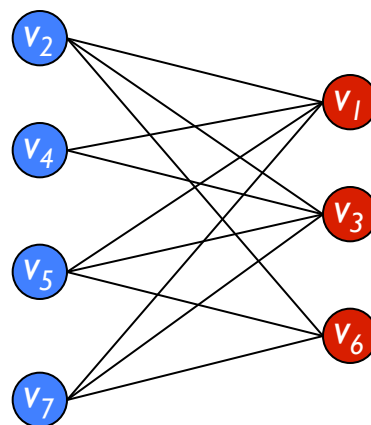
easier if the underlying graph is bipartite (matching)

tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



*a bipartite graph  $G$*

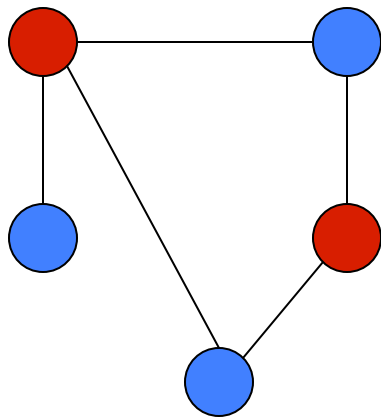


*another drawing of  $G$*

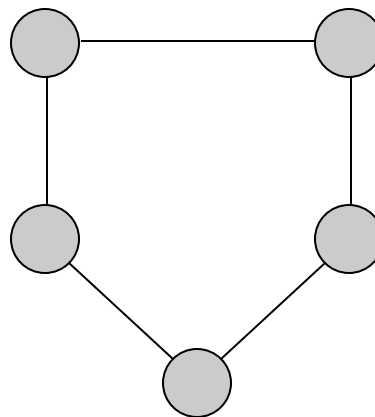
# An Obstruction to Bipartiteness

Lemma. If a graph  $G$  is bipartite, it cannot contain an odd length cycle.

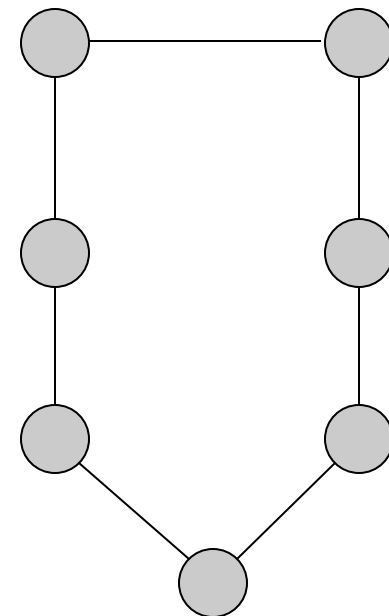
Pf. Impossible to 2-color the odd cycle, let alone  $G$ .



*bipartite  
(2-colorable)*



*not bipartite  
(not 2-colorable)*

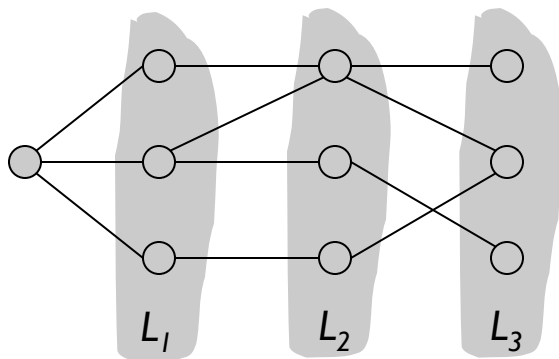


*not bipartite  
(not 2-colorable)*

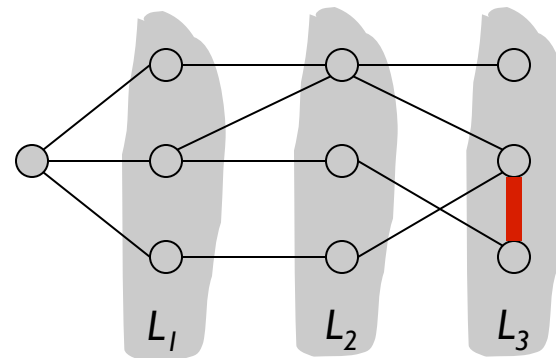
# Bipartite Graphs

Lemma. Let  $G$  be a connected graph, and let  $L_0, \dots, L_k$  be the layers produced by BFS starting at node  $s$ . Exactly one of the following holds.

- (i) No edge of  $G$  joins two nodes of the same layer, and  $G$  is bipartite.
- (ii) An edge of  $G$  joins two nodes of the same layer, and  $G$  contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)



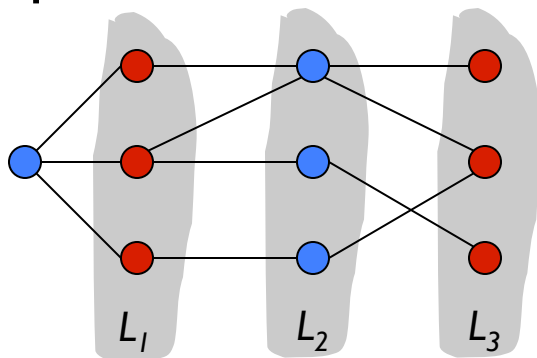
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Pf. (i)

Suppose no edge joins two nodes in the same layer.  
By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:

red = nodes on odd levels,  
blue = nodes on even levels.

# Bipartite Graphs

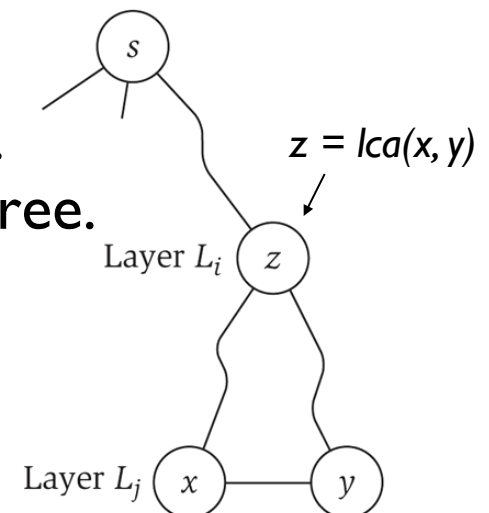
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- (ii) An edge of  $G$  joins two nodes of the same layer, and  $G$  contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

Suppose  $(x, y)$  is an edge &  $x, y$  in same level  $L_j$ .  
 Let  $z =$  their lowest common ancestor in BFS tree.  
 Let  $L_i$  be level containing  $z$ .

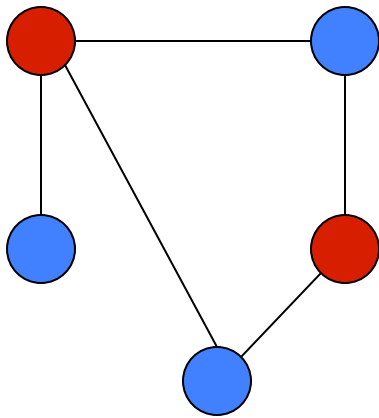
Consider cycle that takes edge from  $x$  to  $y$ ,  
 then tree from  $y$  to  $z$ , then tree from  $z$  to  $x$ .  
 Its length is  $\underbrace{1}_{(x,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$ , which is odd.



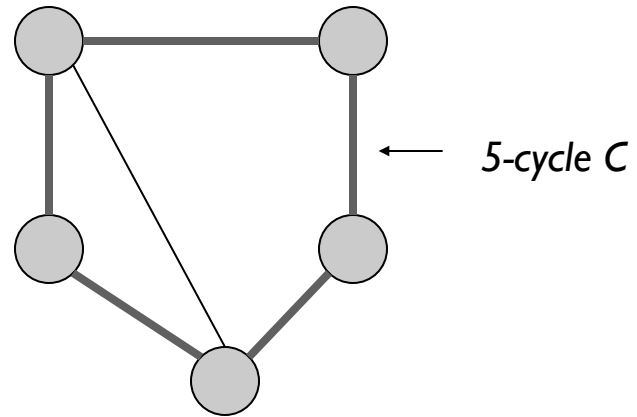
# Obstruction to Bipartiteness

Cor: A graph  $G$  is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic—it *finds* a coloring or odd cycle.



*bipartite*  
*(2-colorable)*



*not bipartite*  
*(not 2-colorable)*

## 3.6 DAGs and Topological Ordering

This should be review of 331 material

I won't lecture on it, but you should read book/slides to be sure it makes sense, with emphasis on correctness, analysis.

# Precedence Constraints

Precedence constraints. Edge  $(v_i, v_j)$  means task  $v_i$  must occur before  $v_j$ .

## Many Applications

Course prerequisites: course  $v_i$  must be taken before  $v_j$

Compilation: must compile module  $v_i$  before  $v_j$

Computing workflow: output of job  $v_i$  is input to job  $v_j$

Manufacturing or assembly: sand it before you paint it...

Spreadsheet evaluation order: if A7 is "`=A6+A5+A4`", evaluate 4,5,6 first

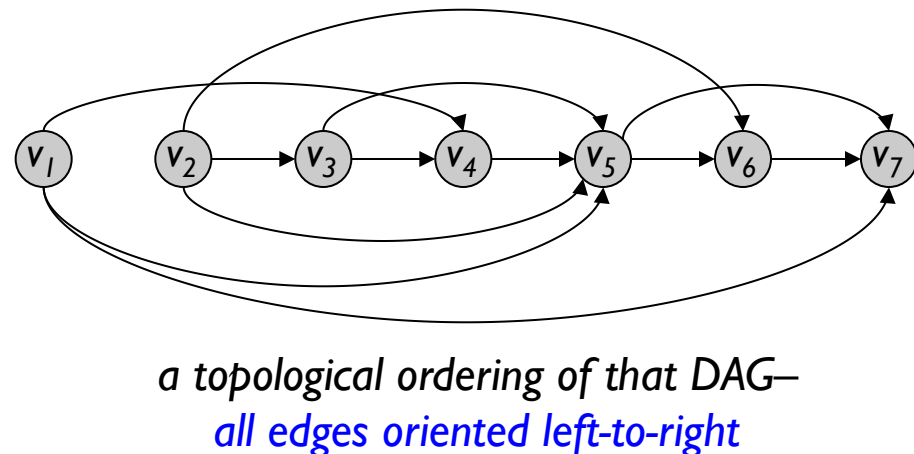
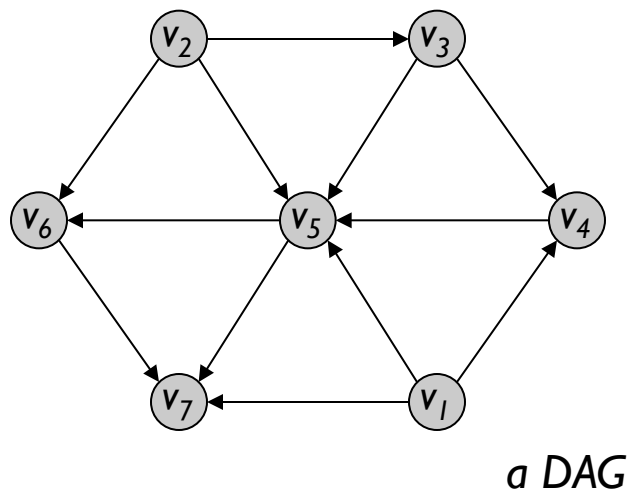
# Directed Acyclic Graphs

Def. A **DAG** is a directed acyclic graph, i.e., one that contains no directed cycles.

Ex. Precedence constraints: edge  $(v_i, v_j)$  means  $v_i$  must precede  $v_j$ .

Def. A topological order of a directed graph  $G = (V, E)$  is an ordering of its nodes as  $v_1, v_2, \dots, v_n$  so that for every edge  $(v_i, v_j)$  we have  $i < j$ .

E.g.,  $\forall$  edge  $(v_i, v_j)$ , finish  $v_i$  before starting  $v_j$



# Directed Acyclic Graphs

Lemma. If  $G$  has a topological order, then  $G$  is a DAG.

if all edges go  $L \rightarrow R$ ,  
you can't loop back  
to close a cycle

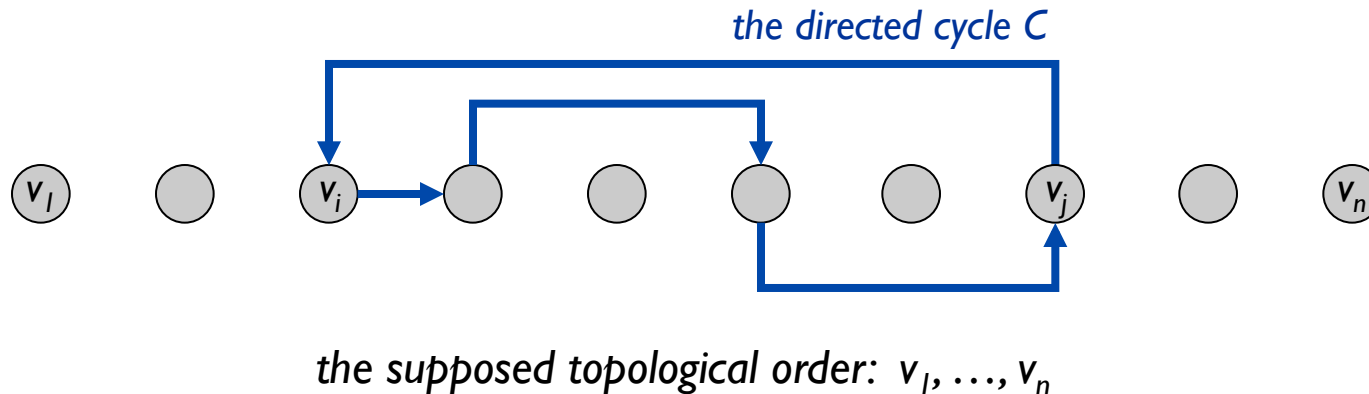
Pf. (by contradiction)

Suppose that  $G$  has a topological order  $v_1, \dots, v_n$   
and that  $G$  also has a directed cycle  $C$ .

Let  $v_i$  be the lowest-indexed node in  $C$ , and let  $v_j$  be the node just  
before  $v_i$ ; thus  $(v_j, v_i)$  is an edge.

By our choice of  $i$ , we have  $i < j$ .

On the other hand, since  $(v_j, v_i)$  is an edge and  $v_1, \dots, v_n$  is a topological  
order, we must have  $j < i$ , a contradiction.



# Directed Acyclic Graphs

Lemma (above).

If  $G$  has a topological order, then  $G$  is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?



# Directed Acyclic Graphs

Lemma. If  $G$  is a DAG, then  $G$  has a node with no incoming edges.

Pf. (by contradiction)

Suppose that  $G$  is a DAG and every node has at least one incoming edge. Let's see what happens.

Pick any node  $v$ , and begin following edges *backward* from  $v$ . Since  $v$  has at least one incoming edge  $(u, v)$  we can walk backward to  $u$ .

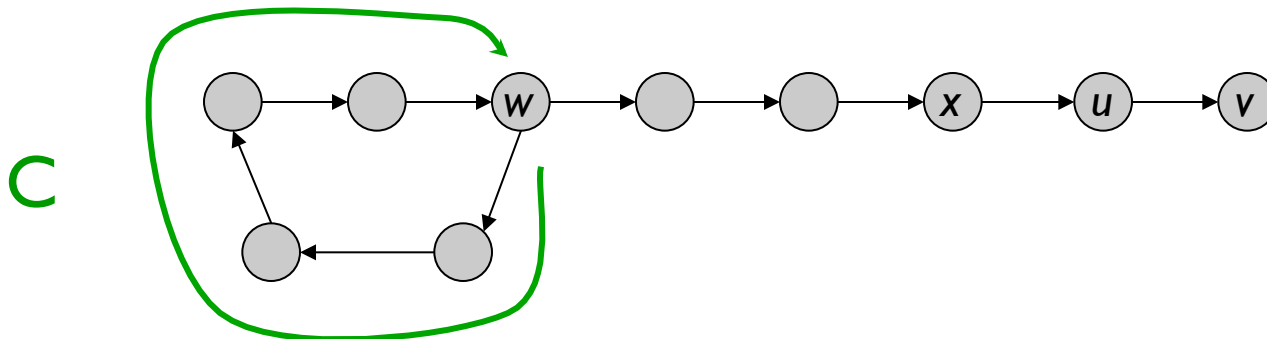
Then, since  $u$  has at least one incoming edge  $(x, u)$ , we can walk backward to  $x$ .

Repeat until we visit a node, say  $w$ , twice.

Why must this happen?

Let  $C$  be the sequence of nodes encountered

between successive visits to  $w$ .  $C$  is a cycle, contradicting acyclicity.



# Directed Acyclic Graphs

Lemma. If  $G$  is a DAG, then  $G$  has a topological ordering.

Pf. (by induction on  $n$ )

Base case: true if  $n = 1$ .

Given DAG on  $n > 1$  nodes, find a node  $v$  with no incoming edges.

$G - \{v\}$  is a DAG, since deleting  $v$  cannot create cycles.

By inductive hypothesis,  $G - \{v\}$  has a topological ordering.

Place  $v$  first in topological ordering; then append nodes of  $G - \{v\}$  in topological order. This is valid since  $v$  has no incoming edges. ▀

---

To compute a topological ordering of  $G$ :

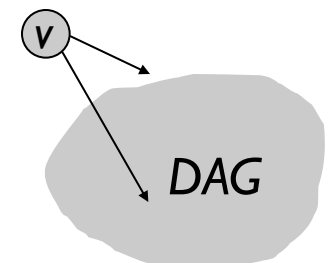
Find a node  $v$  with no incoming edges and order it first

Delete  $v$  from  $G$

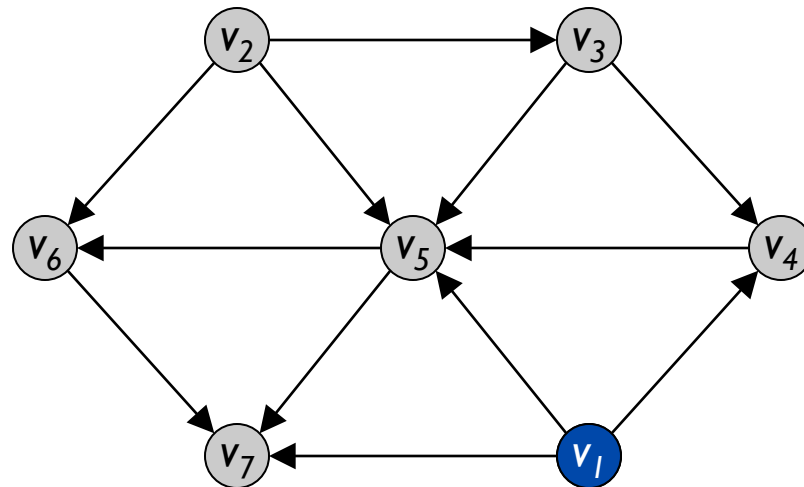
Recursively compute a topological ordering of  $G - \{v\}$

and append this order after  $v$

---

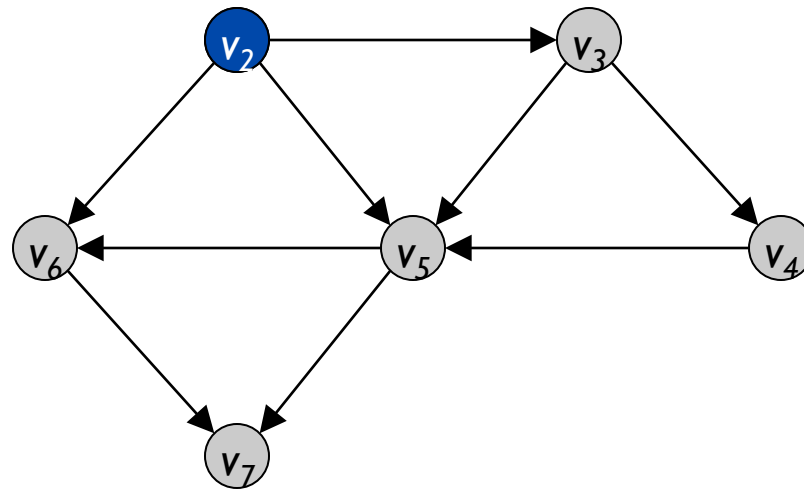


# Topological Ordering Algorithm: Example



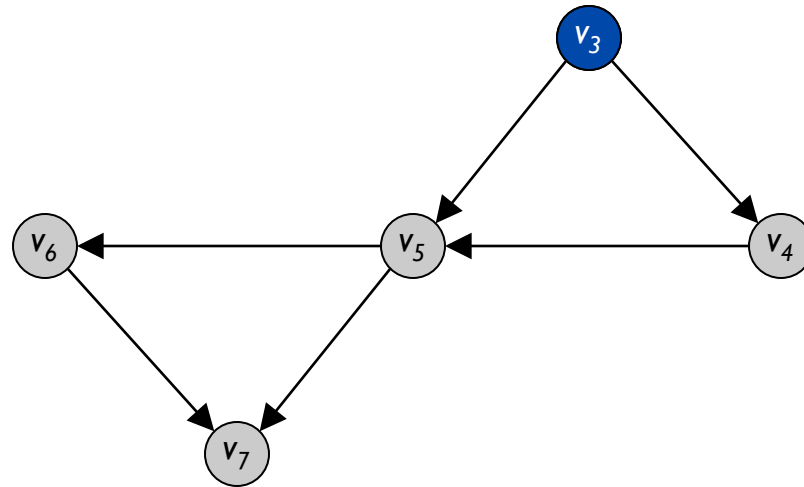
*Topological order:*

# Topological Ordering Algorithm: Example



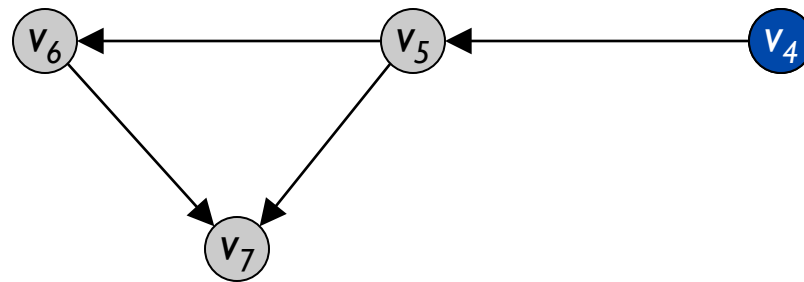
*Topological order:  $v_1$*

# Topological Ordering Algorithm: Example



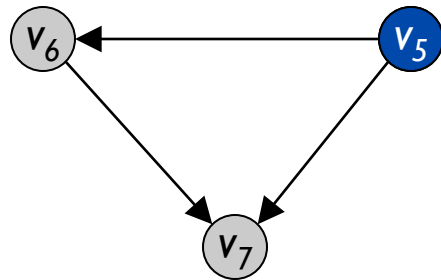
*Topological order:  $v_1, v_2$*

# Topological Ordering Algorithm: Example



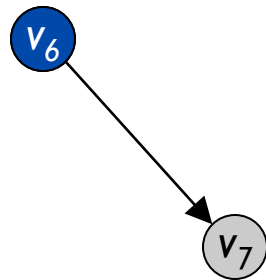
*Topological order:  $v_1, v_2, v_3$*

# Topological Ordering Algorithm: Example



*Topological order:*  $v_1, v_2, v_3, v_4$

# Topological Ordering Algorithm: Example



*Topological order:*  $v_1, v_2, v_3, v_4, v_5$

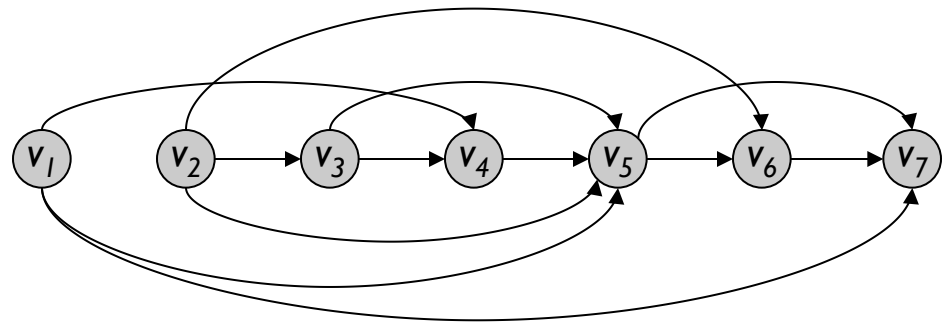
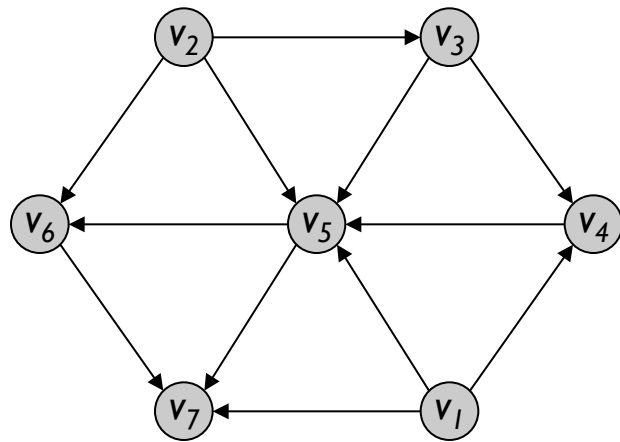


# Topological Ordering Algorithm: Example



*Topological order:*  $v_1, v_2, v_3, v_4, v_5, v_6$

# Topological Ordering Algorithm: Example



*Topological order:  $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ .*

# Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w)

S = S ∪ {w} for all w with count[w]==0

} O(m + n)

Main loop:

while S not empty

    remove some v from S

    make v next in topo order

    for all edges from v to some w

        count[w]--

        if count[w] == 0 then add w to S

} O(1) per node  
O(1) per edge

why does it terminate?

Correctness: clear, I hope

what if G has cycle?

Time: O(m + n) (assuming edge-list representation of graph)

nested loops: why not n•m?

