CSE 421: Introduction to Algorithms

Complexity and Representative Problems

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Measuring efficiency: The RAM model

- RAM = Random Access Machine
- Time ≈ # of instructions executed in an ideal assembly language
  - each simple operation (+,*,-,=,if,call) takes one time step
  - each memory access takes one time step

Complexity analysis

- Problem size N
  - **Worst-case complexity**: **max** # steps algorithm takes on any input of size **N**
  - **Best-case complexity: min** # steps algorithm takes on any input of size **N**
  - **Average-case complexity**: **avg** # steps algorithm takes on inputs of size **N**

Stable Matching

- Problem size
  - $N=2n^2$ words
    - $2n$ people each with a preference list of length n
  - $2n^2 \log n$ bits
    - specifying an ordering for each preference list takes $n \log n$ bits

- Brute force algorithm
  - Try all $n!$ possible matchings

- Gale-Shapley Algorithm
  - n^2 iterations, each costing constant time
    - For each man an array listing the women in preference order
    - For each woman an array listing the preferences indexed by the names of the men
    - An array listing the current partner (if any) for each woman
    - An array listing the preference index of the last woman each man proposed to (if any)

Complexity

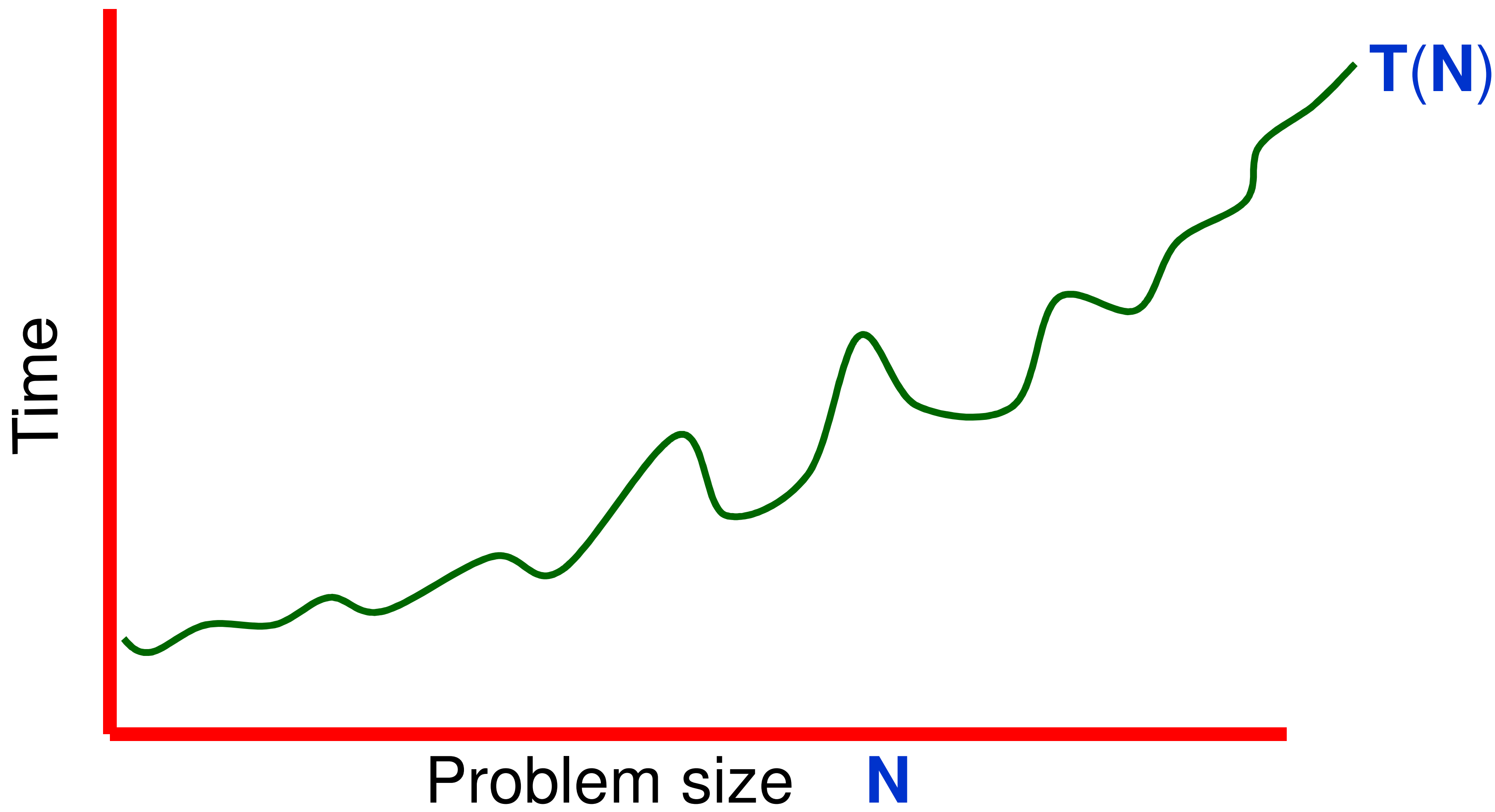
- The complexity of an algorithm associates a number **T(N)**, the worst/average-case/best time the algorithm takes, with each problem size **N**.