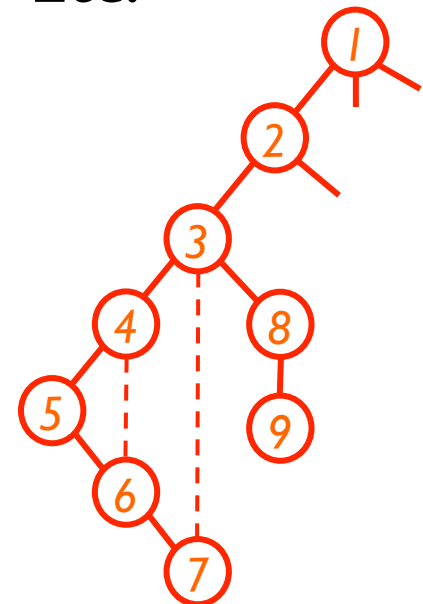
# Depth-First Search

# Depth-First Search

Follow the first path you find as far as you can go

When you reach a dead end, back up to last unexplored edge, then go as far you can. Etc.

Naturally implemented using recursion or a stack



# DFS(v) – Recursive version

Global Initialization:

```
for all nodes v, v.dfs# = -1 // mark v "undiscovered"  
dfscounter = 0
```

DFS(v):

```
v.dfs# = dfscounter++ // v "discovered", number it  
for each edge (v,x)  
    if (x.dfs# = -1) // tree edge (x previously undiscovered)  
        DFS(x)  
    else ... // code for back-, fwd-, parent-  
              // edges, if needed; mark v  
              // "completed," if needed
```

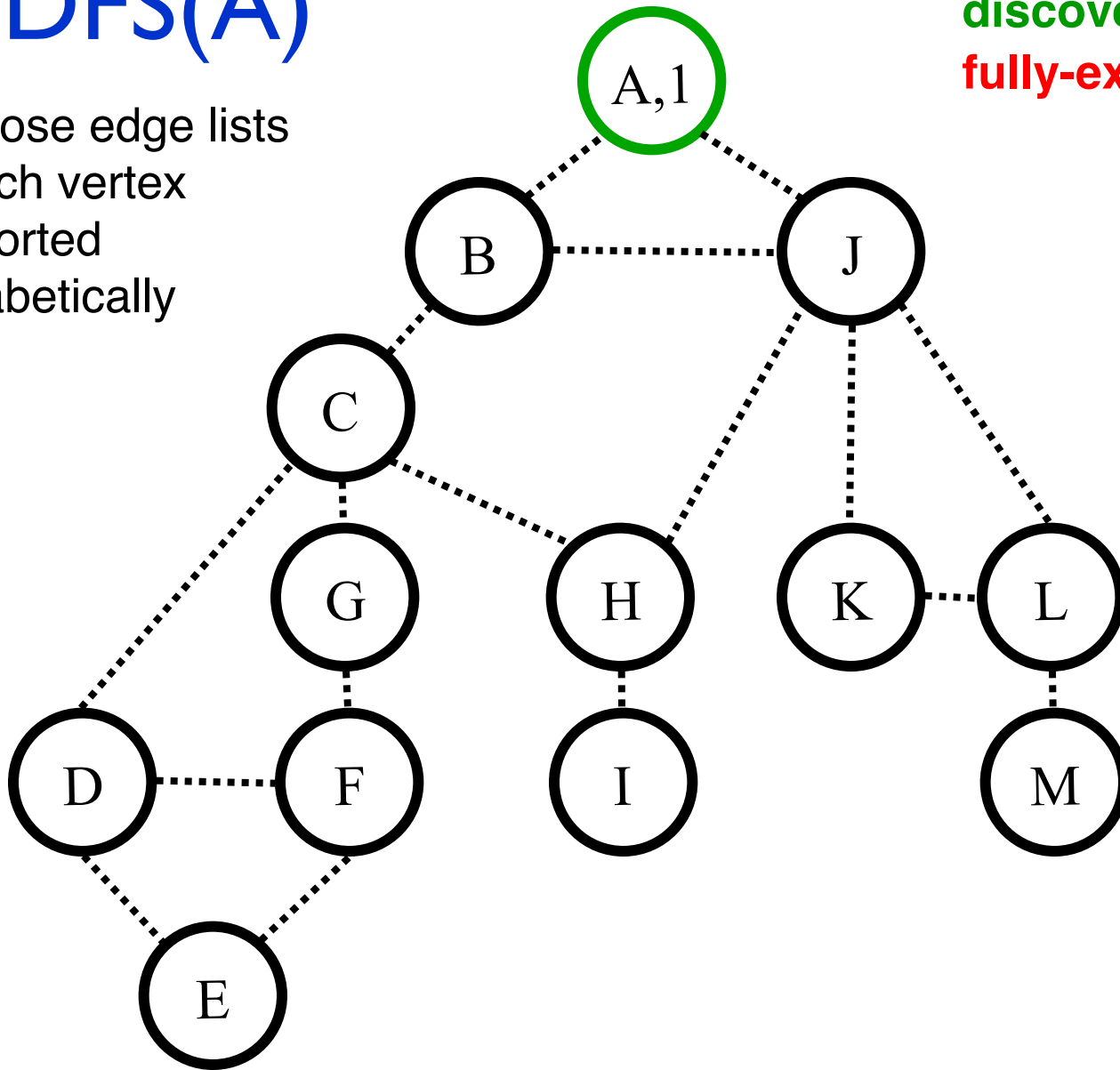
# Why fuss about trees (again)?

BFS tree  $\neq$  DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – *only descendant/ancestor*

Proof below

# DFS(A)

Suppose edge lists  
at each vertex  
are sorted  
alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

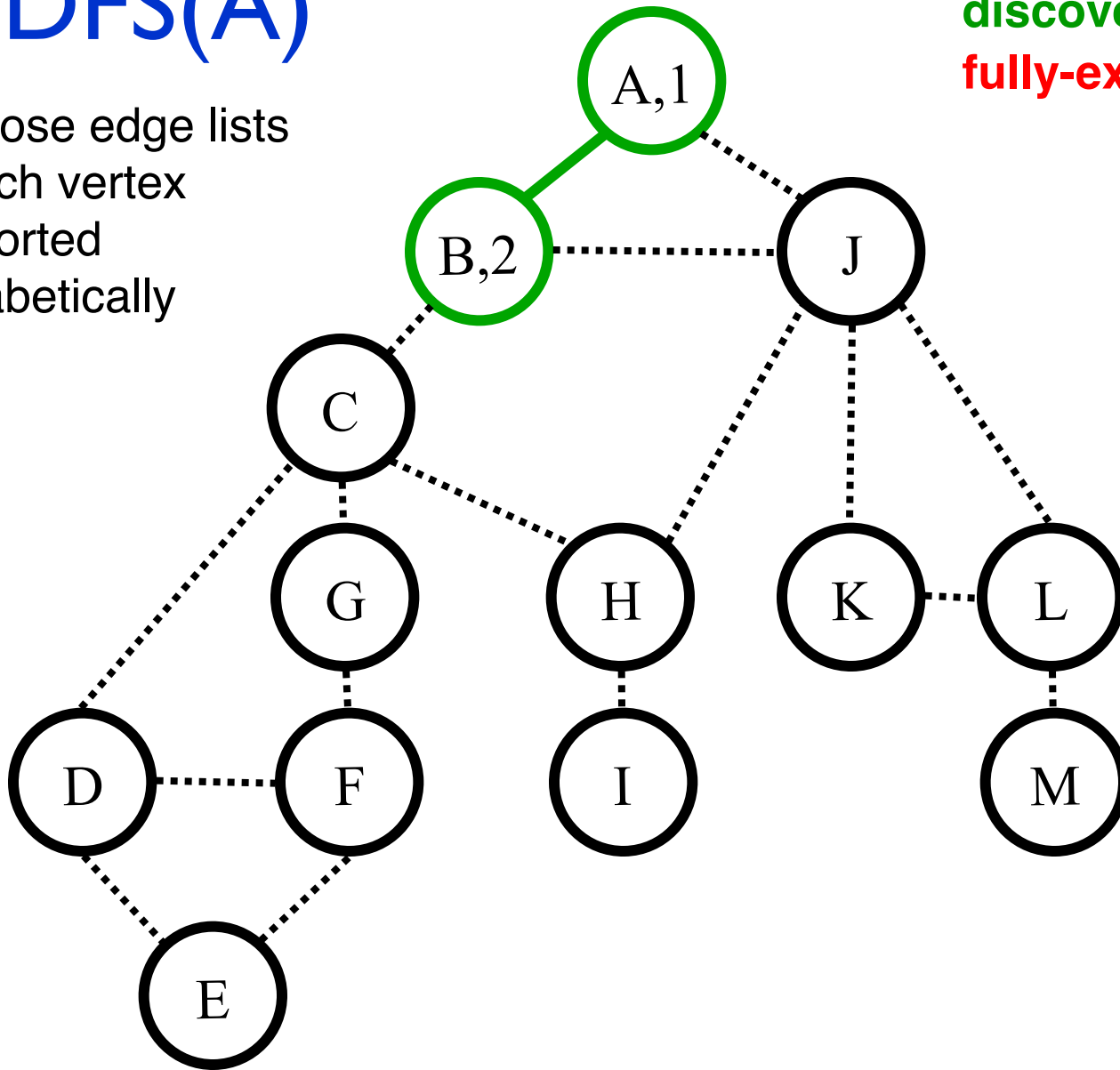
Call Stack  
(Edge list):

A (B,J)



# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

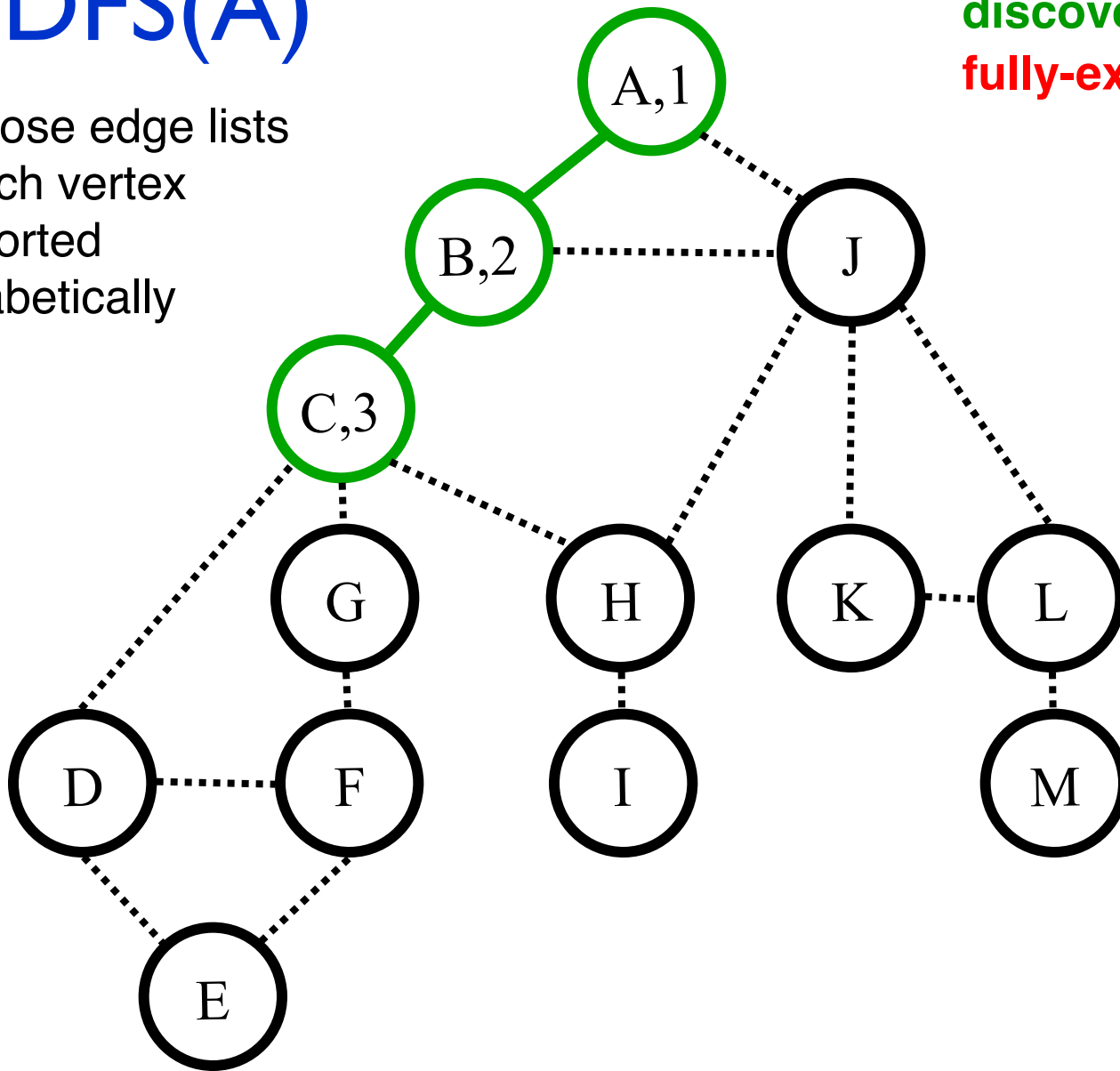
Call Stack:  
(Edge list)

A (~~B~~,J)

B (A,C,J)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

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Call Stack:  
(Edge list)

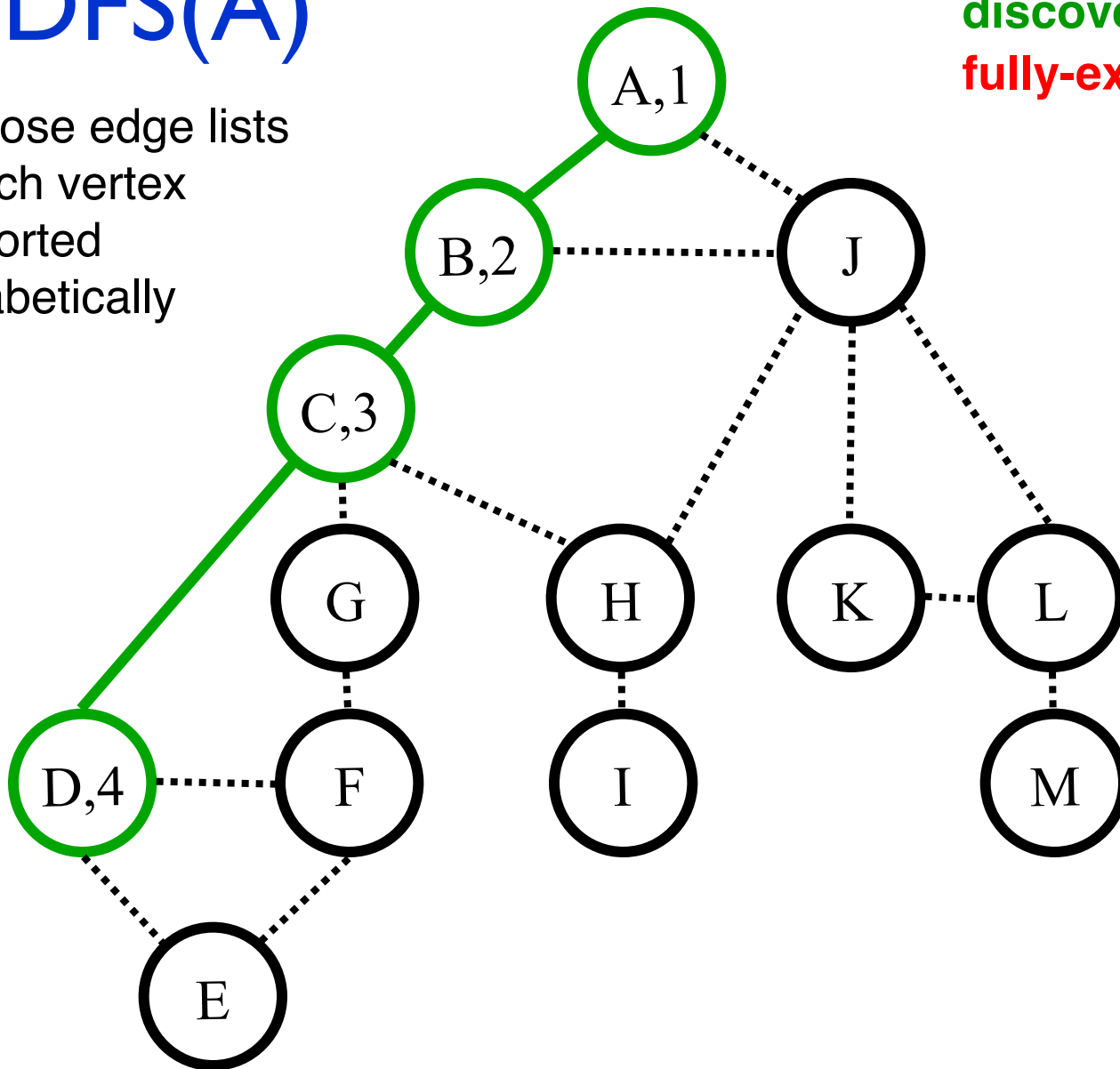
A (~~B~~,J)

B (~~A~~,~~C~~,J)

C (B,D,G,H)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

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**discovered**

**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)

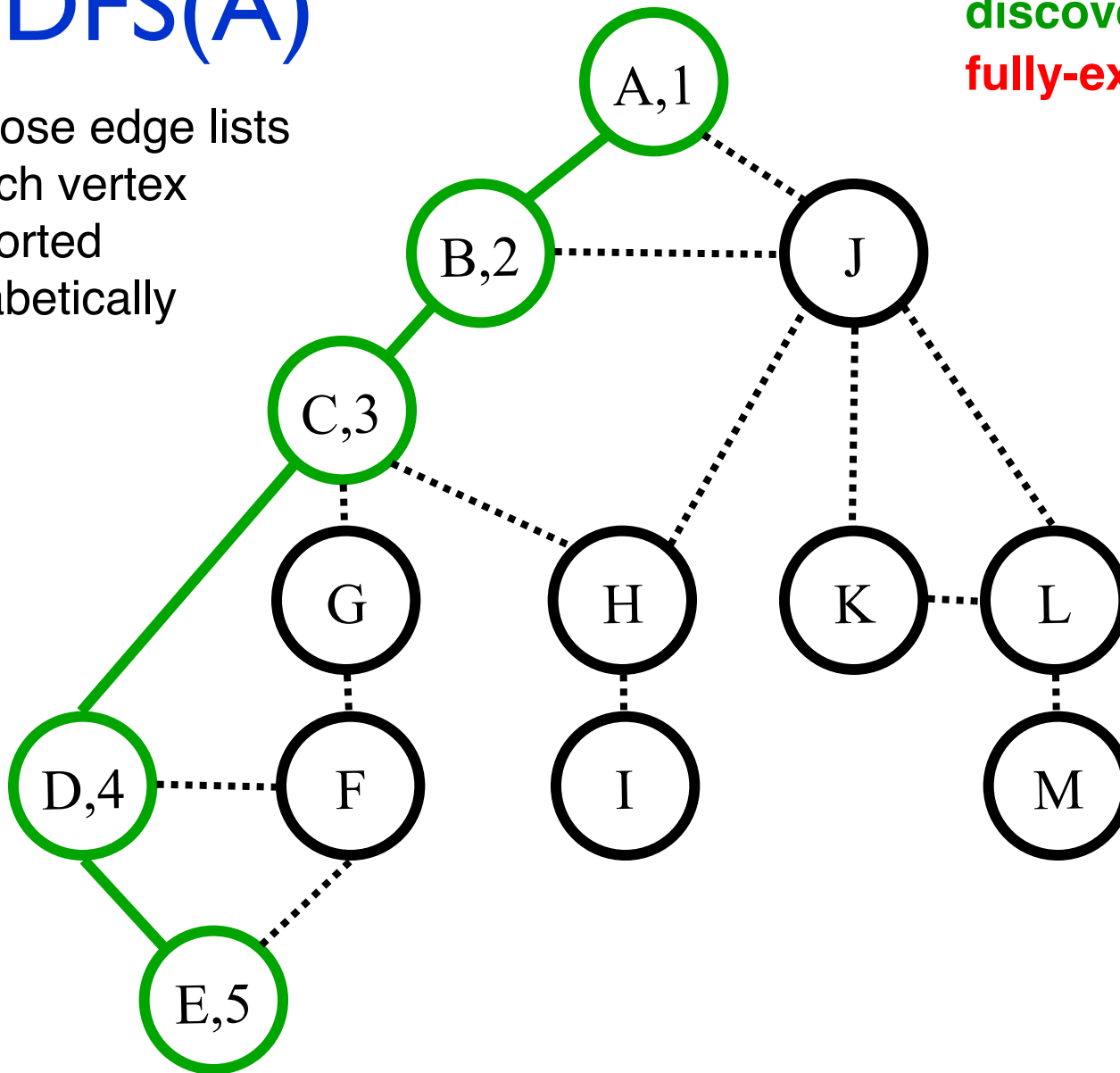
B (~~A~~,~~C~~,J)

C (~~B~~,~~D~~,G,H)

D (C,E,F)

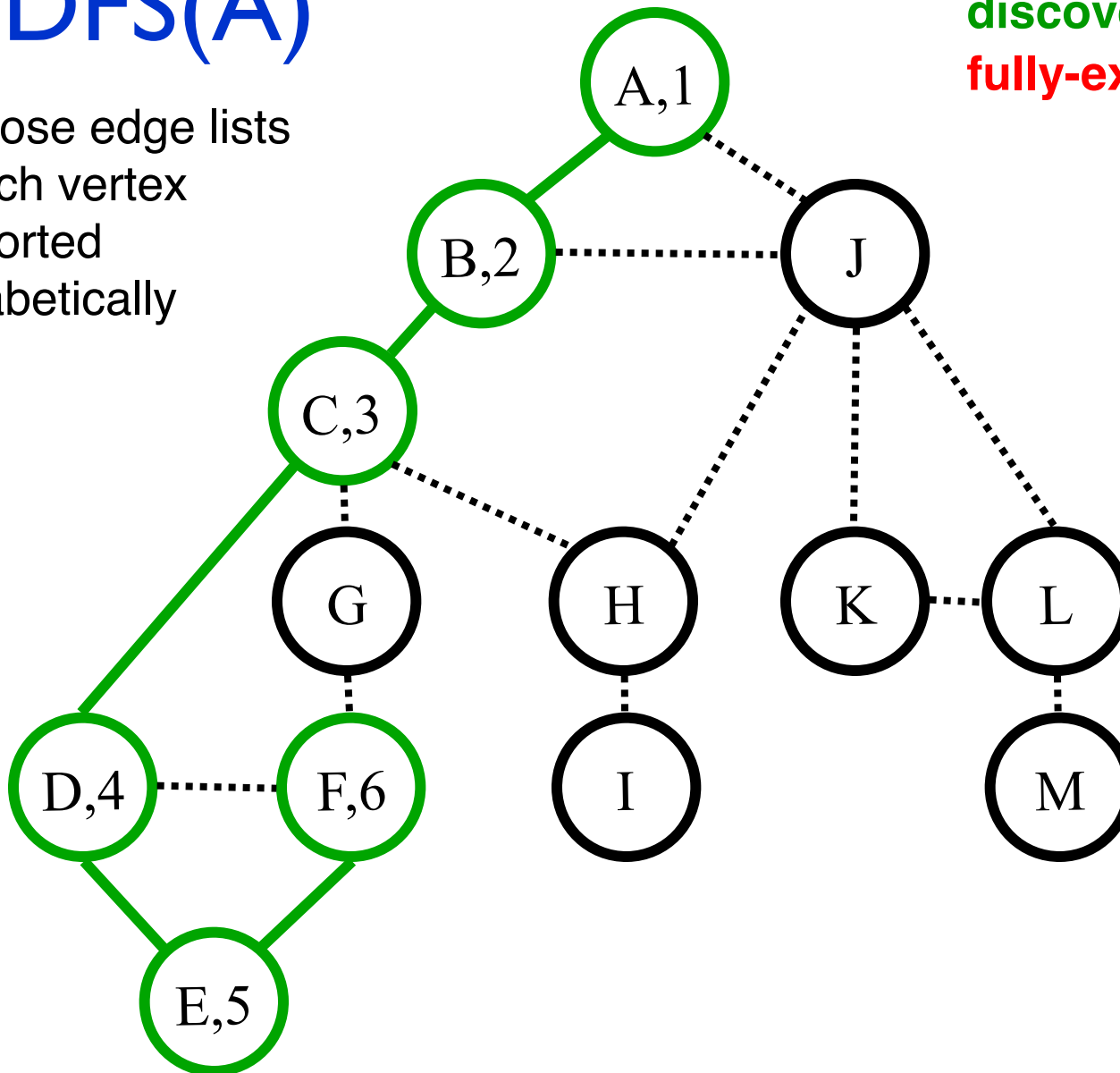
# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

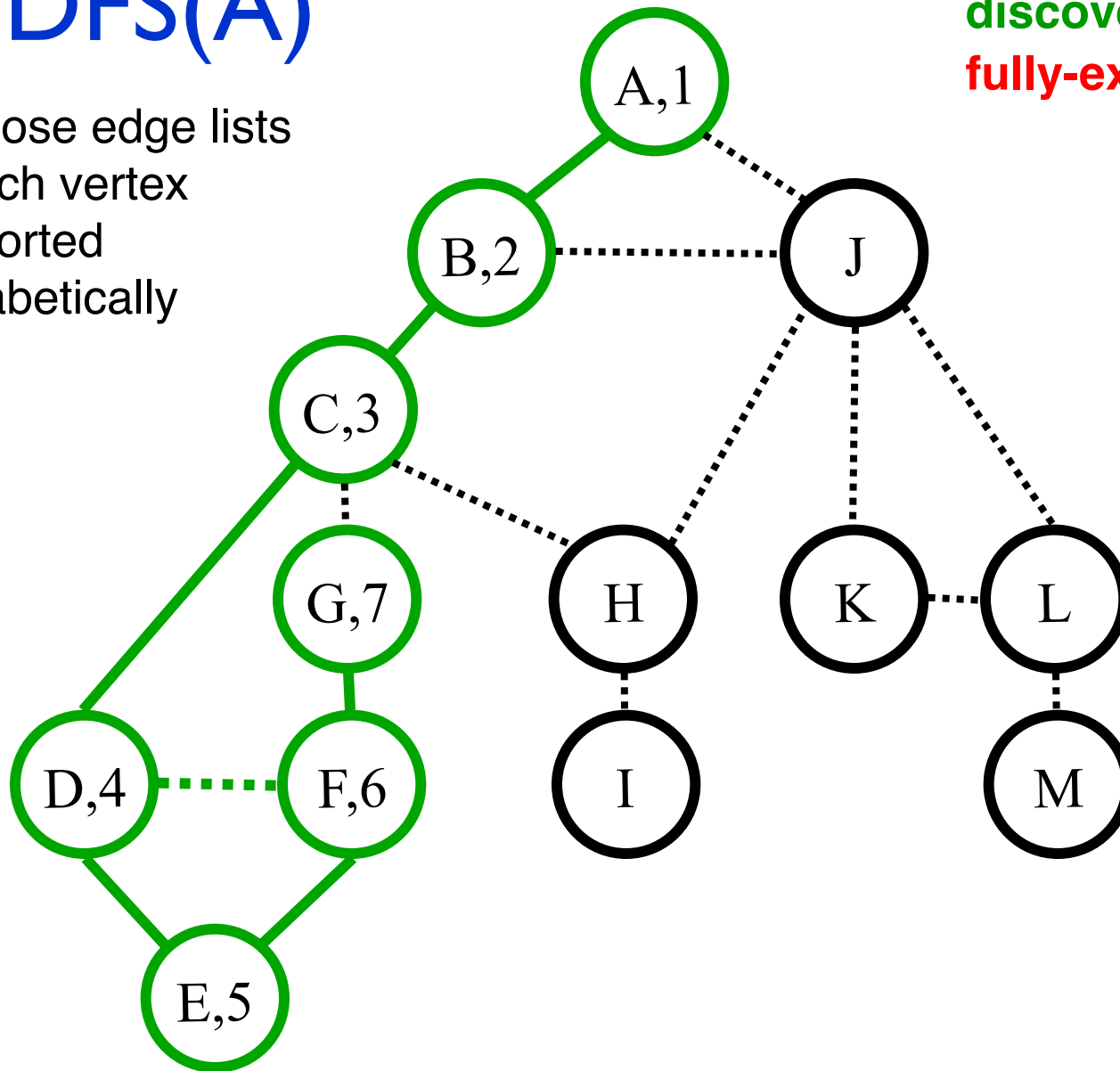
**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,G,H)  
D (~~C~~,~~E~~,F)  
E (~~D~~,~~F~~)  
F (D,E,G)

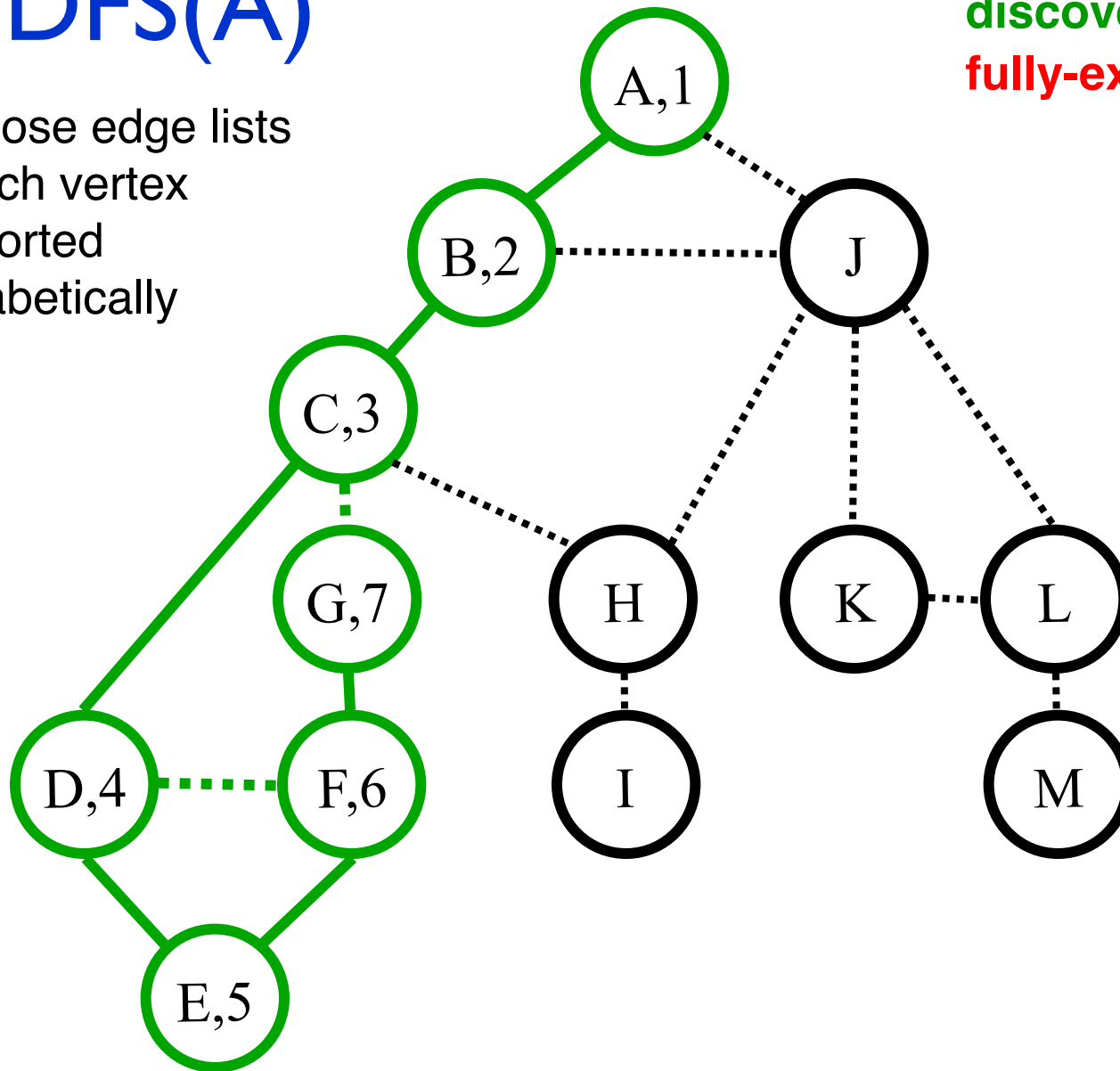
# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



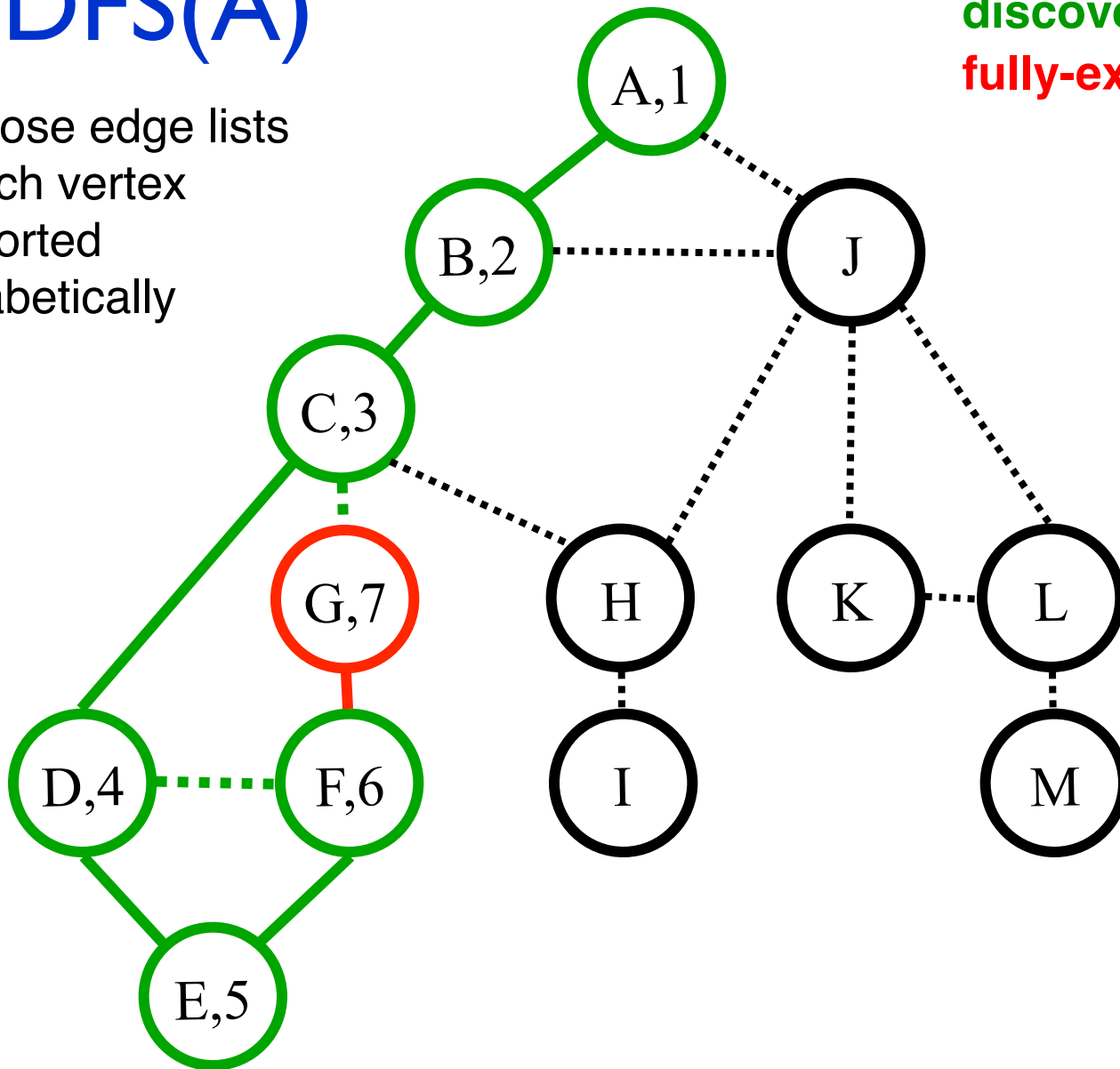
# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,G,H)  
D (~~C~~,~~E~~,F)  
E (~~D~~,~~F~~)  
F (~~D~~,~~E~~,~~G~~)



# DFS(A)

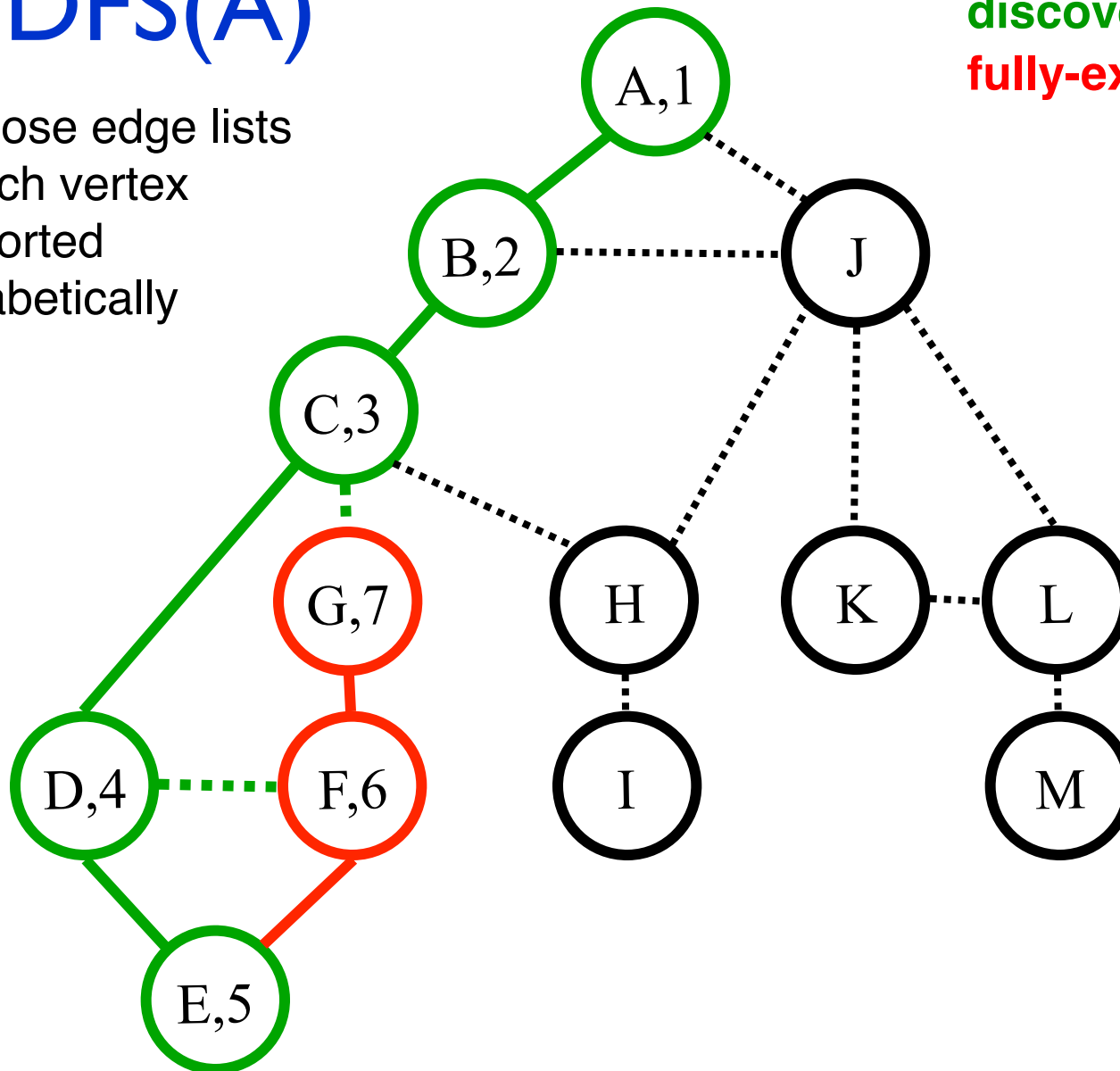
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**

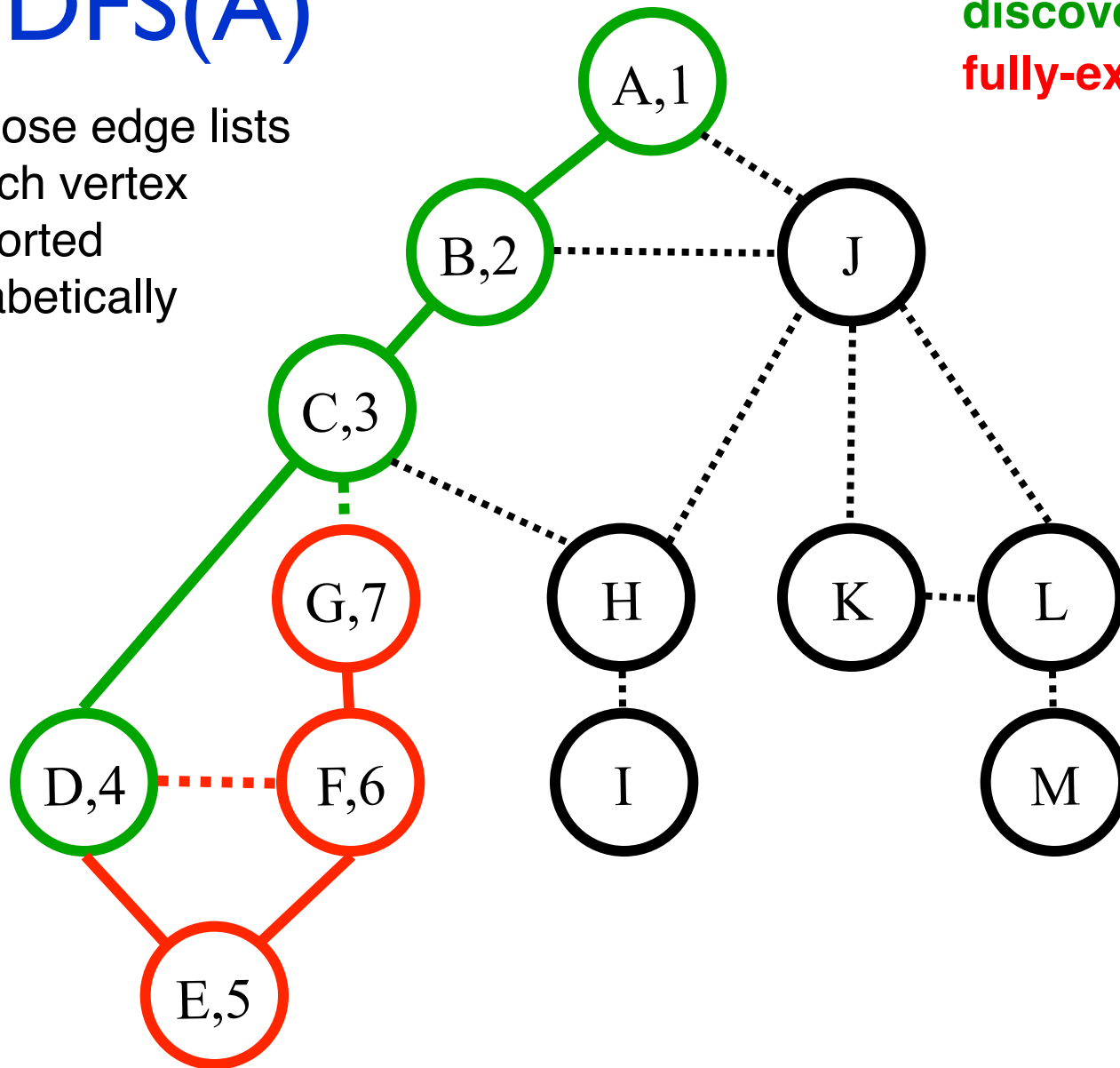


Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,G,H)  
D (~~C~~,~~E~~,F)  
E (~~D~~,~~F~~)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,G,H)  
D (~~C~~,~~E~~,~~F~~)

# DFS(A)

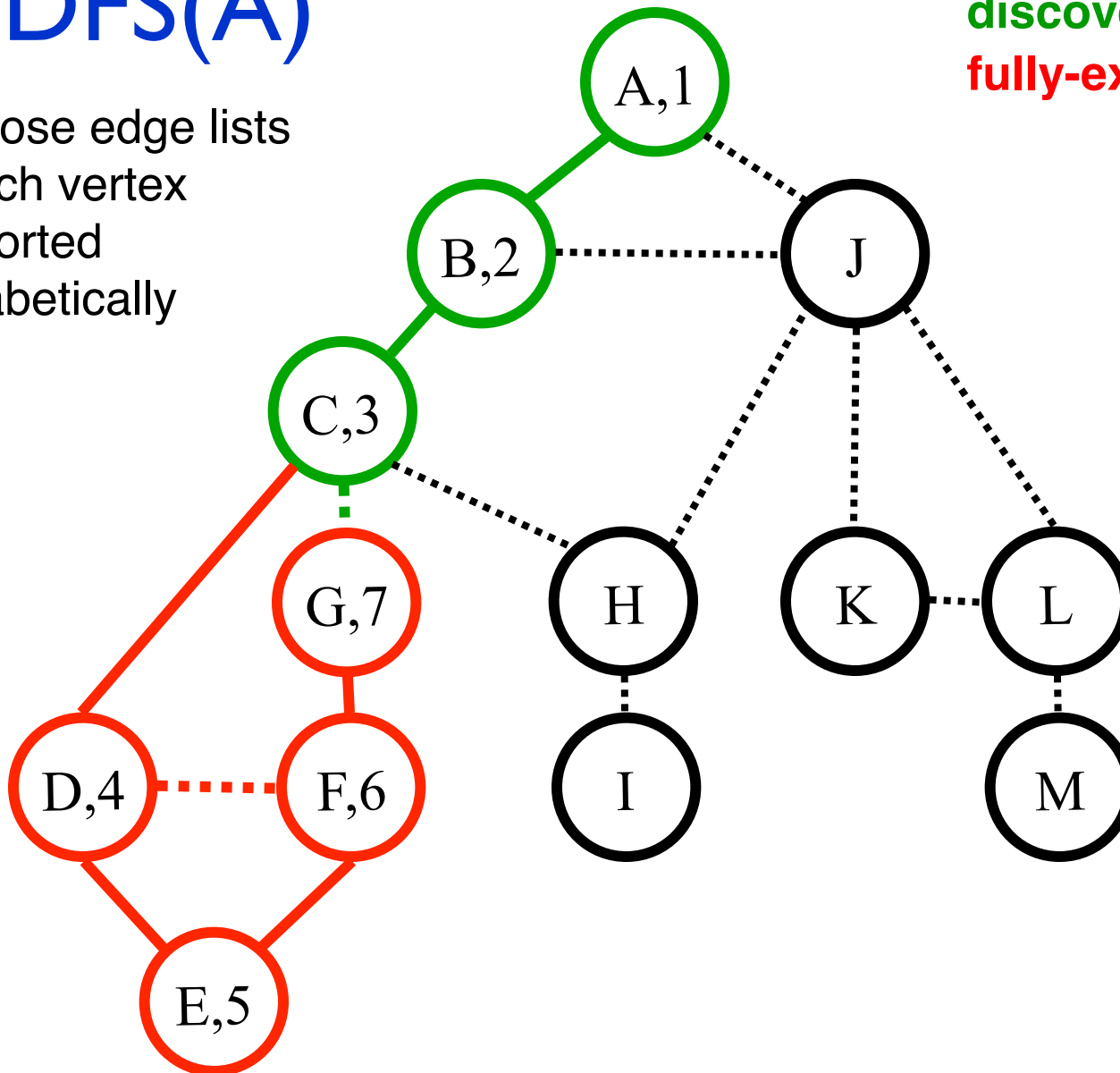
Suppose edge lists at each vertex are sorted alphabetically

Color code:

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**fully-explored**

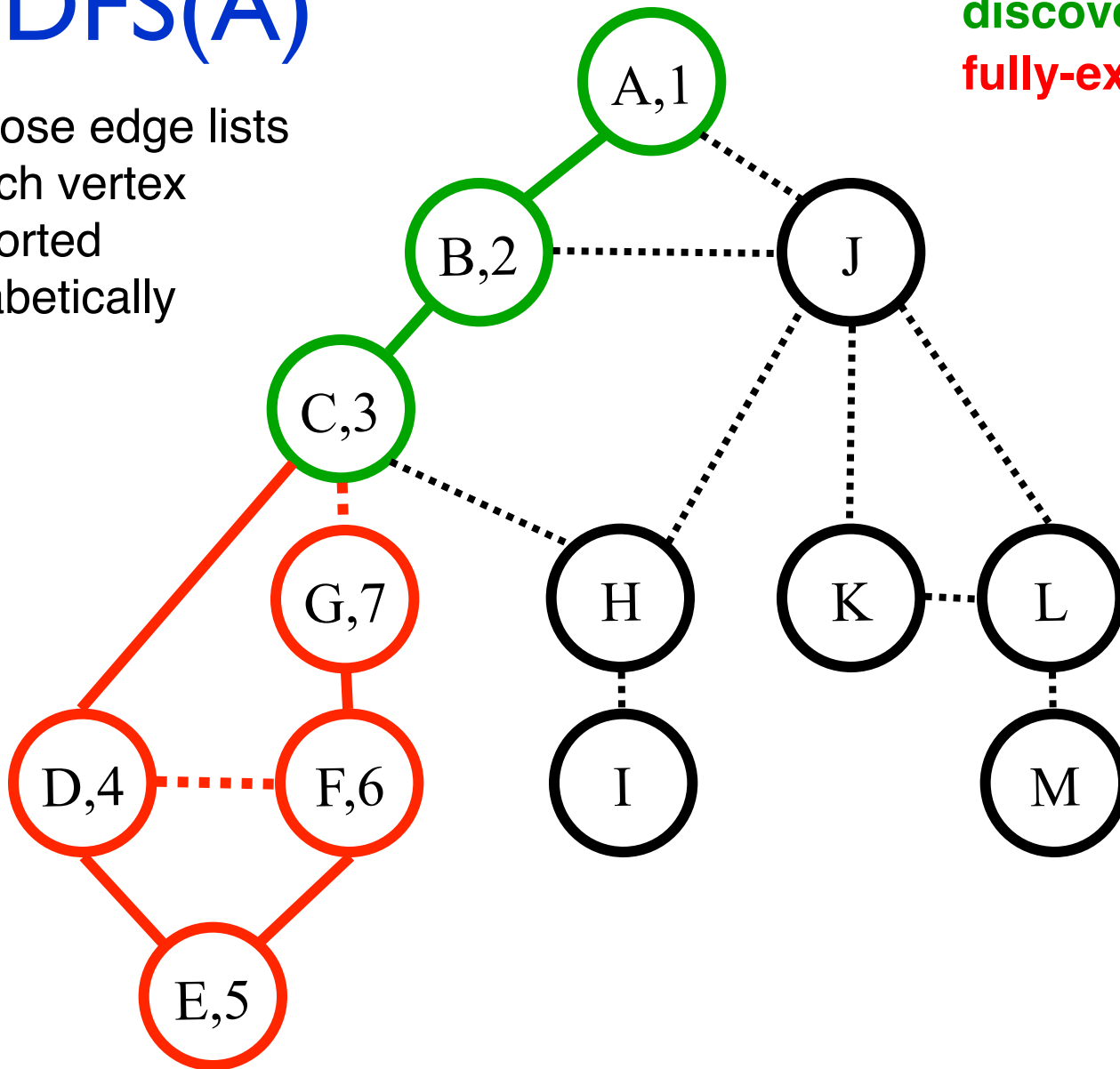


Call Stack:  
(Edge list)

A (~~B~~, J)  
B (~~A~~, ~~C~~, J)  
C (~~B~~, ~~D~~, G, H)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

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**discovered**

**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~, J)

B (~~A~~, ~~C~~, J)

C (~~B~~, ~~D~~, ~~G~~, H)