- Mathematically,
  - **T** is a function that maps positive integers giving problem size to positive real numbers giving number of steps.

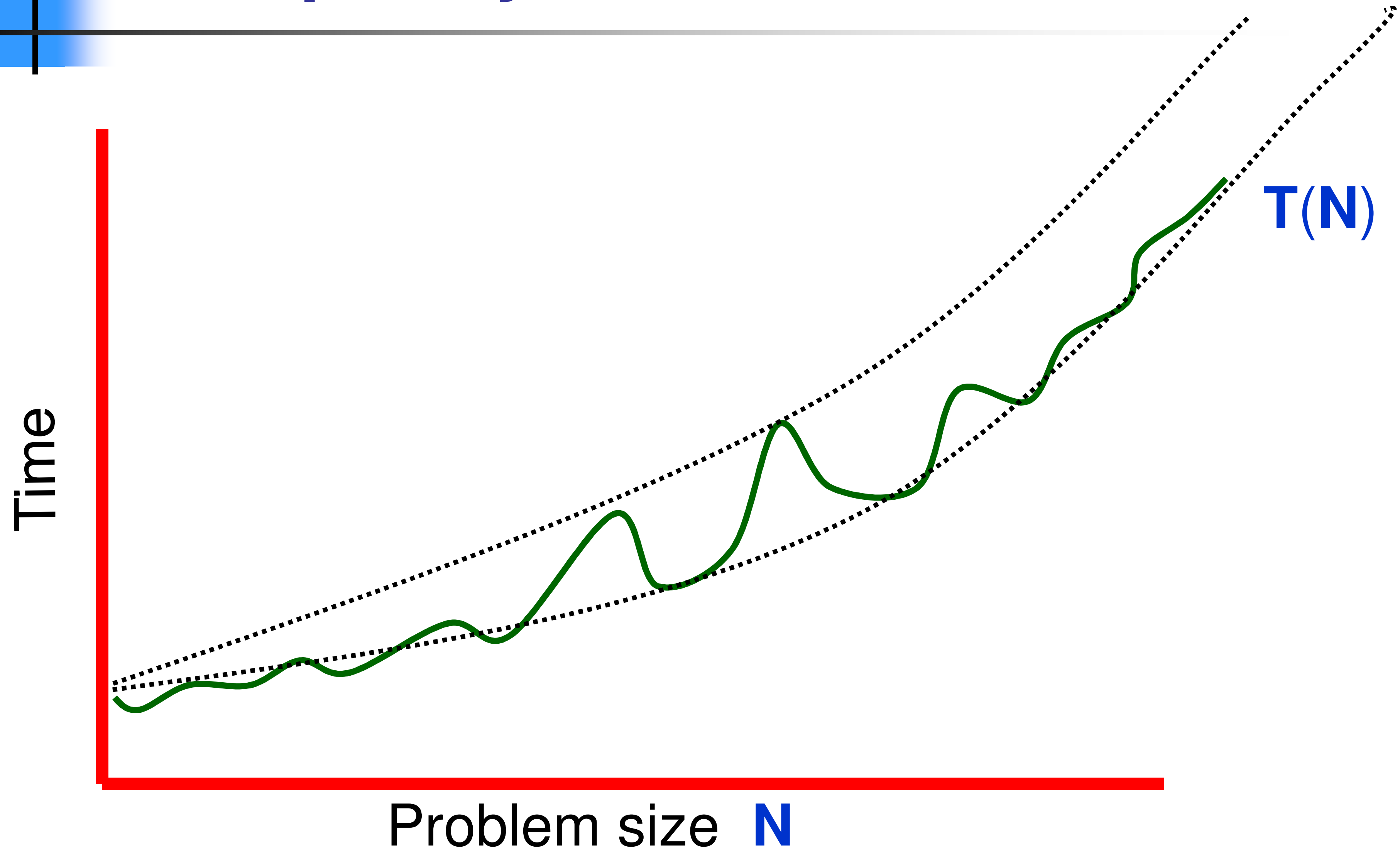
Efficient = Polynomial Time

- Polynomial time
  - Running time $T(N) \leq cN^k+d$ for some $c,d,k \geq 0$
- Why polynomial time?
  - If problem size grows by at most a constant factor then so does the running time
    - E.g. $T(2N) \leq c(2N)^k+d \leq 2^k(cN^k+d)$
    - Polynomial-time is exactly the set of running times that have this property
  - Typical running times are small degree polynomials, mostly less than N^3, at worst N^6, not N^{100}

Complexity



Complexity



O-notation etc

- Given two positive functions **f** and **g**
  - **f(N)** is **O(g(N))** iff there is a constant **c**>**0** so that **f(N)** is eventually always ≤ **c g(N)**
  - **f(N)** is **o(g(N))** iff the ratio **f**(**N**)/**g**(**N**) goes to **0** as **N** gets large
  - **f(N)** is **Ω(g(N))** iff there is a constant ε>**0** so that **f(N)** is ≥ ε **g(N)** for infinitely many values of **N**
  - **f(N)** is **Θ(g(N))** iff **f(N)** is **O(g(N))** and **f(N)** is **Ω(g(N))**

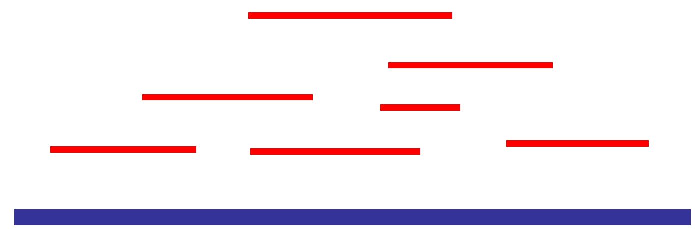
Note: The definition of Ω is the same as "**f**(**N**) is **not o**(**g**(**N**))"

Administrative

- Reading
  - Chapter 2, start Chapter 3 by Wednesday
- Homework 1
  - Posted on website.
  - Due Friday at 4:00 on Canvas.
- Office hours:
  - Me: MW 5:20-5:50, W 3:30-4:20
  - TAs: M 2:30, Tu 11:00, W 12:00, 2:30,
Th 10:30, 12:30, 2:00

5 Representative Problems

- Interval Scheduling
  - Single resource

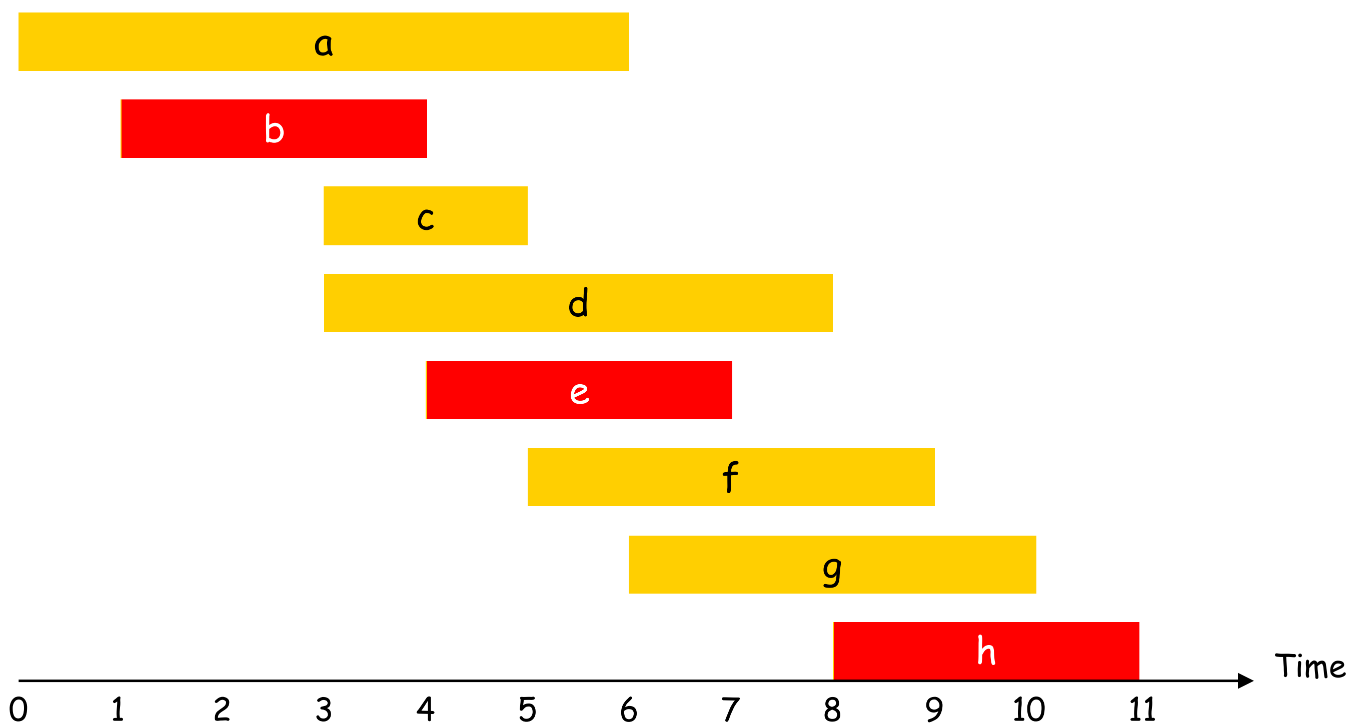


  - Reservation requests
    - Of form “Can I reserve it from start time **s** to finish time **f**?”
      - **s** < **f**

Interval Scheduling

- Input. Set of jobs with start times and finish times.
- Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



Interval scheduling

- Formally
  - Requests $1,2,\ldots,n$
    - request i has start time s_i and finish time $f_i > s_i$
  - Requests i and j are **compatible** iff either
    - request i is for a time entirely before request j