# DFS(A)

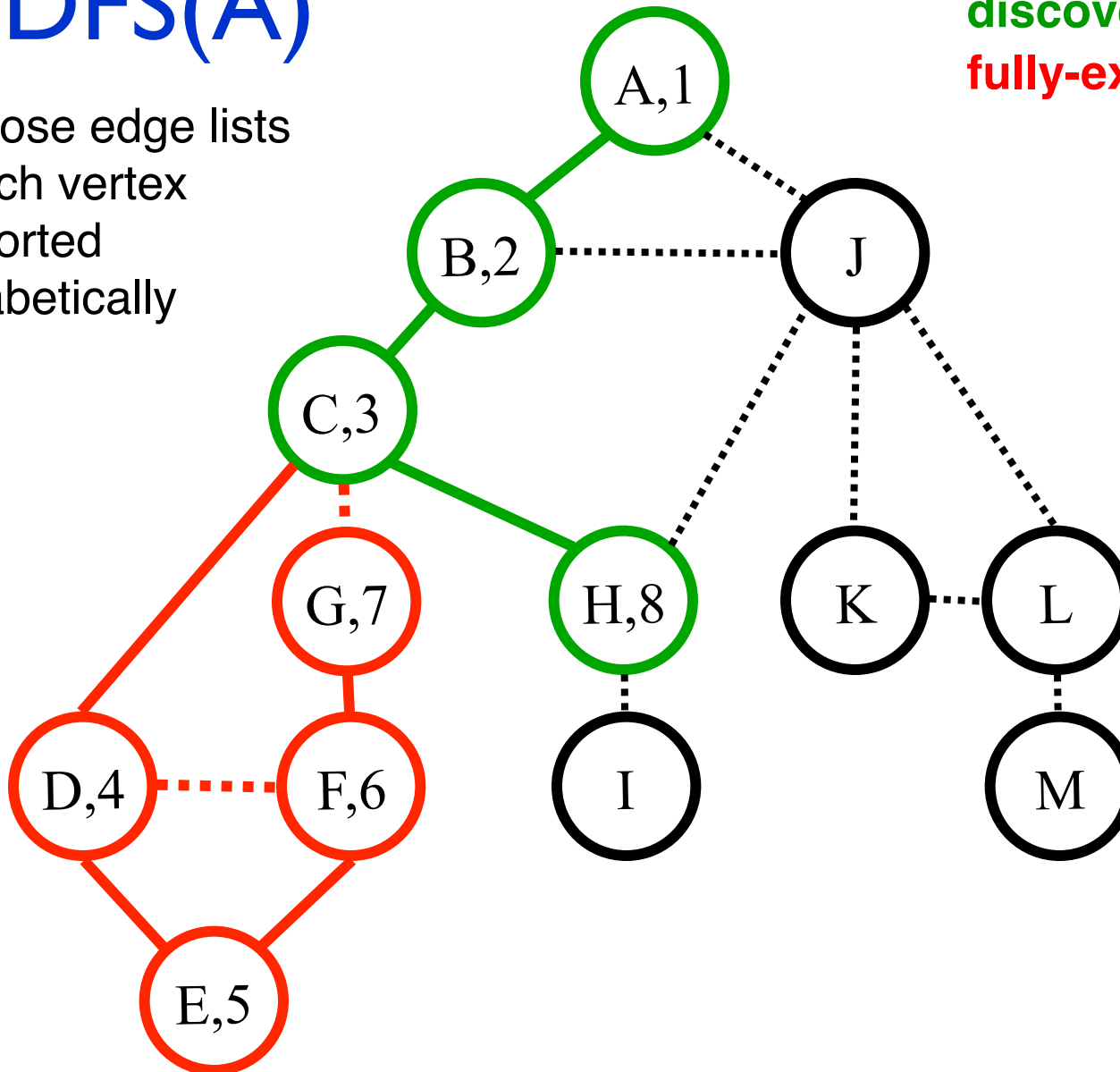
Suppose edge lists at each vertex are sorted alphabetically

Color code:

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**fully-explored**

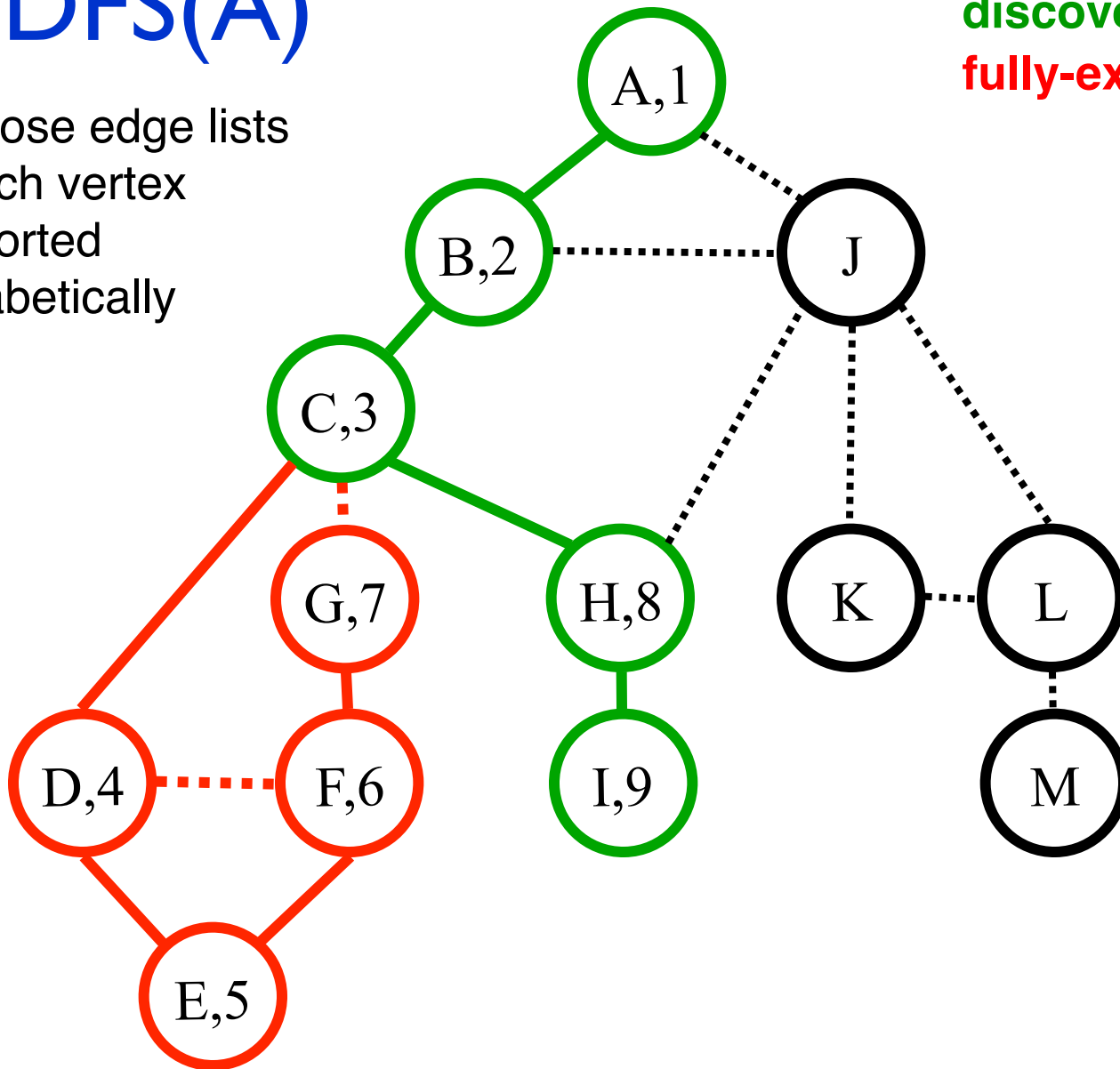


Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,~~H~~)  
H (C,I,J)

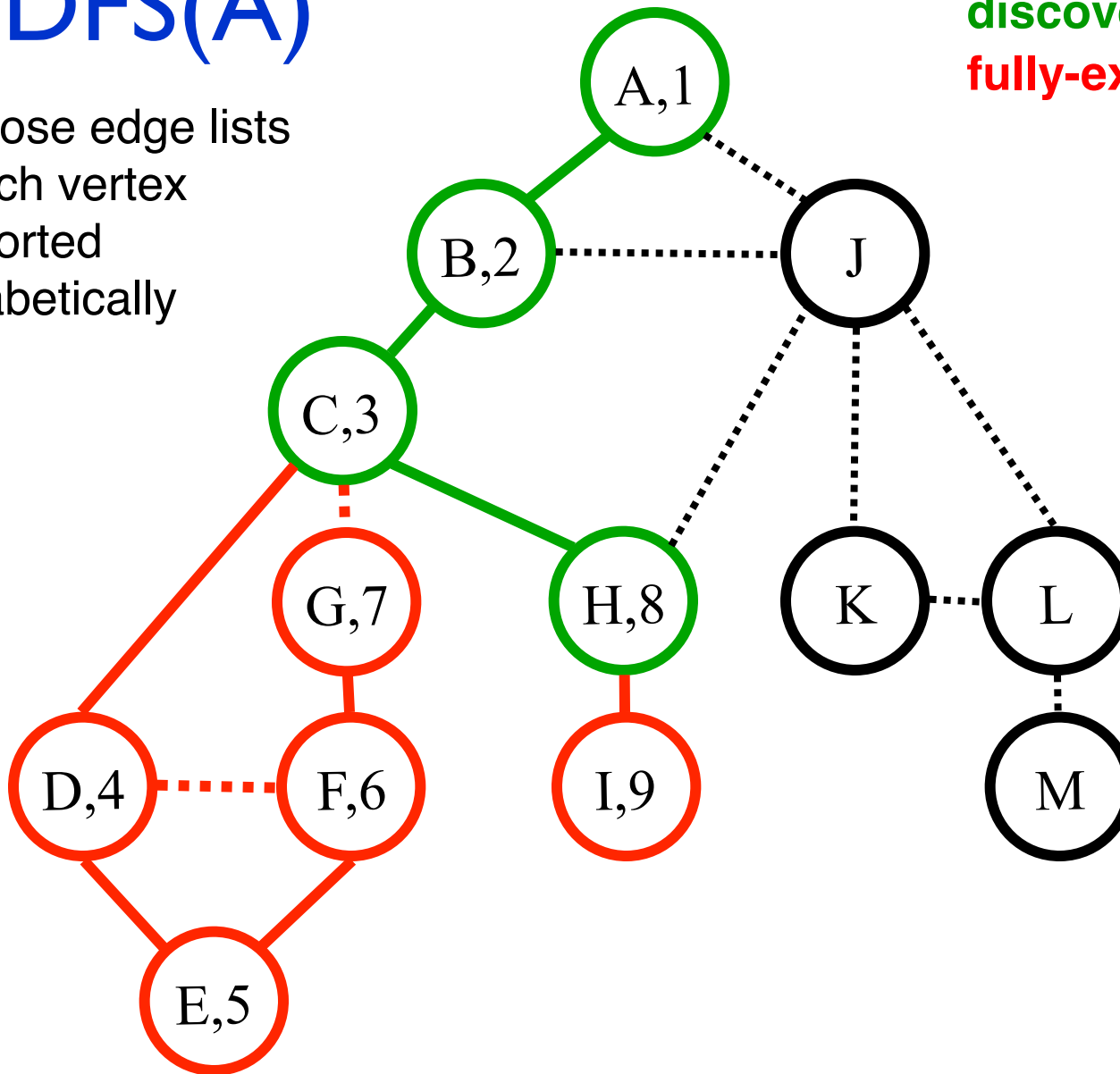
# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

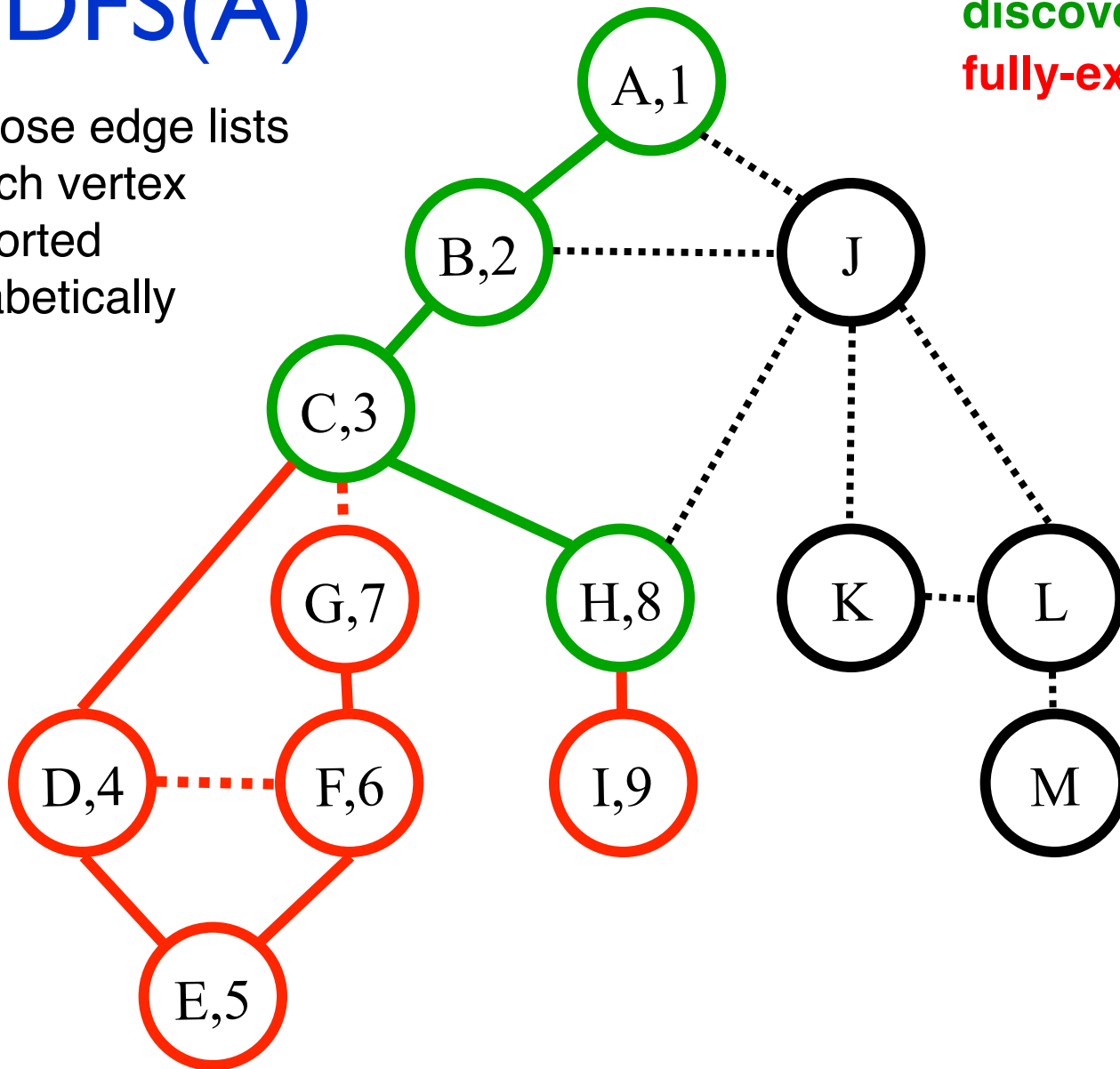
**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
I (~~H~~)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

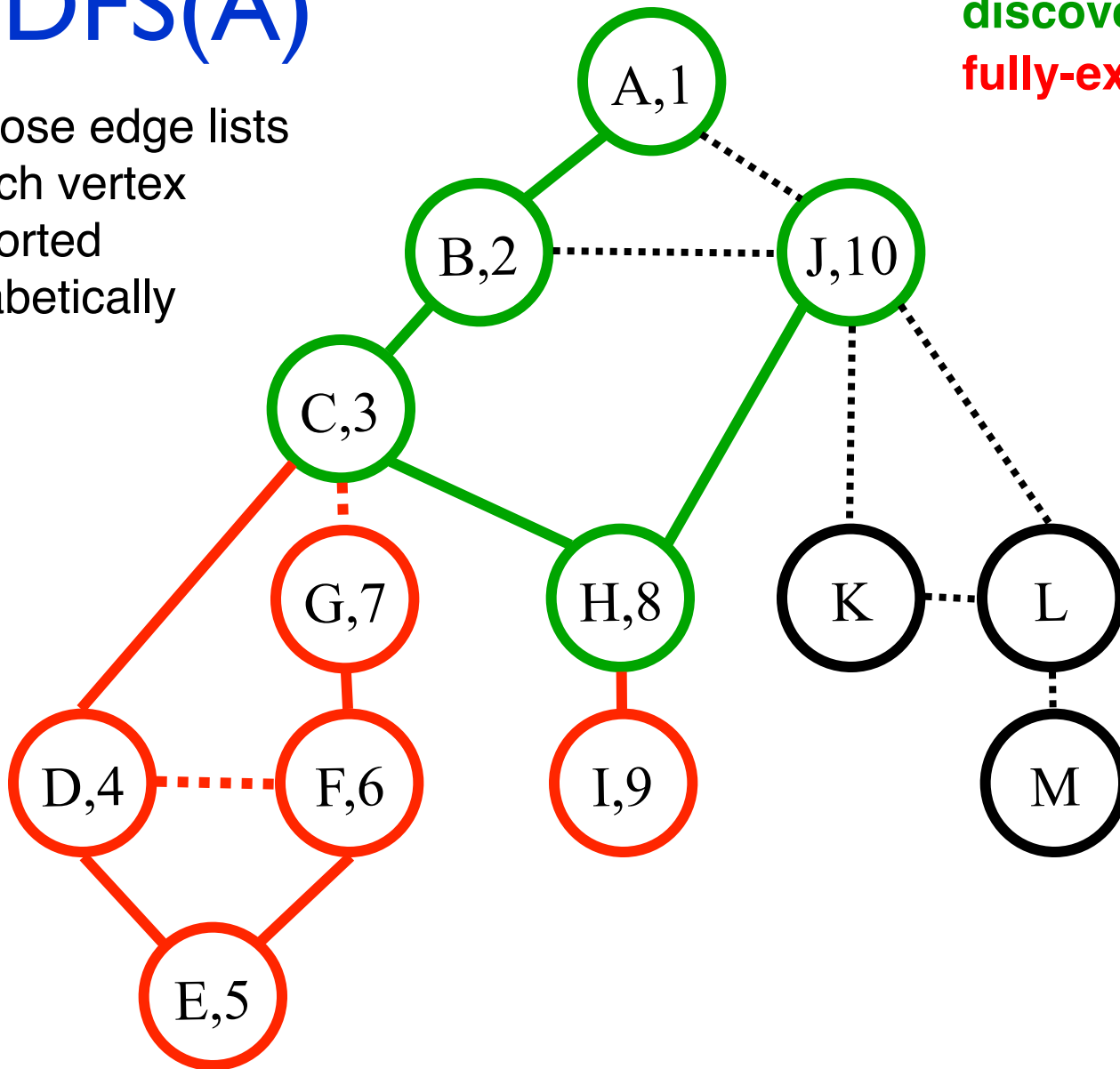
Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,~~H~~)  
H (~~C~~,~~I~~,J)



# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

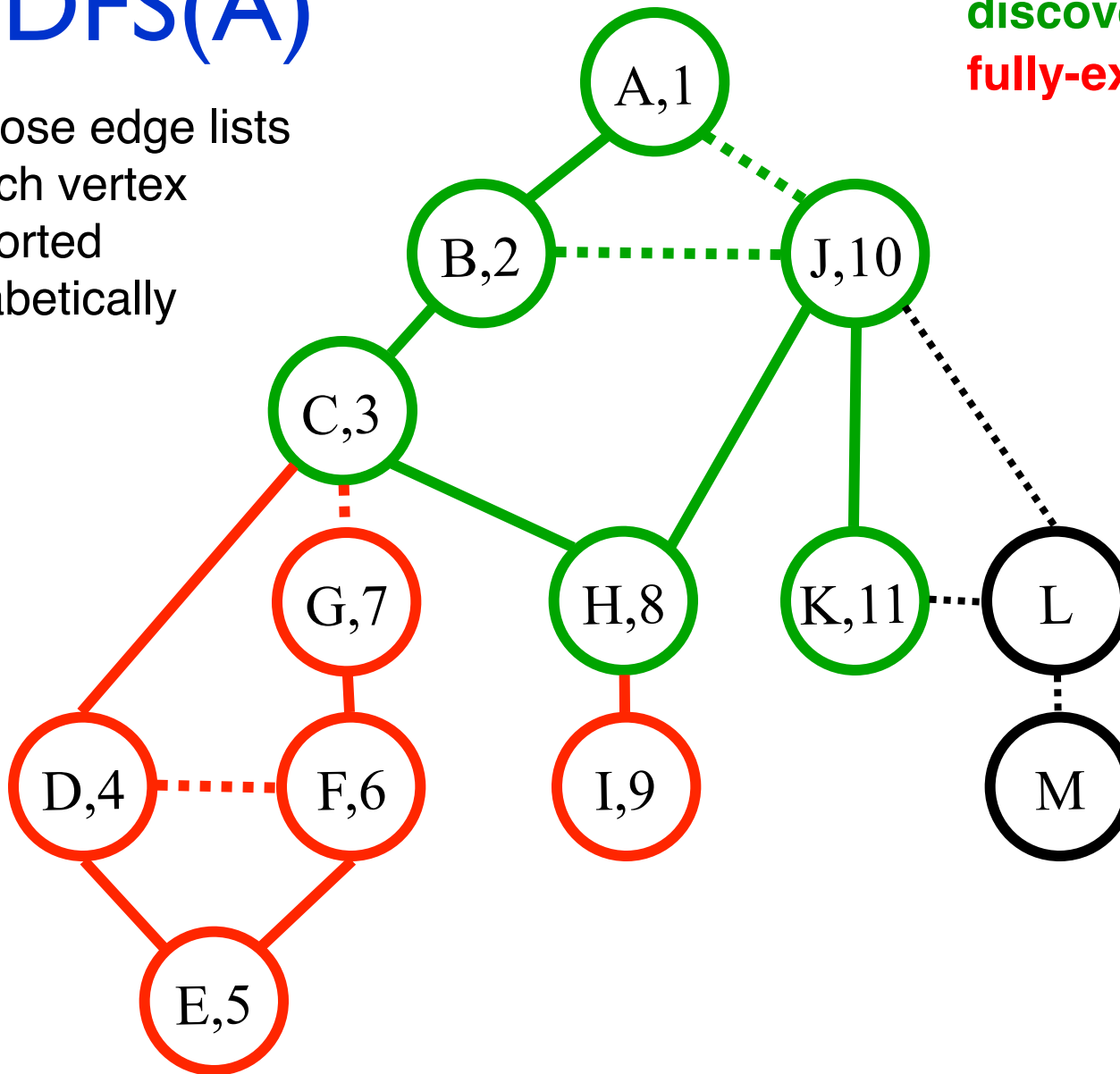
**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
J (A,B,H,K,L)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
J (~~A~~,~~B~~,~~H~~,~~K~~,L)  
K (J,L)

# DFS(A)

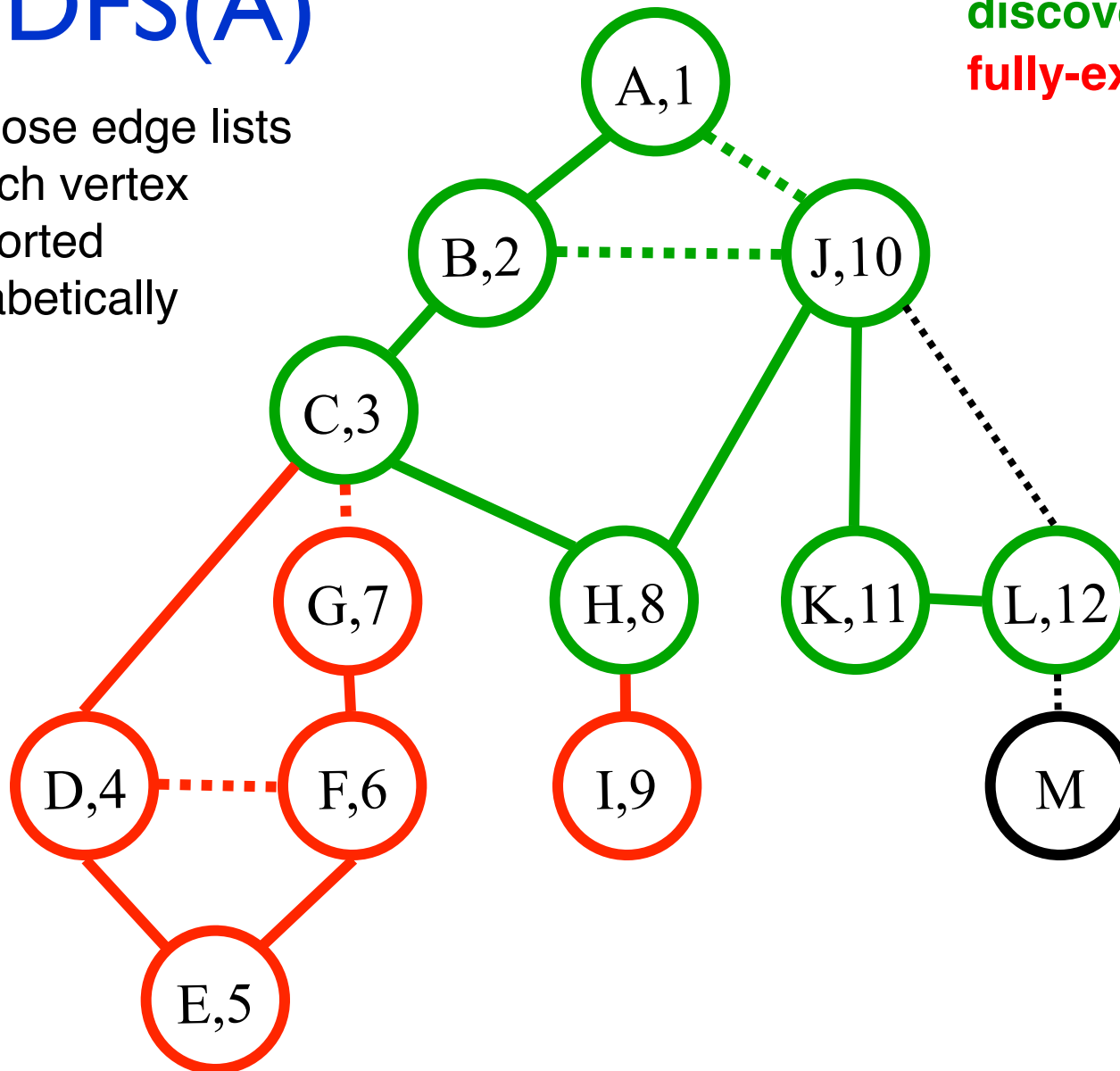
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
J (~~A~~,~~B~~,~~H~~,~~K~~,L)  
K (~~J~~,~~L~~)  
L (J,K,M)

# DFS(A)

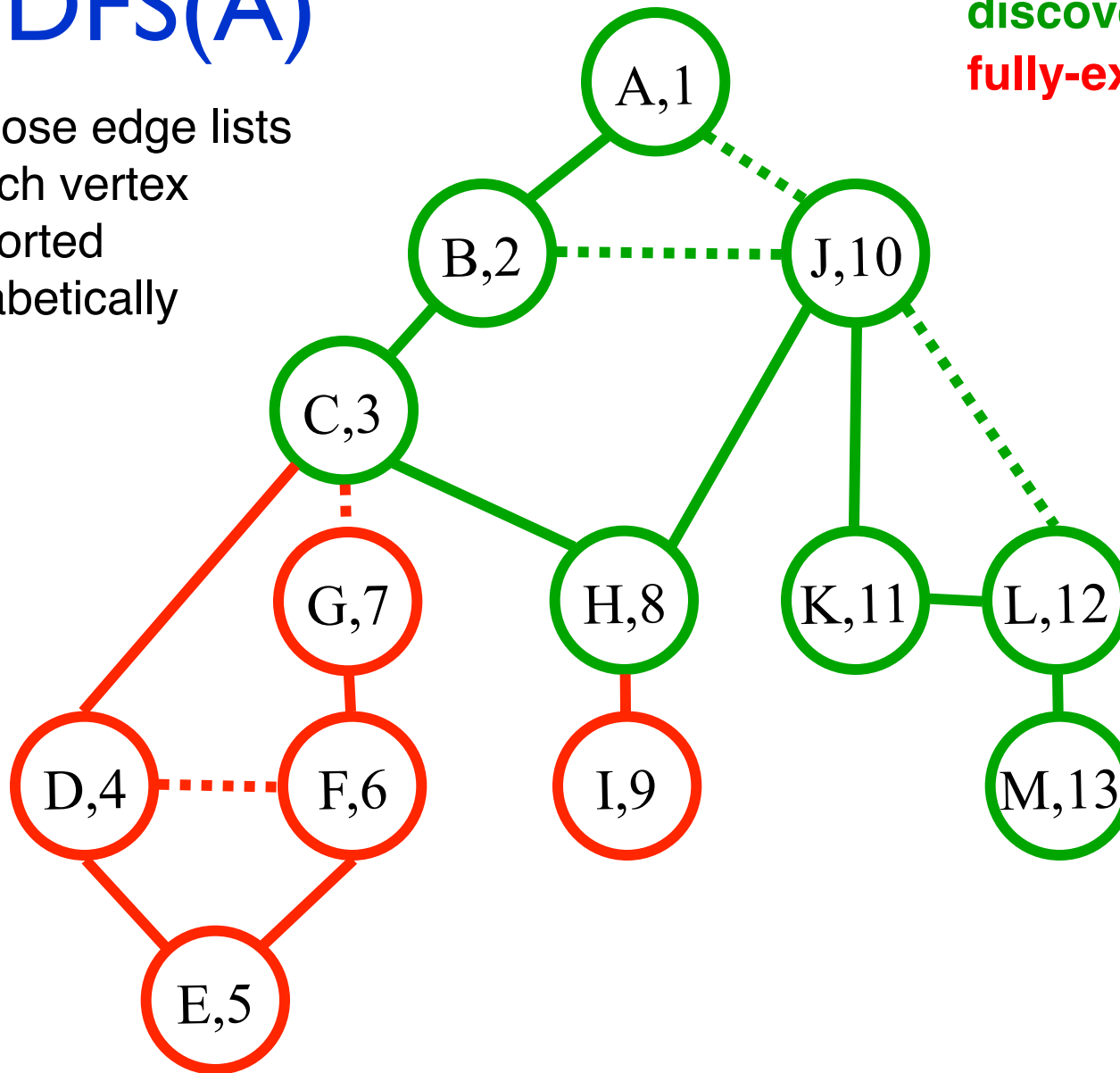
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,~~J~~)  
B (~~A~~,~~C~~,~~J~~)  
C (~~B~~,~~D~~,~~G~~,~~H~~)  
H (~~C~~,~~I~~,~~J~~)  
J (~~A~~,~~B~~,~~H~~,~~K~~,~~L~~)  
K (~~J~~,~~L~~)  
L (~~J~~,~~K~~,~~M~~)  
M(L)

# DFS(A)

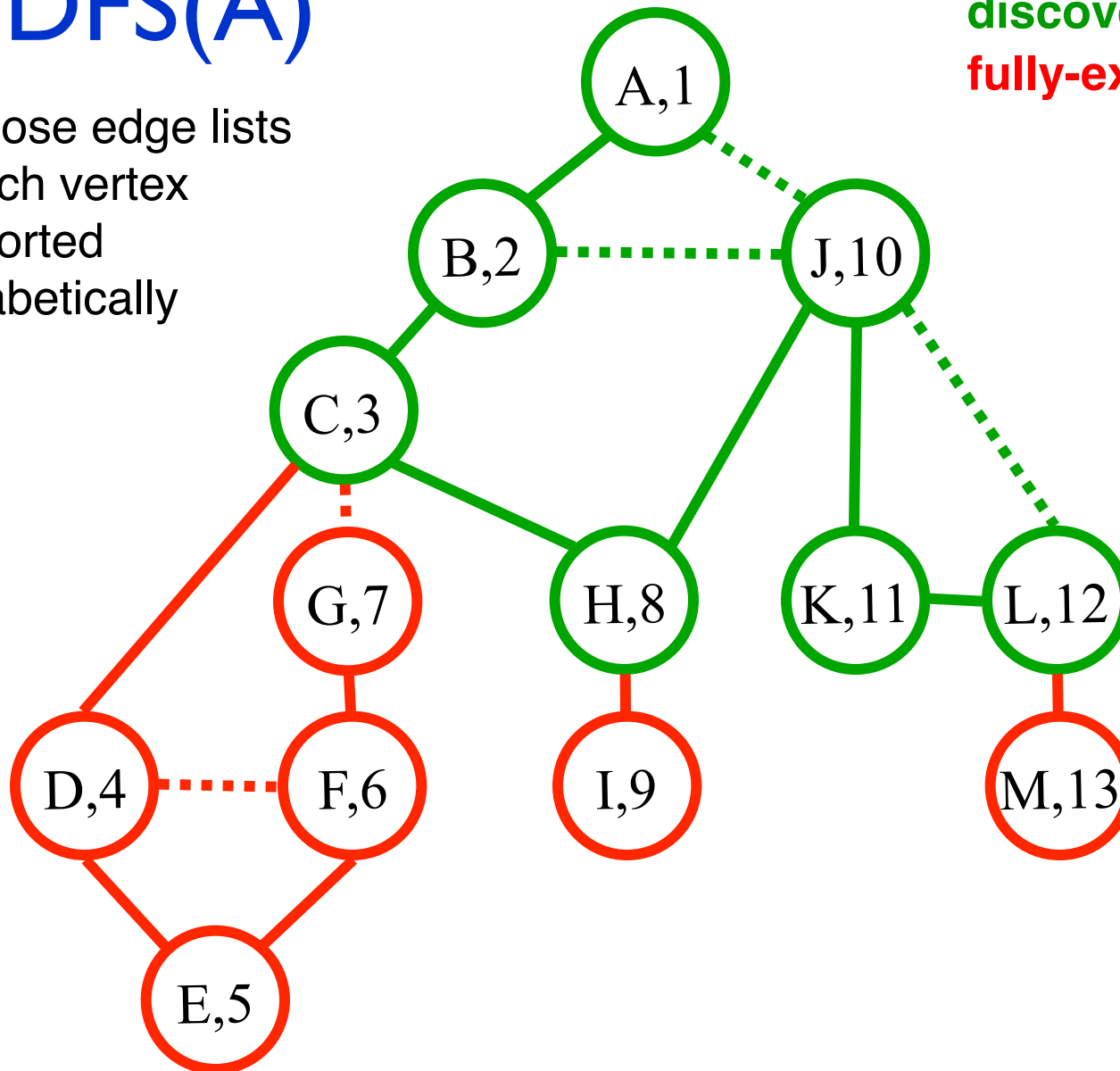
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
J (~~A~~,~~B~~,~~H~~,~~K~~,L)  
K (~~J~~,L)  
L (~~J~~,~~K~~,M)

# DFS(A)

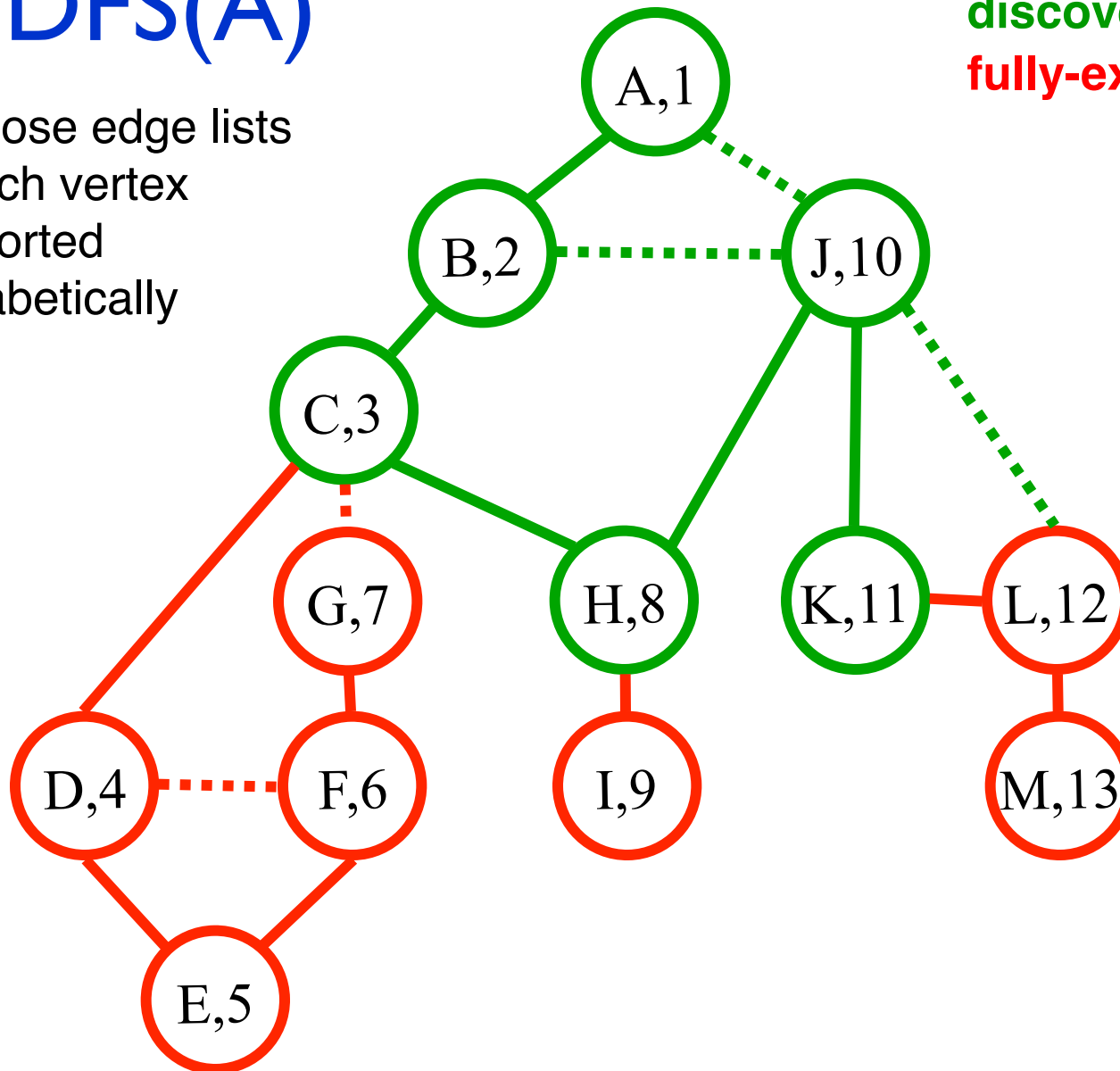
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,I,J)  
J (~~A~~,~~B~~,~~H~~,~~K~~,L)  
K (~~J~~,L)

# DFS(A)

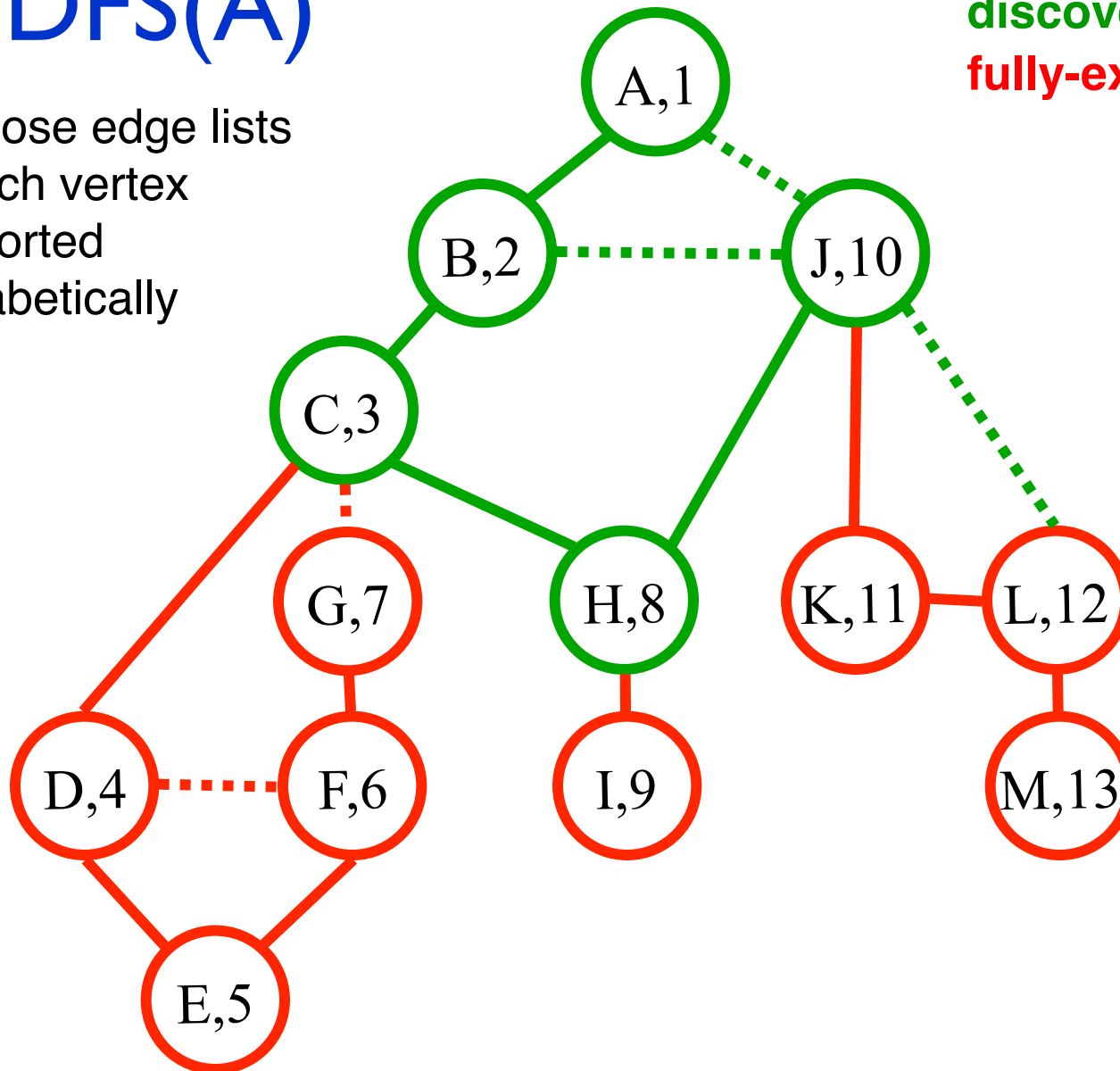
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,H)  
H (~~C~~,~~I~~,J)  
J (~~A~~,~~B~~,~~H~~,~~K~~,L)

# DFS(A)

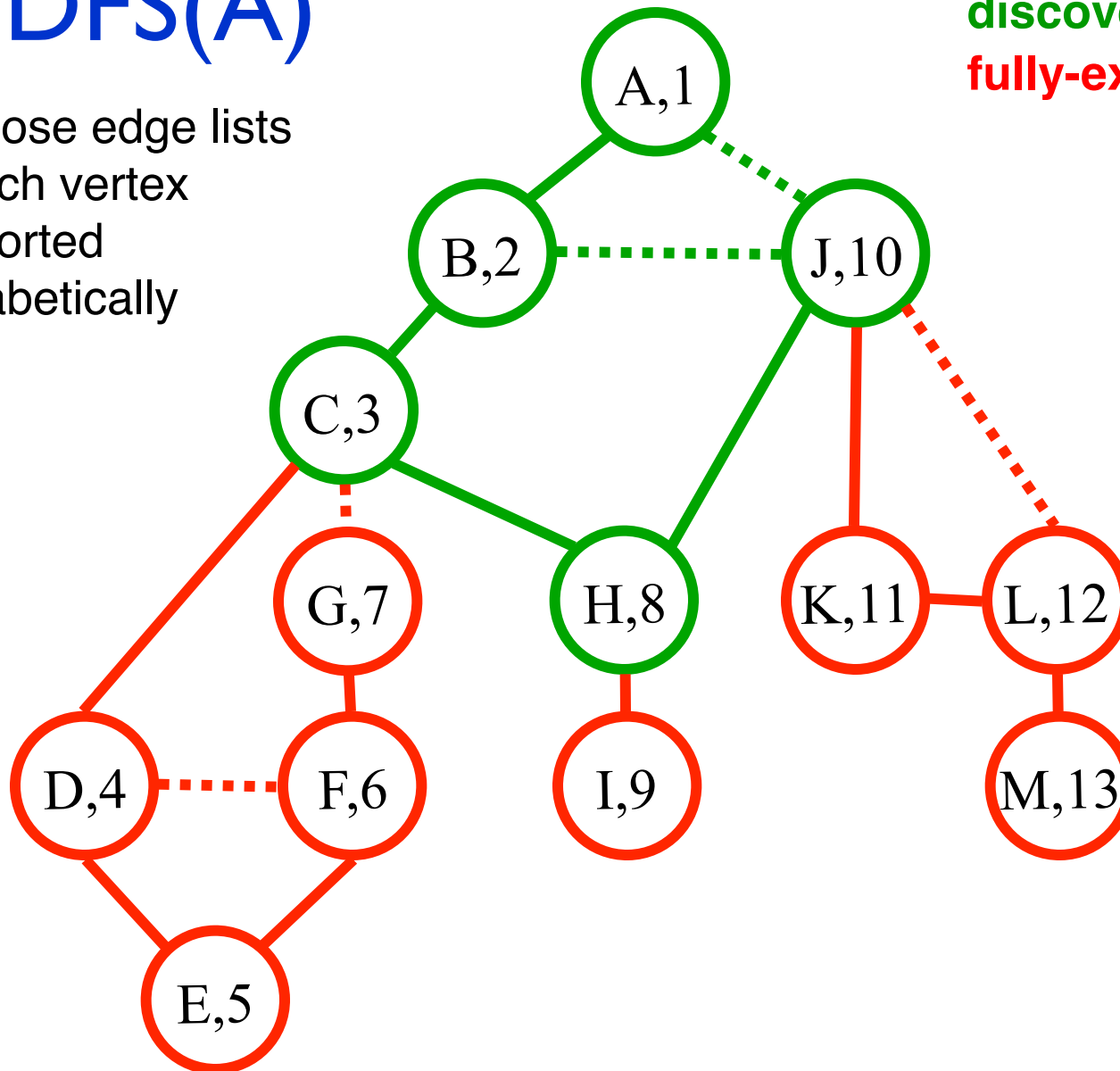
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack: (Edge list)
A ( <del>B</del> ,J)
B ( <del>A</del> , <del>C</del> ,J)
C ( <del>B</del> , <del>D</del> , <del>G</del> ,H)
H ( <del>C</del> ,I,J)
J ( <del>A</del> , <del>B</del> , <del>H</del> , <del>K</del> ,L)



# DFS(A)

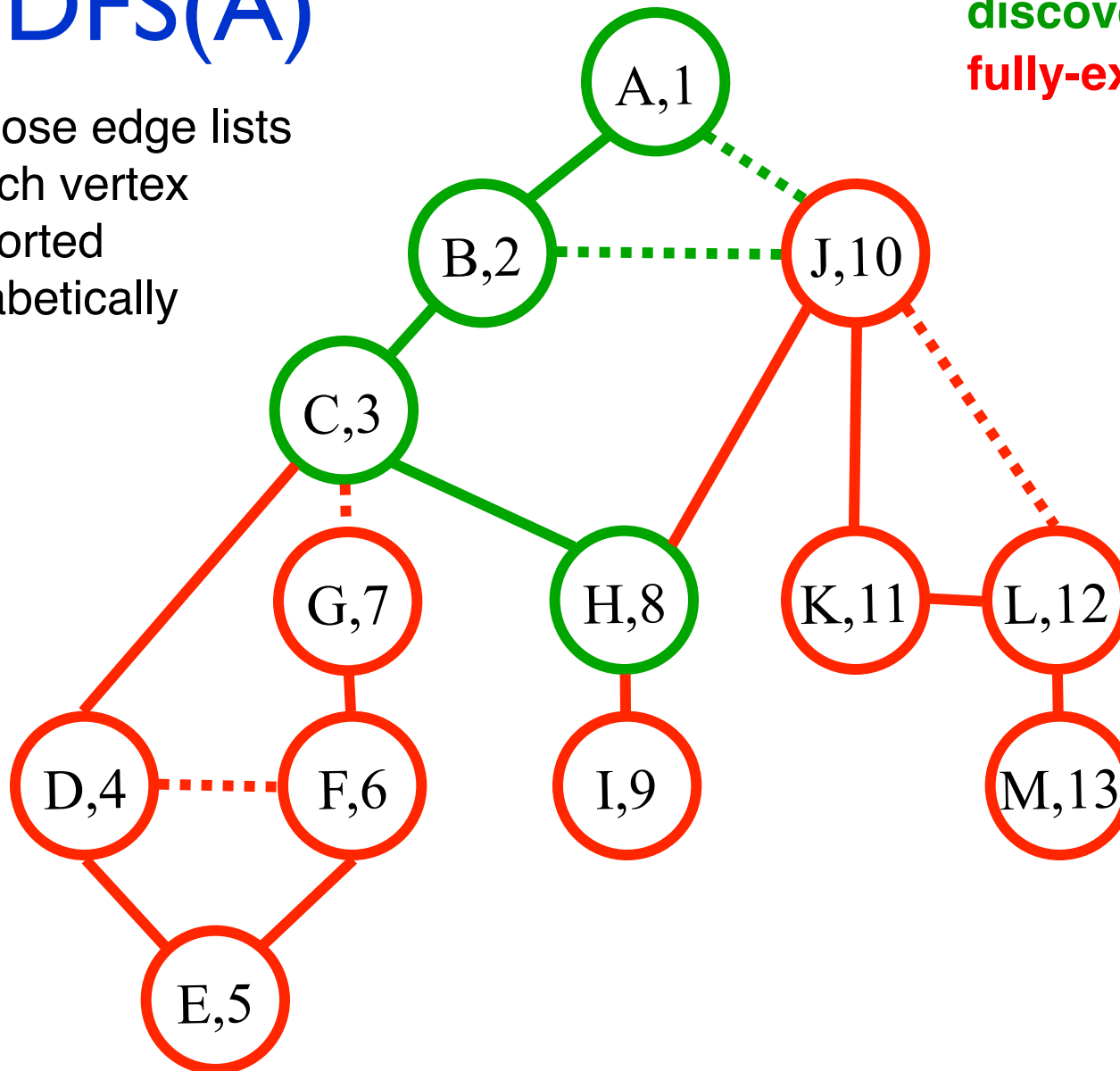
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**



Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,~~H~~)  
H (~~C~~,~~I~~,~~J~~)

# DFS(A)

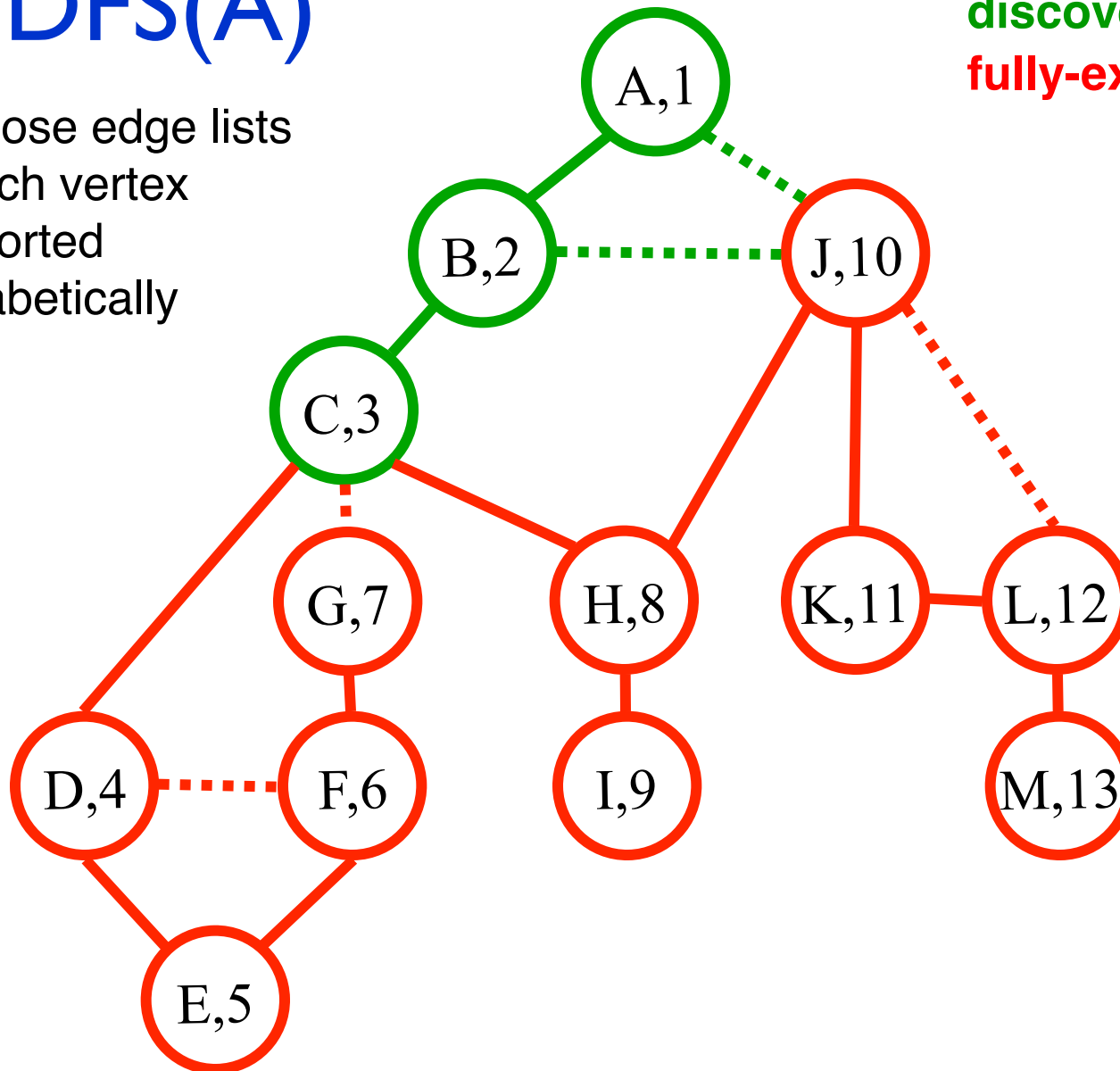
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Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)  
C (~~B~~,~~D~~,~~G~~,~~H~~)