      - $f_i \leq s_j$
    - or, request j is for a time entirely before request i
      - $f_j \leq s_i$
  - Set A of requests is **compatible** iff every pair of requests $i,j \in A$, $i \neq j$ is compatible
  - Goal: Find maximum size subset A of compatible requests

Interval Scheduling

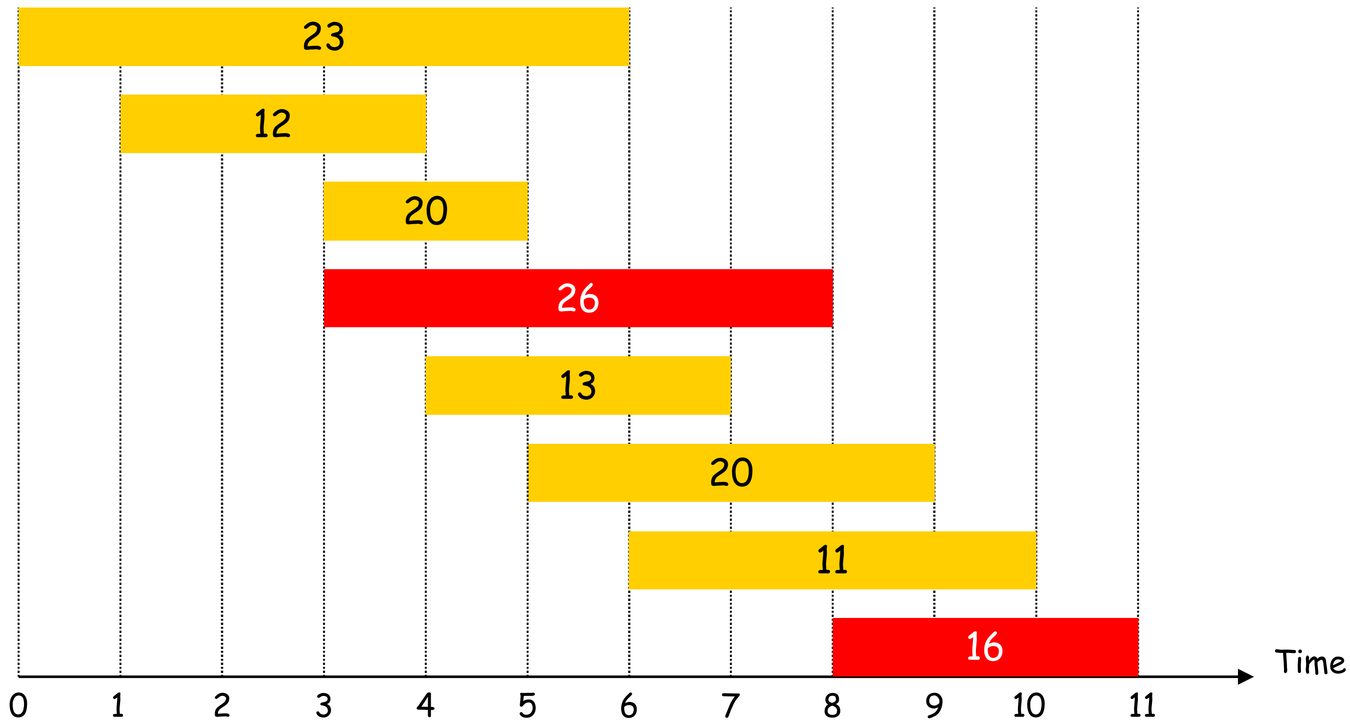
- We'll see that an optimal solution can be found using a "greedy algorithm"
  - Myopic kind of algorithm that seems to have no look-ahead
  - These algorithms only work when the problem has a special kind of structure
  - When they do work they are typically very efficient

Weighted Interval Scheduling

- Same problem as interval scheduling except that each request **i** also has an associated value or weight $\mathbf{w_i}$
  - $\mathbf{w_i}$ might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used

Weighted Interval Scheduling

- Input. Set of jobs with start times, finish times, and weights.
- Goal. Find maximum weight subset of mutually compatible jobs.



Weighted Interval Scheduling

- Ordinary interval scheduling is a special case of this problem
  - Take all $w_i = 1$
- Problem is quite different though
  - E.g. one weight might dwarf all others
- “Greedy algorithms” don’t work
- Solution: “Dynamic Programming”
  - builds up optimal solutions from smaller problems using a compact table to store them

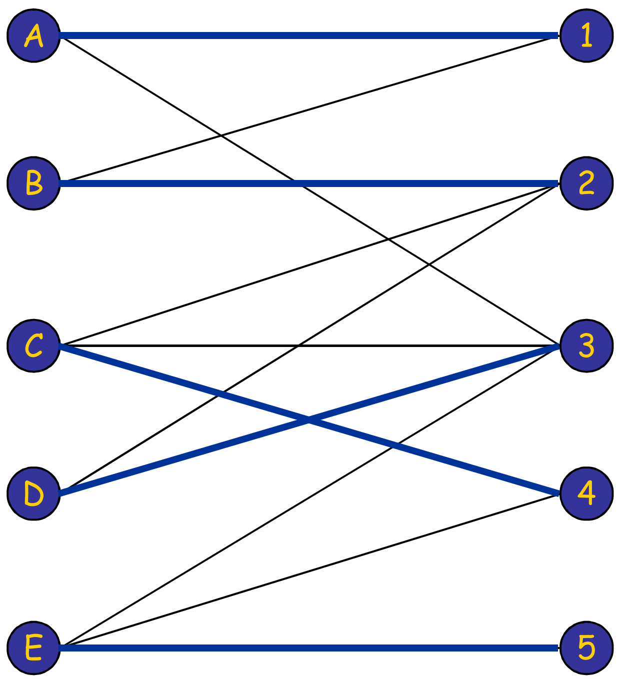
Bipartite Matching

- A graph $G=(V,E)$ is **bipartite** iff
  - V consists of two disjoint pieces X and Y such that every edge e in E is of the form (x,y) where $x \in X$ and $y \in Y$
  - Similar to stable matching situation but in that case all possible edges were present

- $M \subseteq E$ is a matching in G iff no two edges in M share a vertex
  - Goal: Find a matching M in G of maximum possible size

Bipartite Matching

- Input. Bipartite graph.
- Goal. Find maximum cardinality matching.