# DFS(A)

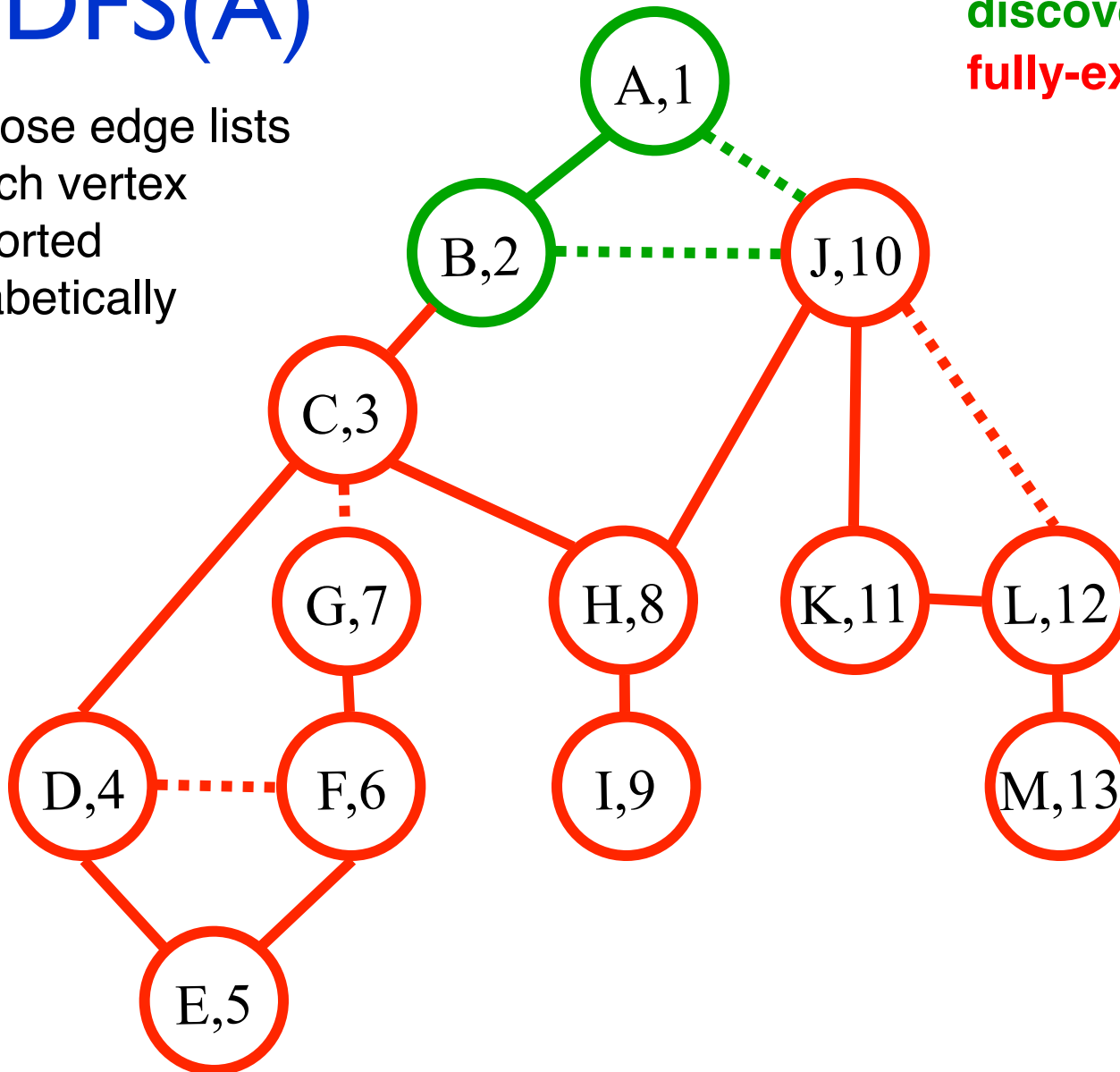
Suppose edge lists at each vertex are sorted alphabetically

Color code:

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Call Stack:  
(Edge list)

A (~~B~~, J)  
B (~~A~~, ~~C~~, J)

# DFS(A)

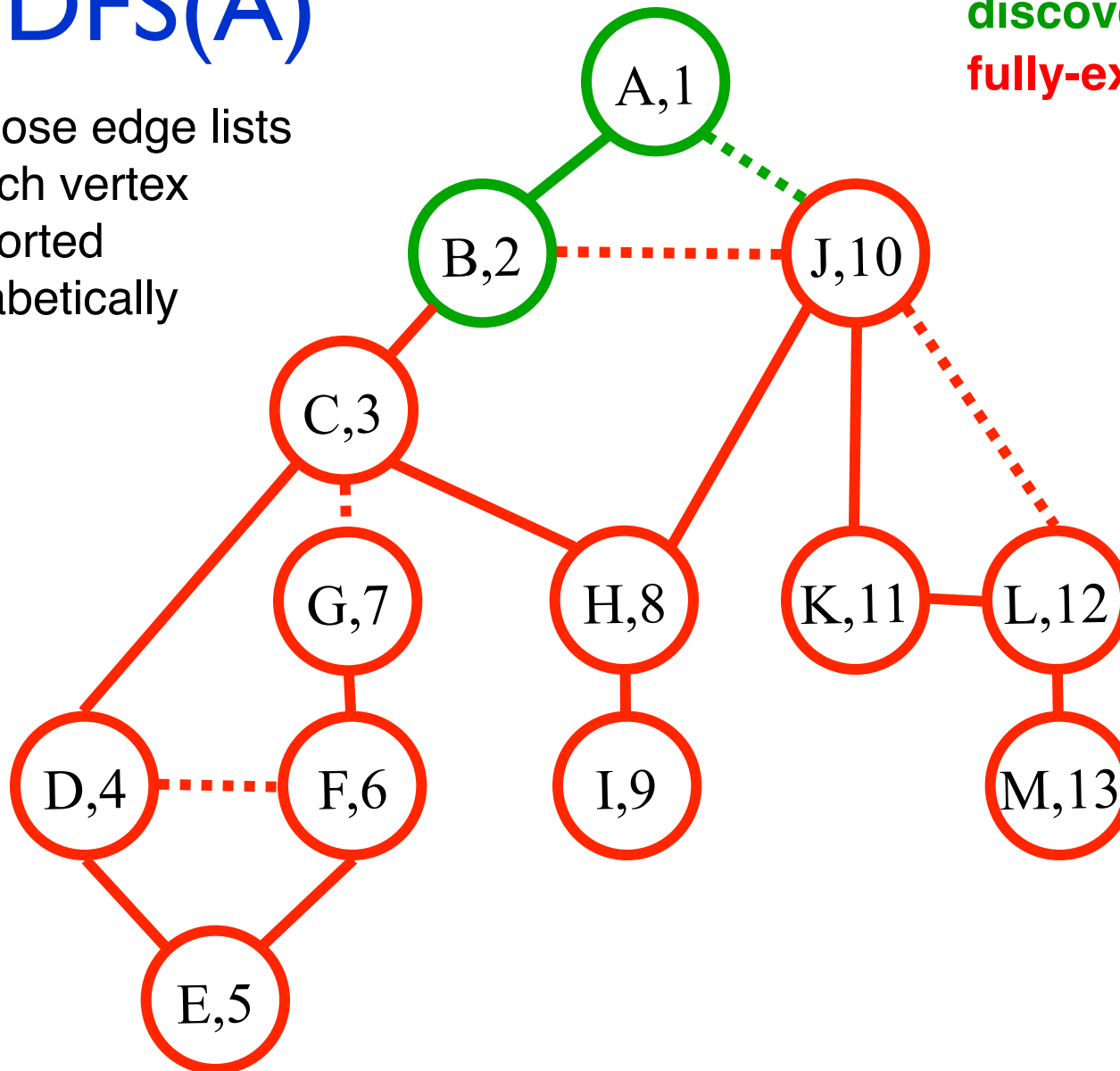
Suppose edge lists at each vertex are sorted alphabetically

Color code:

**undiscovered**

**discovered**

**fully-explored**

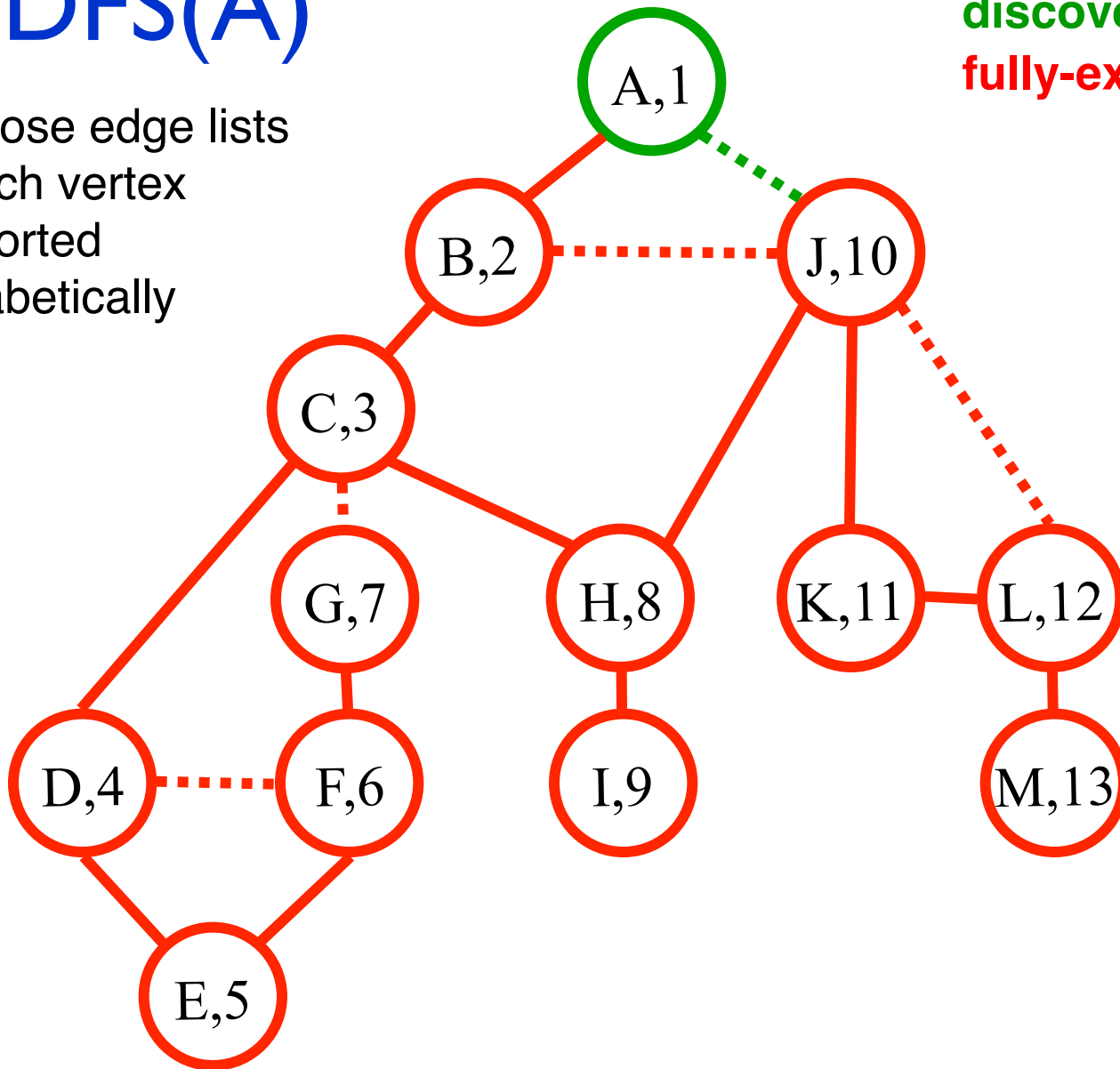


Call Stack:  
(Edge list)

A (~~B~~,J)  
B (~~A~~,~~C~~,J)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



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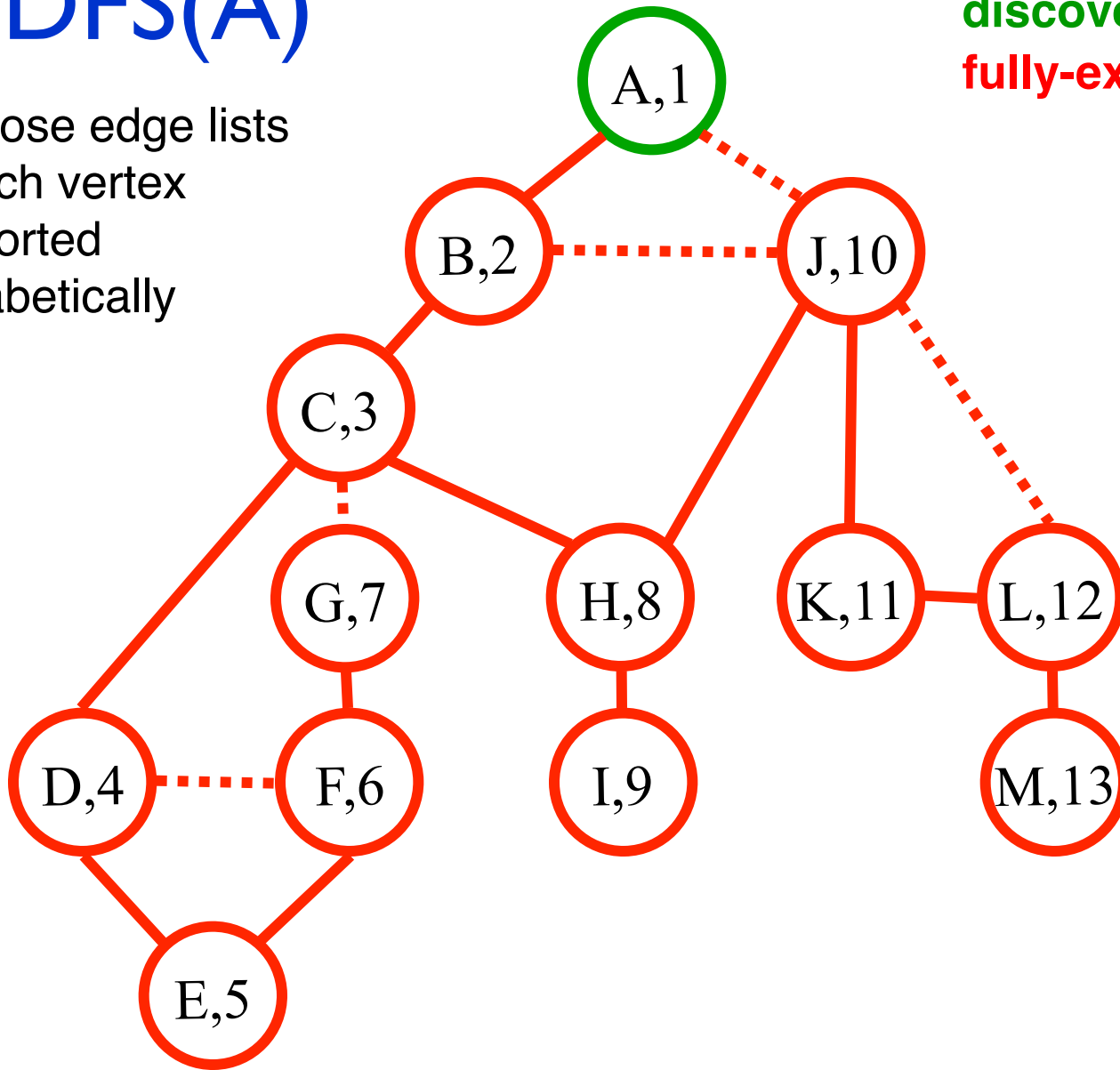
**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~,J)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

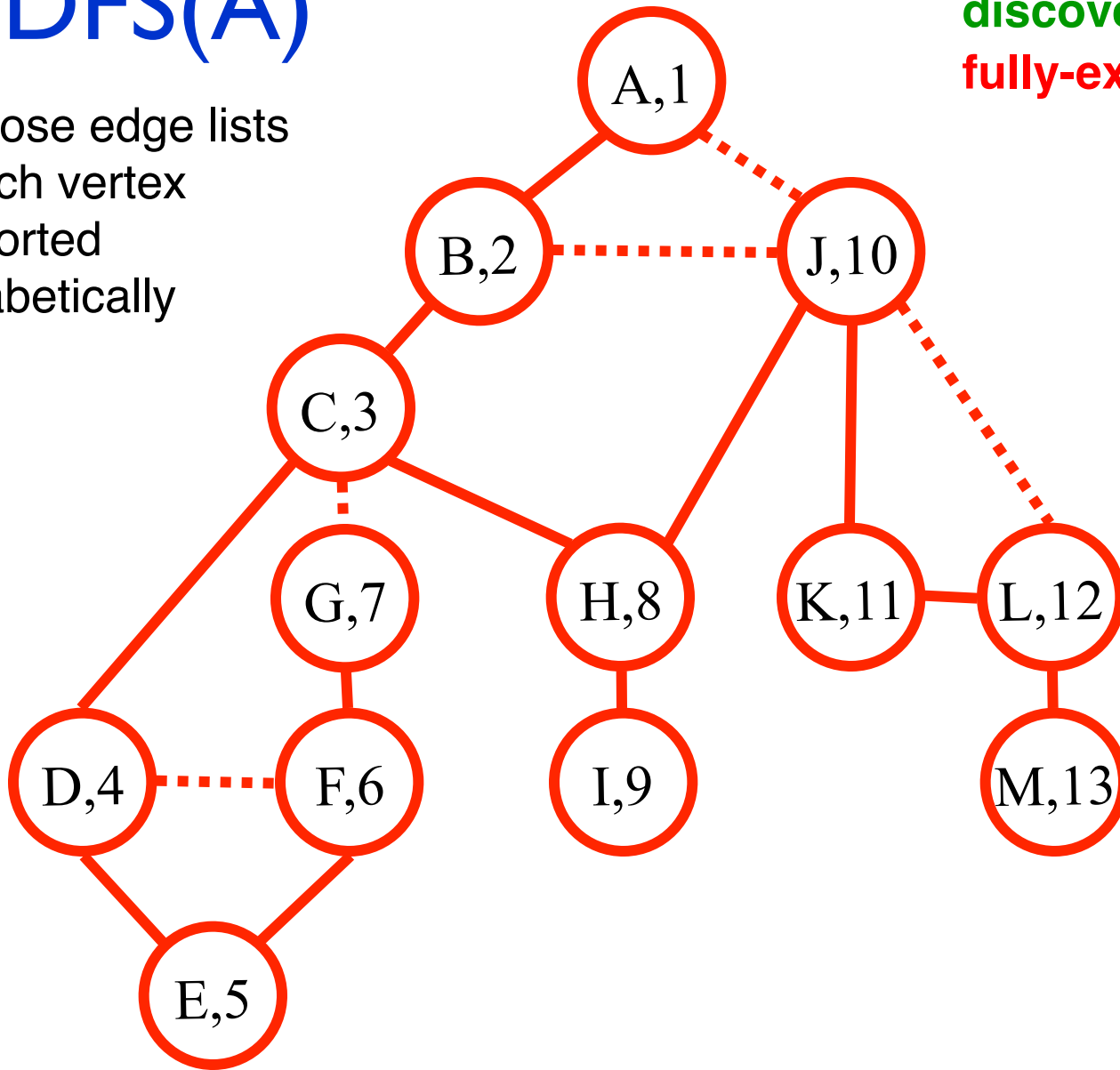
**fully-explored**

Call Stack:  
(Edge list)

A (~~B~~, ~~J~~)

# DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

**undiscovered**

**discovered**

**fully-explored**

Call Stack:  
(Edge list)

TA-DA!!

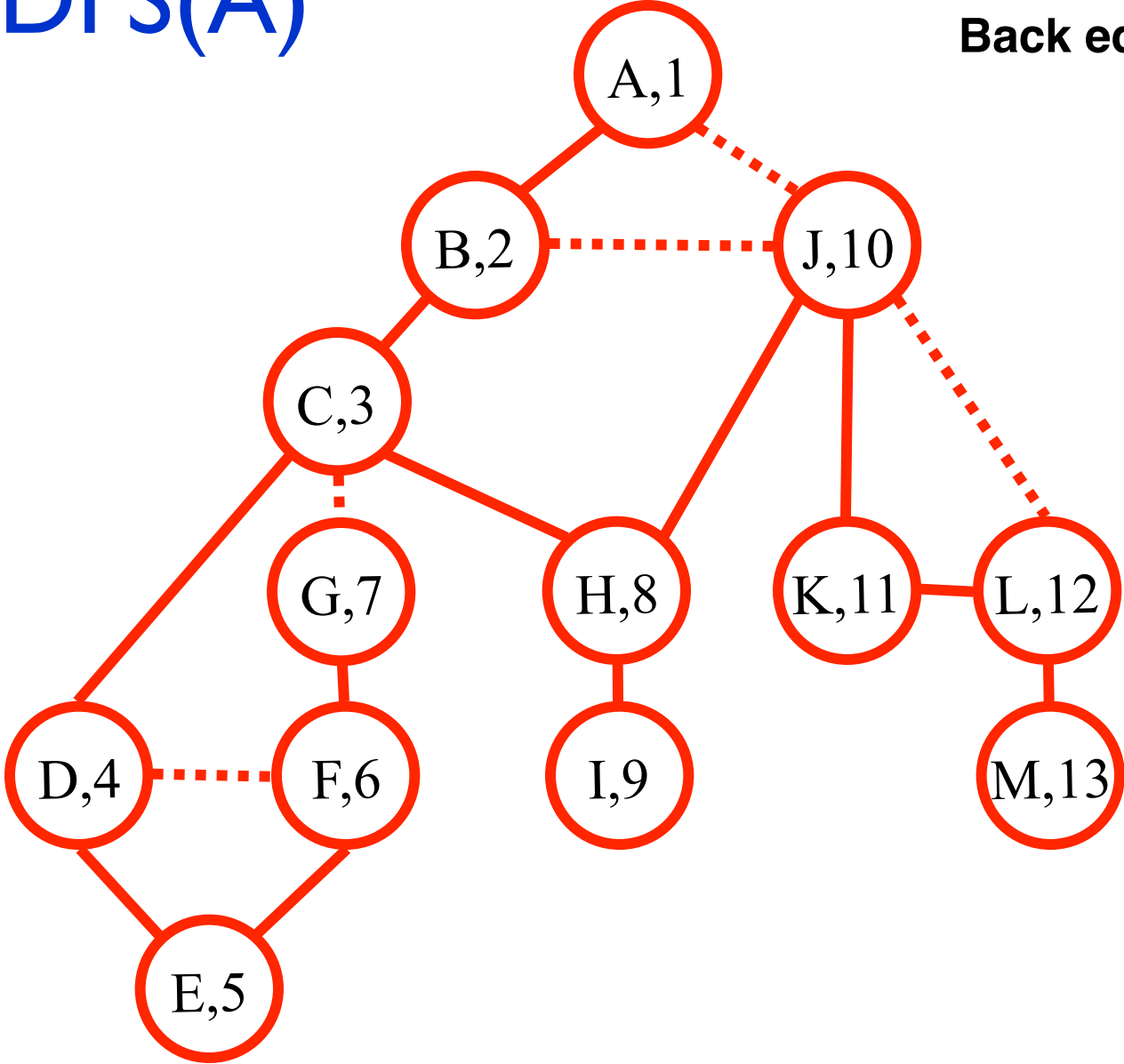
# DFS(A)

Edge code:

Tree edge



Back edge



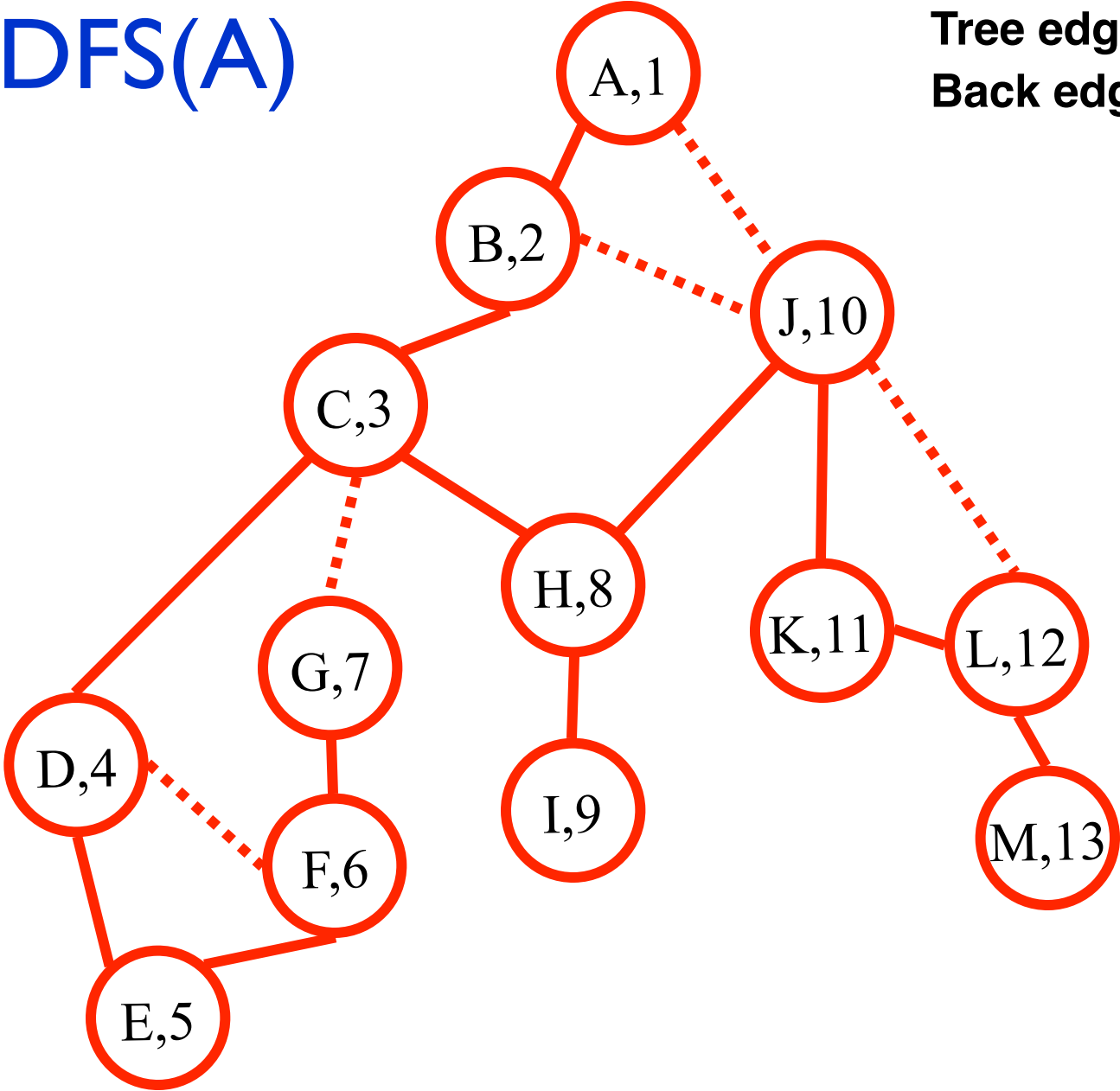


# DFS(A)

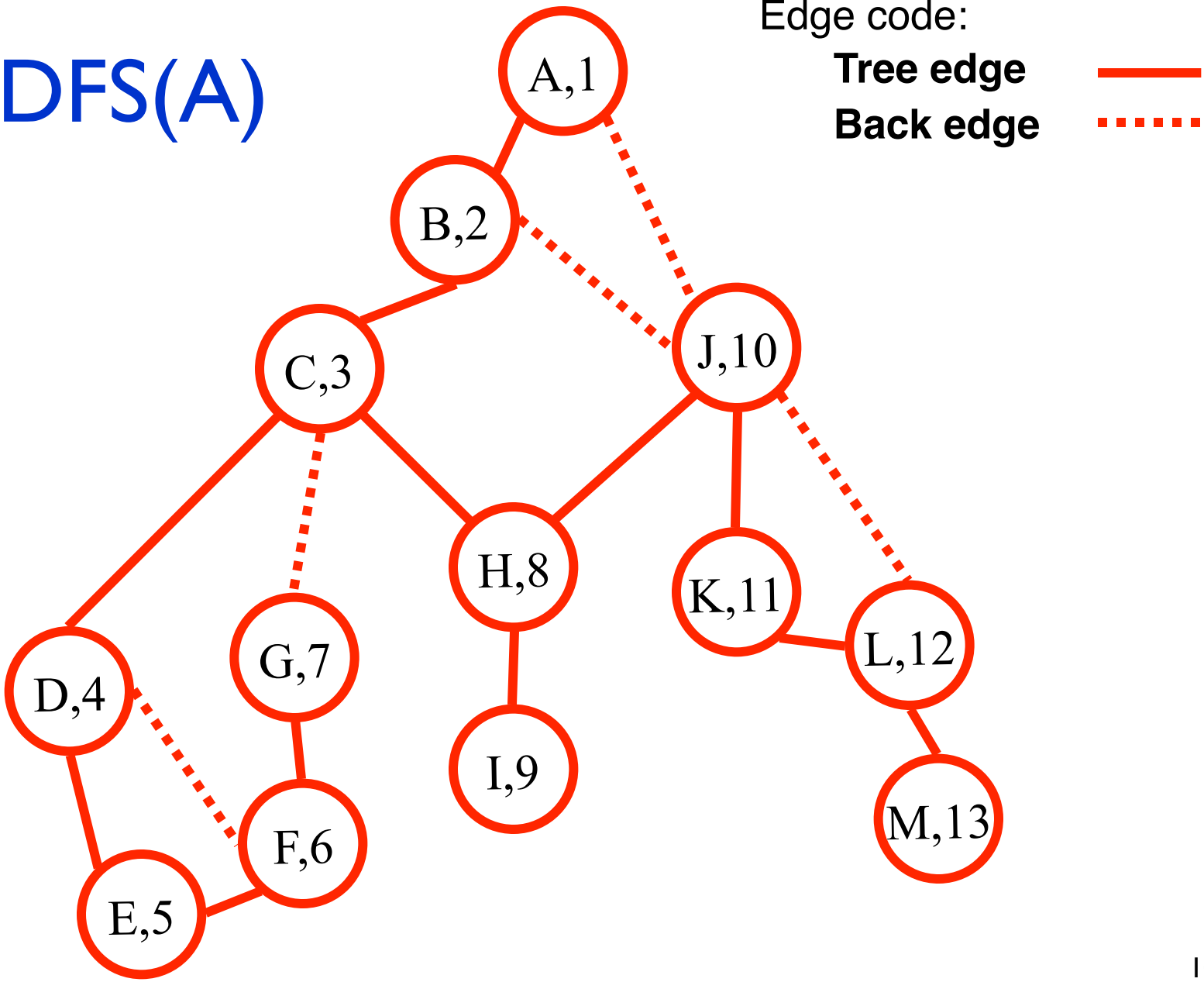
Edge code:

Tree edge

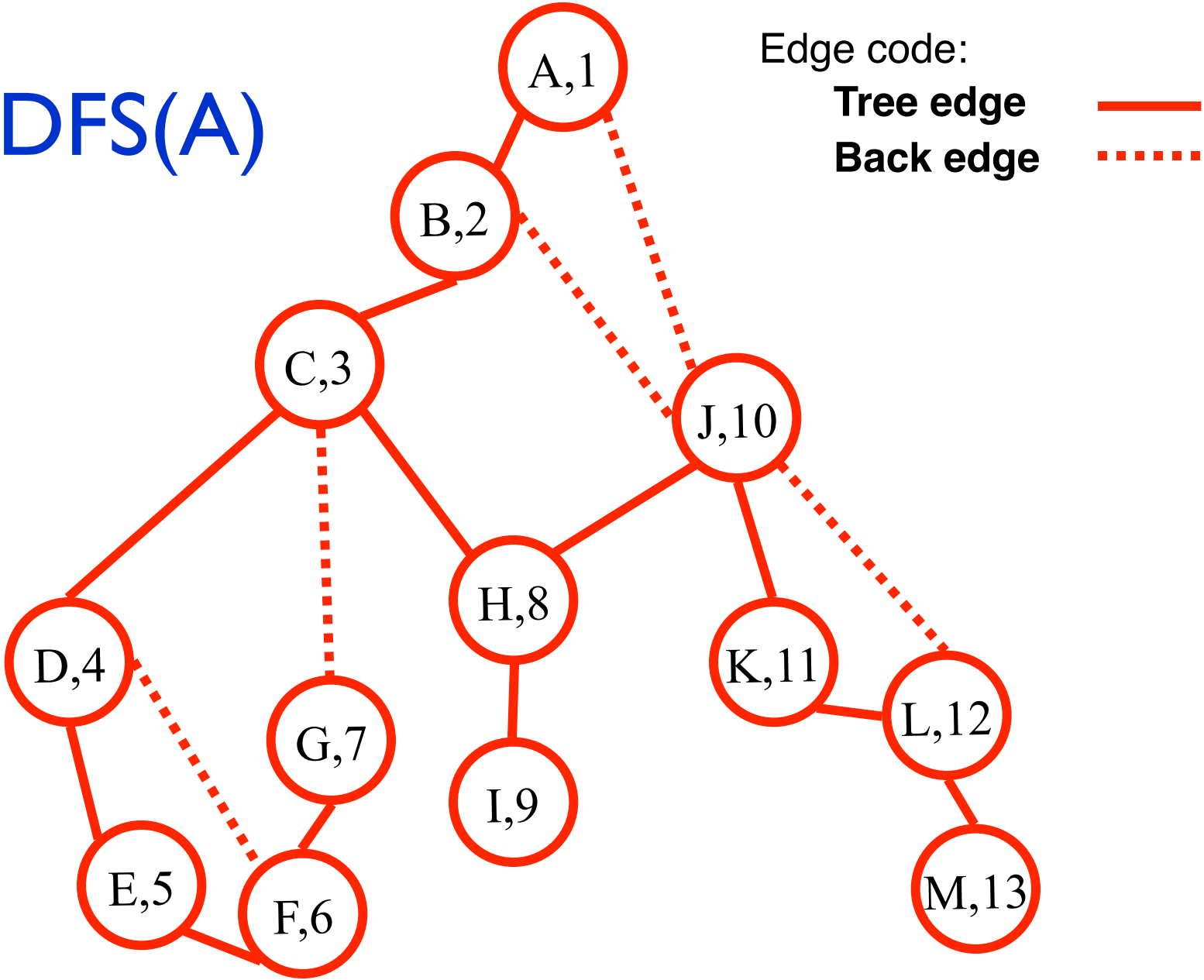
Back edge



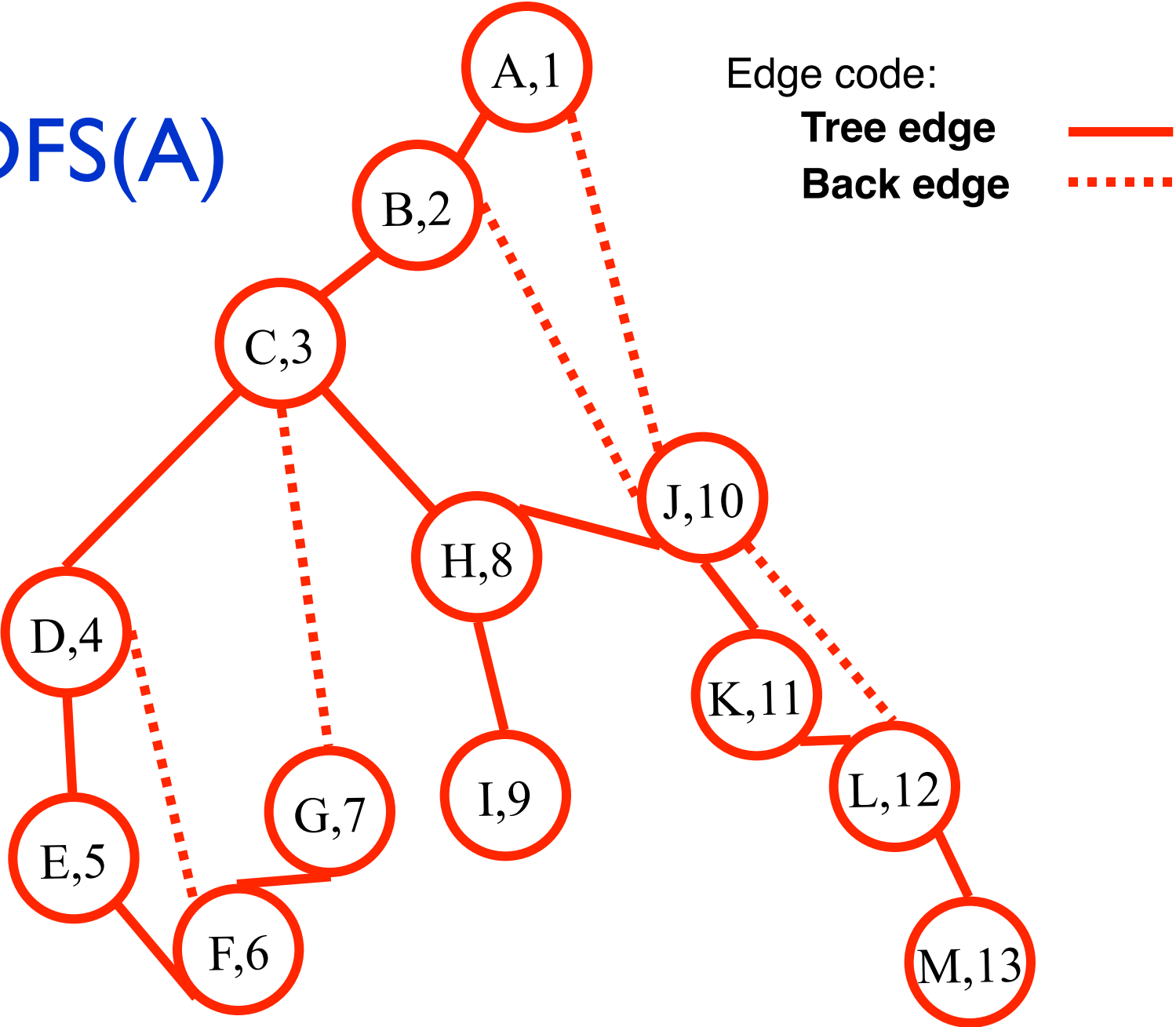
# DFS(A)



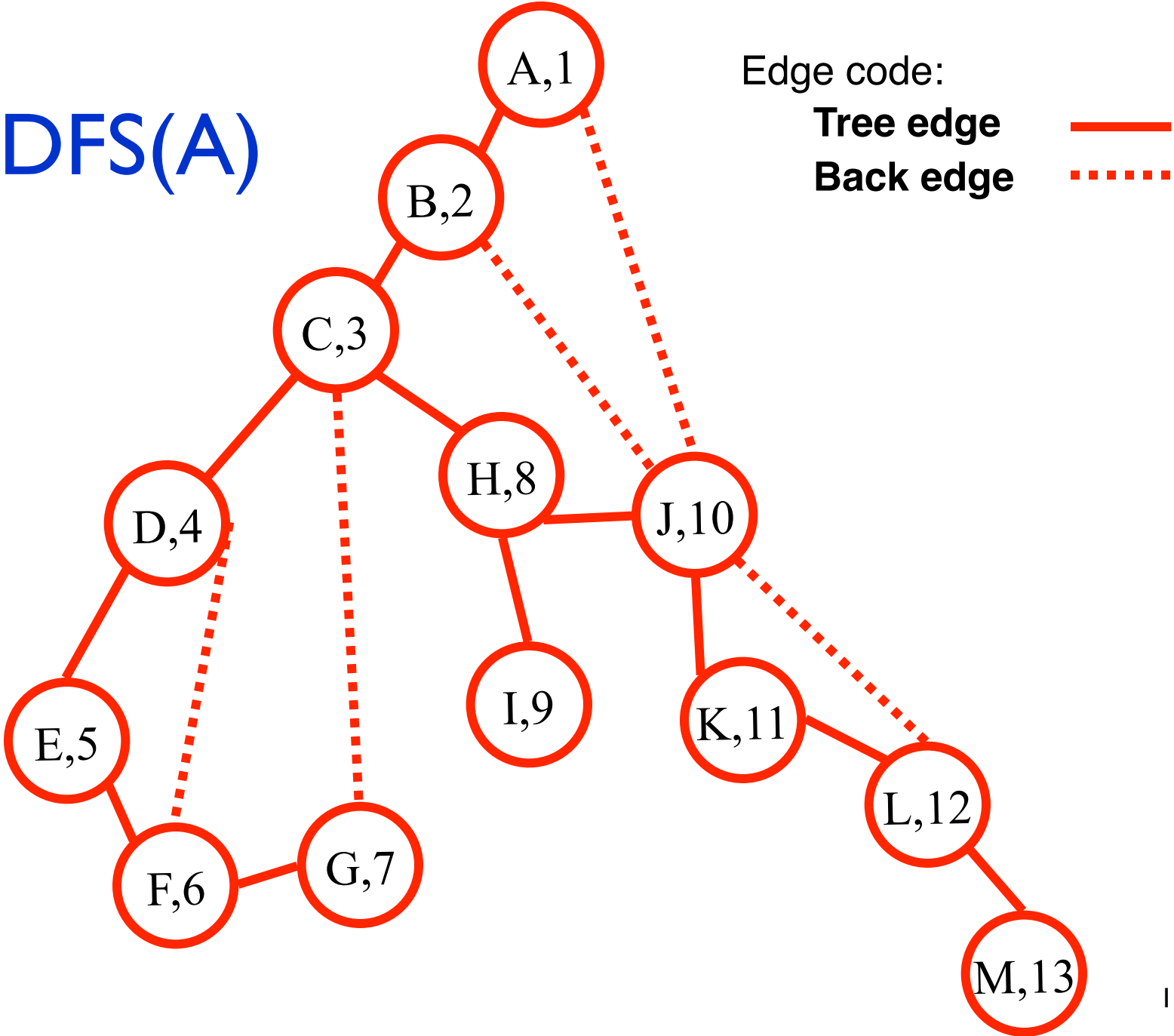
# DFS(A)



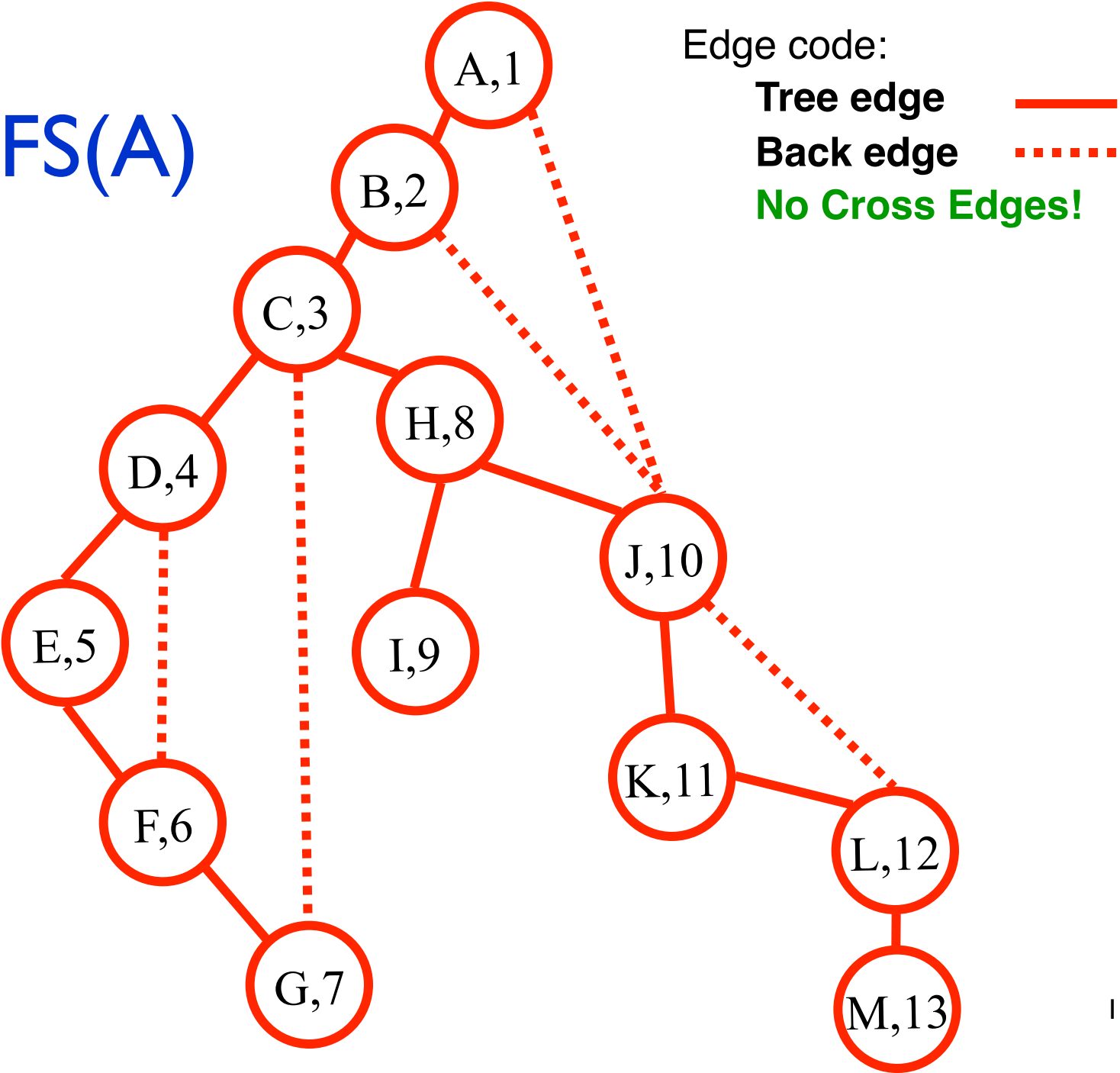
DFS(A)



# DFS(A)



# DFS(A)



# Properties of (Undirected) DFS(v)

Like BFS(v):

DFS(v) visits  $x$  if and only if there is a path in  $G$  from  $v$  to  $x$  (through previously unvisited vertices)

Edges into then-undiscovered vertices define a **tree** – the "depth first spanning tree" of  $G$

Unlike the BFS tree:

the DF spanning tree isn't minimum depth

its levels don't reflect min distance from the root

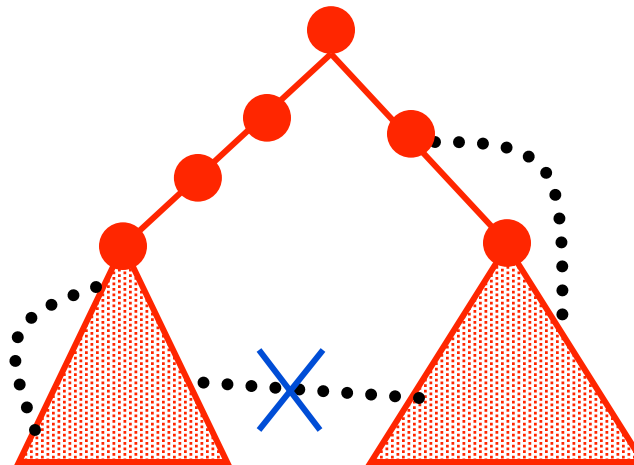
non-tree edges never join vertices on the same or adjacent levels

**BUT...**

# Non-tree edges

All non-tree edges join a vertex and one of its descendants/ancestors in the DFS tree

No cross edges!





# Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor

# A simple problem on trees

*Given:* tree  $T$ , a value  $L(v)$  defined for every vertex  $v$  in  $T$

*Goal:* find  $M(v)$ , the min value of  $L(v)$  anywhere in the subtree rooted at  $v$  (including  $v$  itself).

*How?* Depth first search, using:

$$M(v) = \left. \begin{array}{l} L(v) \\ \min(L(v), \min_{w \text{ a child of } v} M(w)) \end{array} \right\} \begin{array}{l} \text{if } v \text{ is a leaf} \\ \text{otherwise} \end{array}$$

# Application: Articulation Points

A node in an undirected graph is an *articulation point* iff removing it disconnects the graph (or, more generally, increases the number of connected components)

*Articulation*  
(noun): the state  
of being jointed

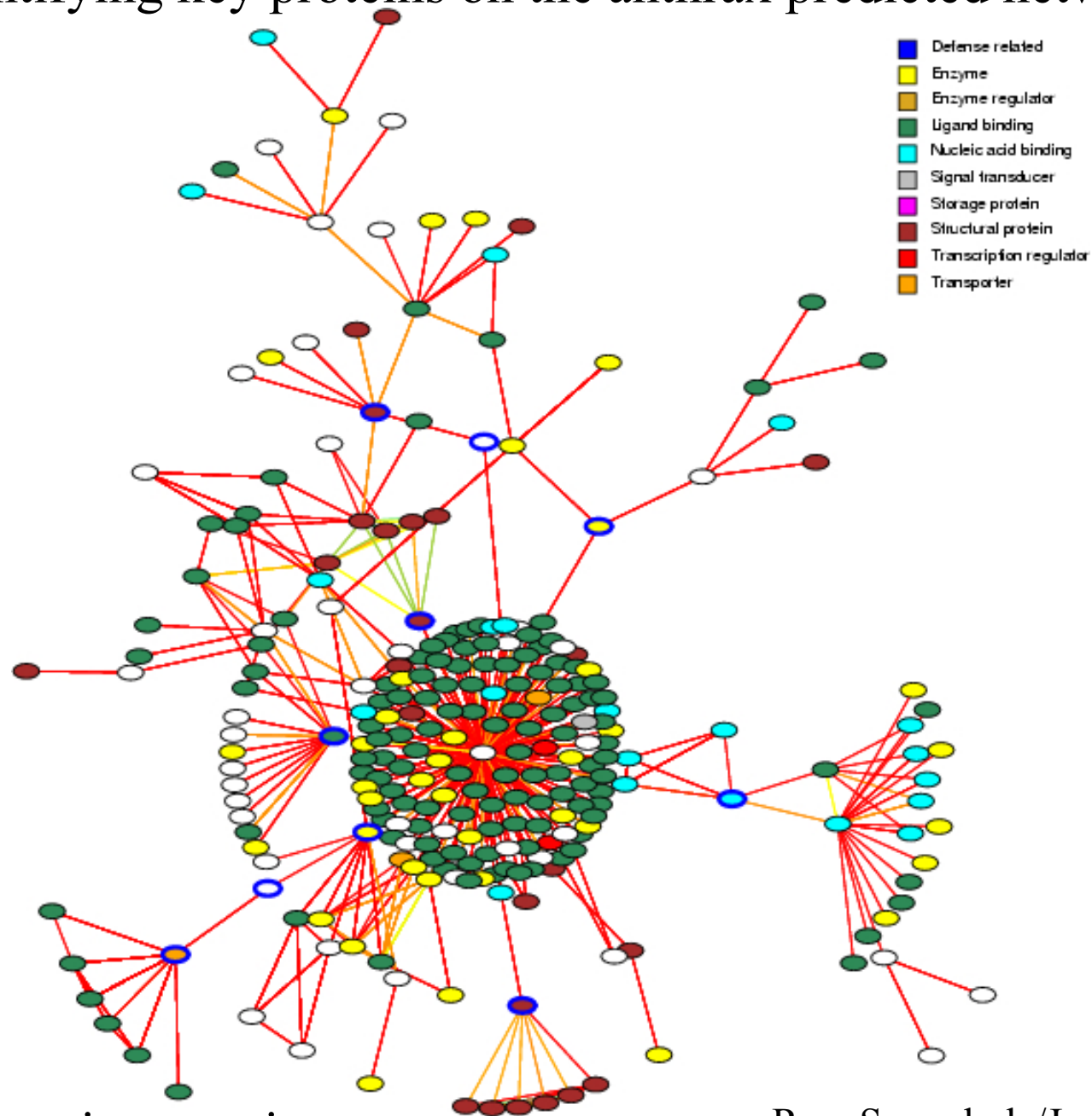
Articulation points represent, e.g.:

vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

bottlenecks to information flow in a network

...

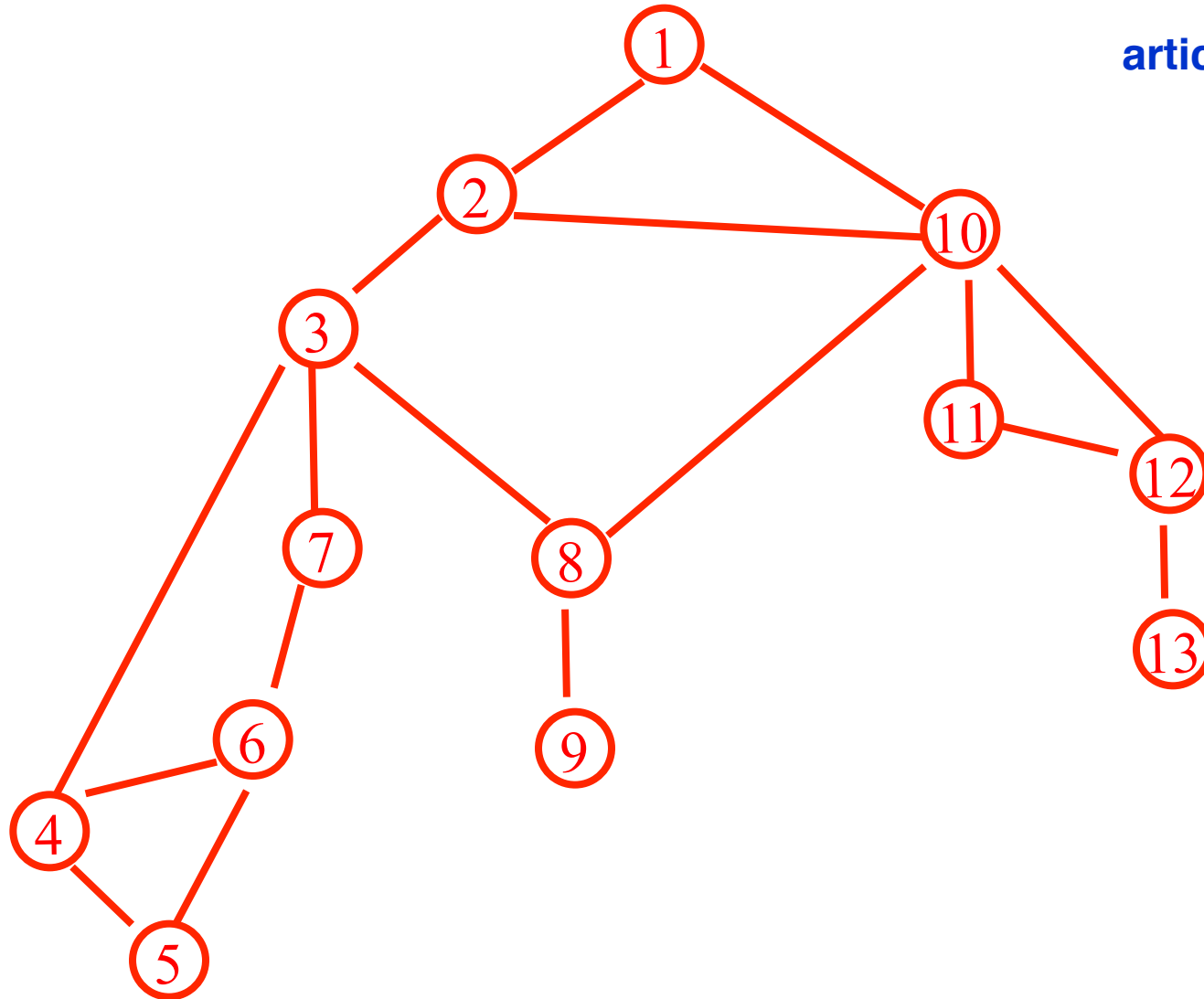
# Identifying key proteins on the anthrax predicted network



Articulation point proteins

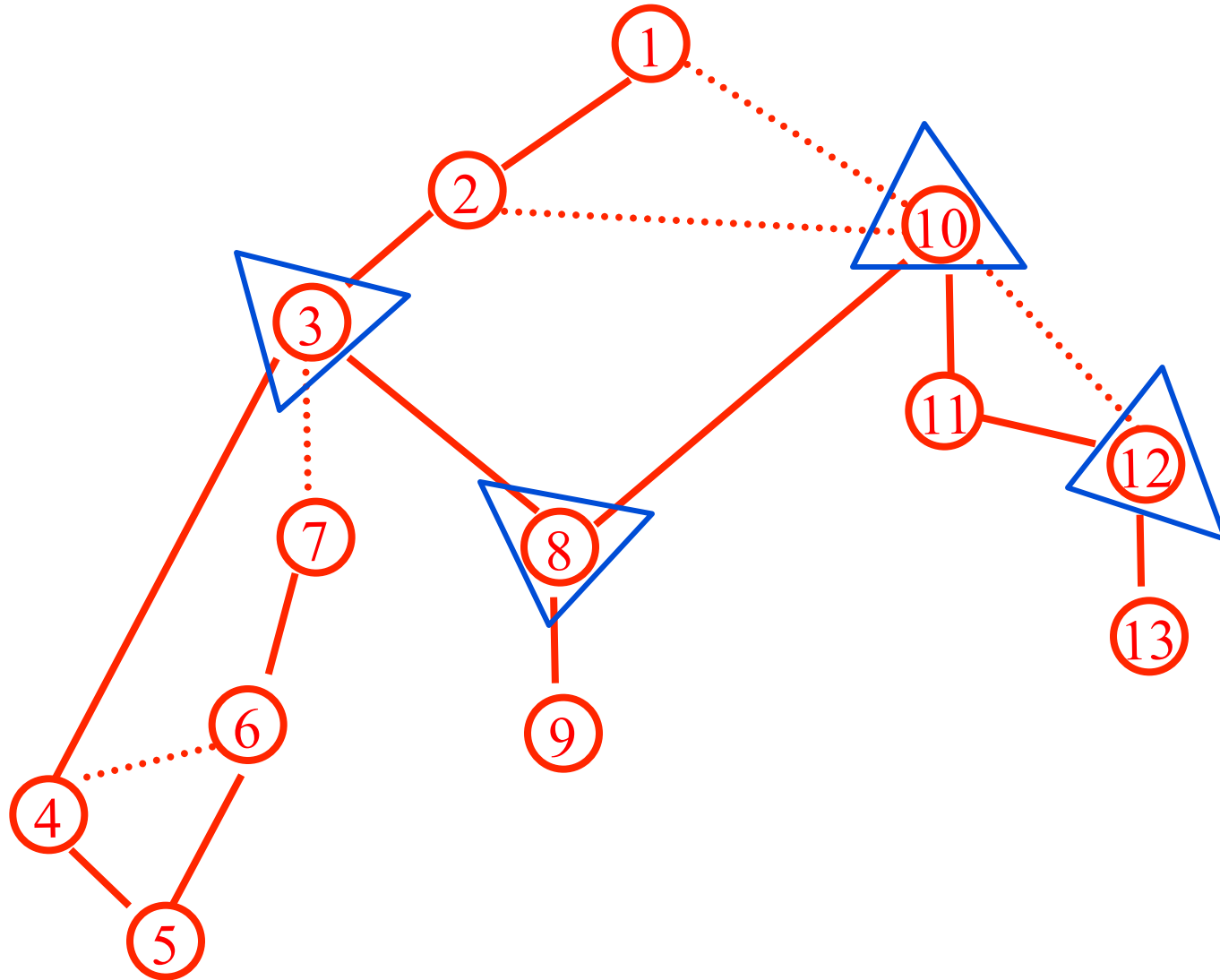
Ram Samudrala/Jason McDermott

# Articulation Points



**articulation point**  
iff its removal  
disconnects  
the graph

# Articulation Points



# Simple Case: Artic. Pts in a tree

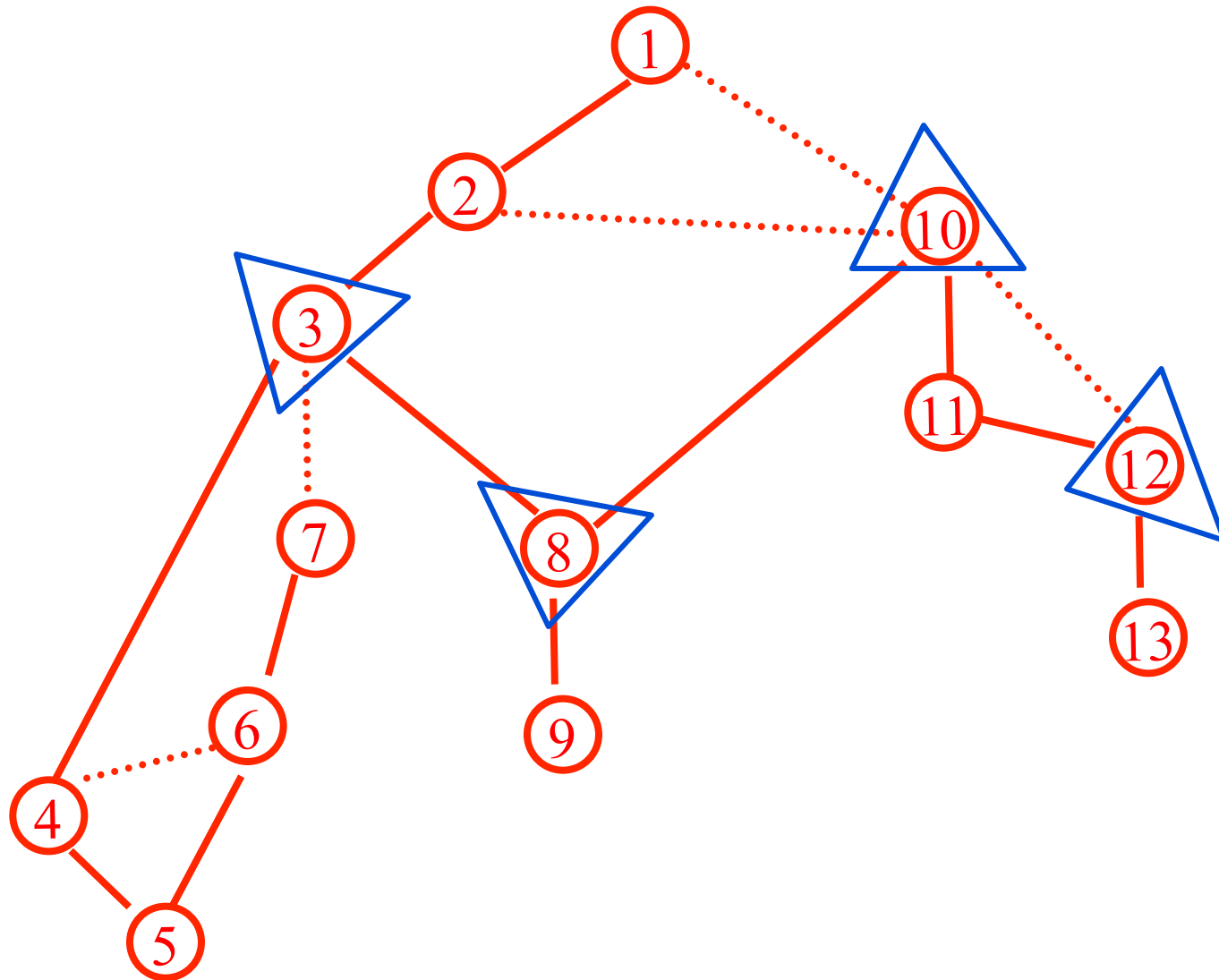
Leaves – never articulation points

Internal nodes – always articulation points

Root – articulation point if and only if it has two or more children

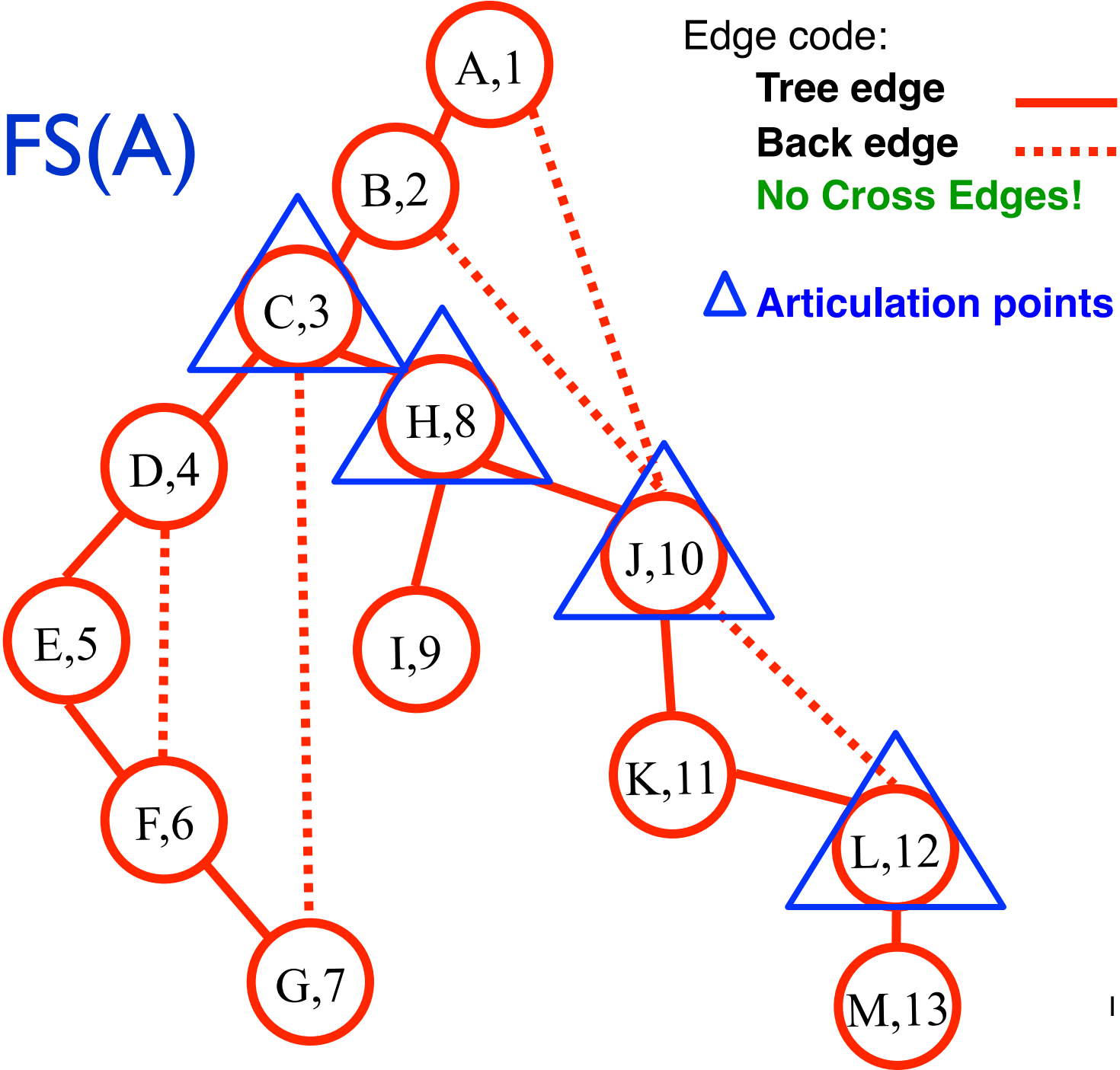
Non-tree: extra edges remove some articulation points (which ones?)

# Articulation Points





# DFS(A)



# Articulation Points from DFS

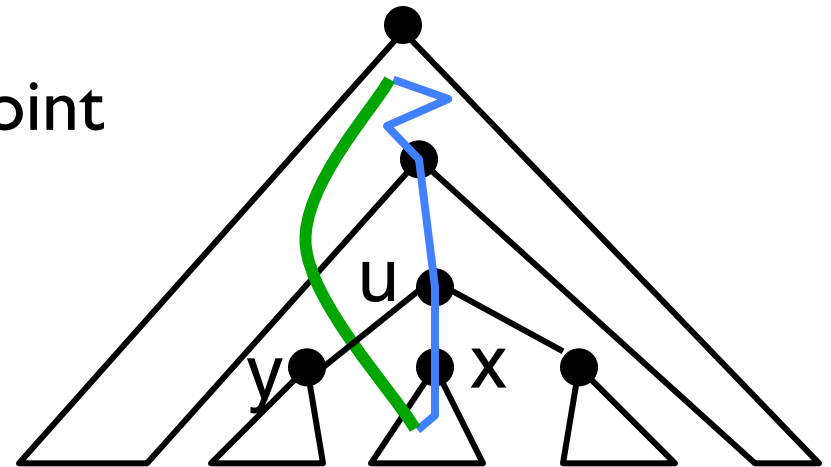
Root node is an articulation point  
iff it has more than one child

Leaf is never an articulation point

Non-leaf, non-root node  
 $u$  is an articulation point



$\exists$  some child  $y$  of  $u$  s.t.  
no non-tree edge goes  
above  $u$  from  $y$  or below



*If  $u$ 's removal does NOT separate  $x$ , there must be an exit from  $x$ 's subtree. How? Via back edge.*

LOW(v) = highest  
exit from v's subtree

# Articulation Points: the "LOW" function

trivial

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or directly connected to a vertex in that subtree by one back edge.

critical

**Key idea 1:** if some child x of v has  $LOW(x) \geq dfs\#(v)$  then v is an articulation point (excl. root)

**Key idea 2:**  $LOW(v) = \min ( \{dfs\#(v)\} \cup \{LOW(w) \mid w \text{ a child of } v\} \cup \{ dfs\#(x) \mid \{v,x\} \text{ is a back edge from } v \} )$

# DFS To Find Articulation Points

Global initialization:  $\text{dfscounter} = 0$ ;  $v.\text{dfs\#} = -1$  for all  $v$ .

DFS( $v$ ):

```
v.dfs# = dfscounter++
```

```
v.low = v.dfs# // initialization
```

```
for each edge {v,x}
```

```
  if (x.dfs# == -1) // x is undiscovered
```

```
    DFS(x)
```

```
    v.low = min(v.low, x.low)
```

```
    if (x.low >= v.dfs#)
```

```
      print "v is art. pt., separating x"
```

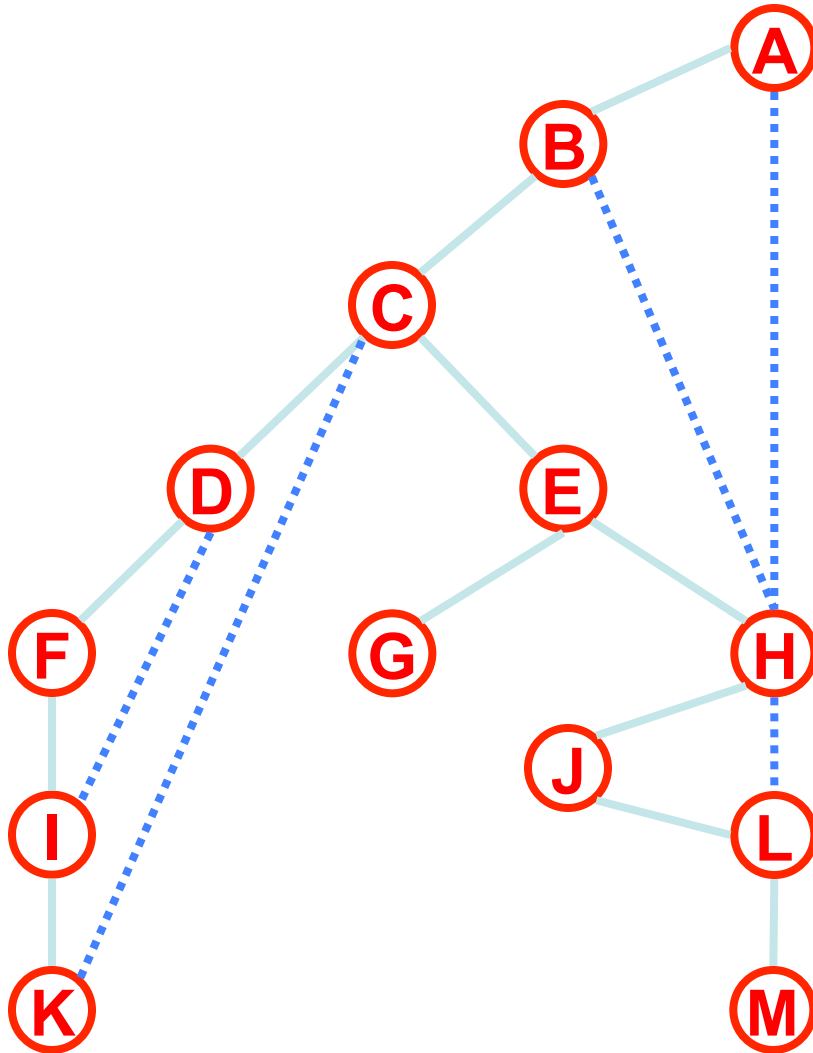
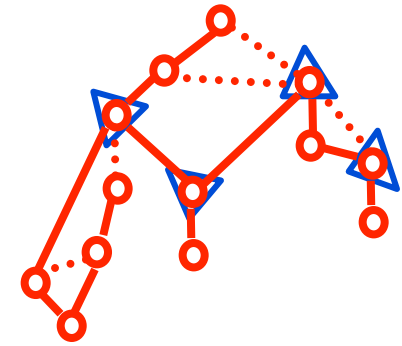
```
    else if (x is not v's parent)
```

```
      v.low = min(v.low, x.dfs#)
```

Except for root. Why?  
What if G is not connected?

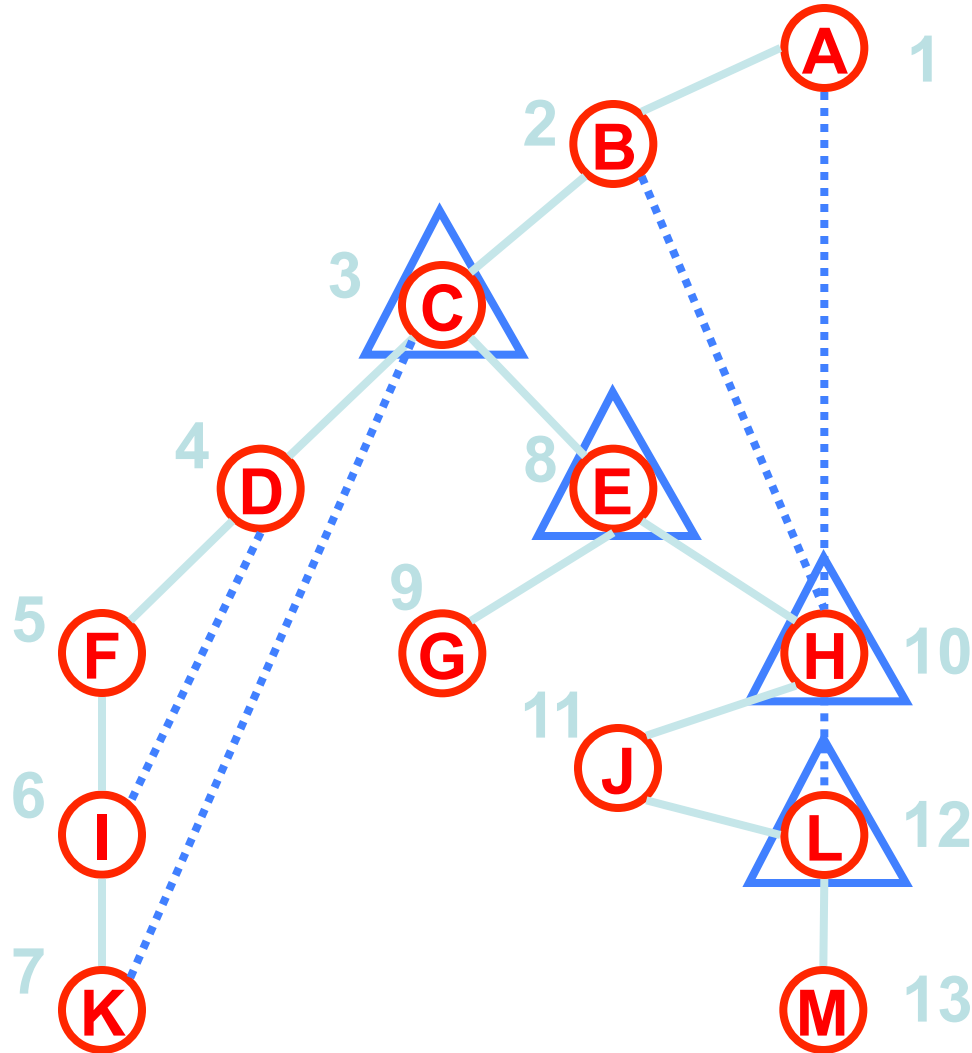
Equiv: "if( {v,x}  
is a back edge)"  
Why?

# Articulation Point



Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

# Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
I	6	3
J	11	10
K	7	3
L	12	10
M	13	13

# Summary

Graphs – abstract relationships among pairs of objects

Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected

Representation – edge list, adjacency matrix

Nodes vs Edges –  $m = O(n^2)$ , often less (sparse/dense)

BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer, tree, global analysis of nested loops

DFS – recursion/stack; all edges ancestor/descendant

Algorithms – connected components, shortest path, bipartiteness, topological sort, articulation points