Bipartite Matching

- Models assignment problems
  - **X** represents jobs, **Y** represents machines
  - **X** represents professors, **Y** represents courses
- If **|X|=|Y|=n**
  - **G** has perfect matching iff maximum matching has size **n**
- **Solution:** polynomial-time algorithm using "augmentation" technique
  - also used for solving more general class of network flow problems

Independent Set

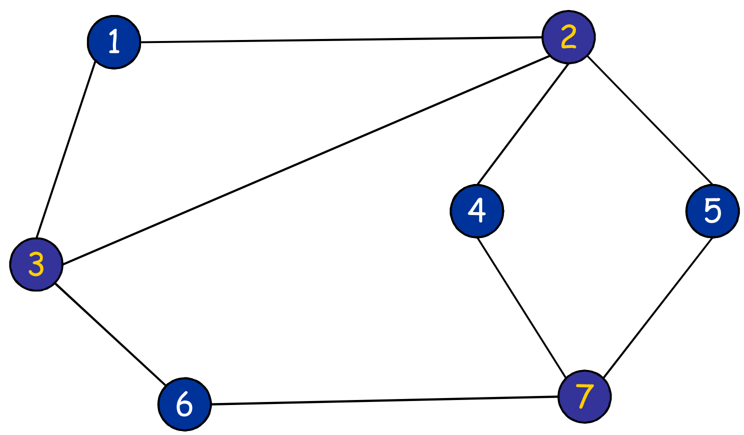
- Given a graph $G=(V,E)$
  - A set $I \subseteq V$ is **independent** iff no two nodes in I are joined by an edge

- Goal: Find an independent subset I in G of maximum possible size

- Models conflicts and mutual exclusion

Independent Set

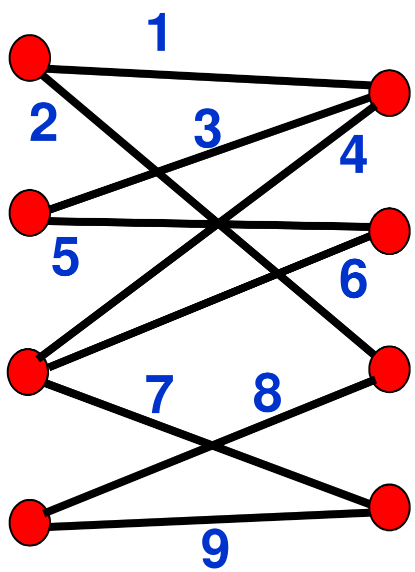
- Input. Graph.
- Goal. Find maximum cardinality independent set.



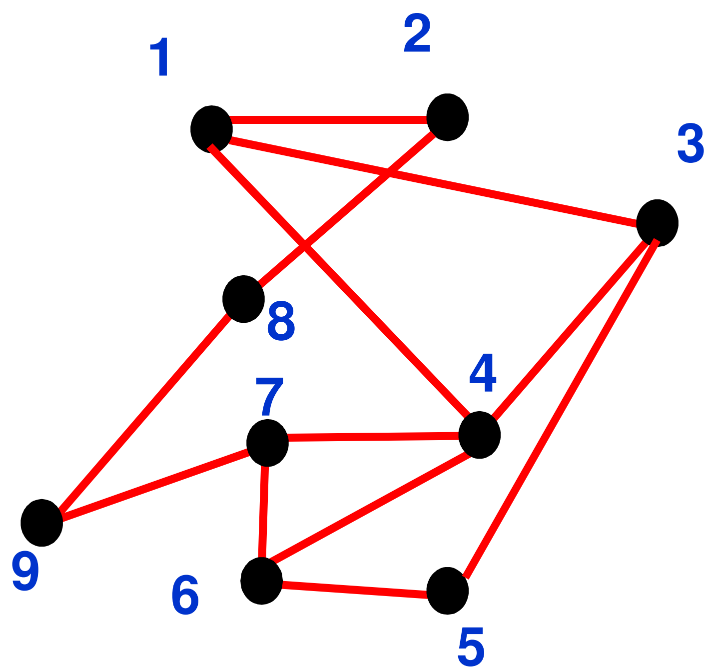
Independent Set

- Generalizes
  - Interval Scheduling
    - Vertices in the graph are the requests
    - Vertices are joined by an edge if they are **not** compatible
  - Bipartite Matching
    - Given bipartite graph **G**=(**V**,**E**) create new graph **G'**=(**V'**,**E'**) where
      - **V'**=**E**
      - Two elements of **V'** (which are edges in **G**) are joined if they share an endpoint in **G**

Bipartite Matching vs Independent Set



G=(U∪V,E)



G'=(V',E')

Independent Set

- No polynomial-time algorithm is known
  - But to convince someone that there was a large independent set all you'd need to do is show it to them
    - they can easily convince themselves that the set is large enough and independent
  - Convincing someone that there isn't one seems much harder
- We will show that Independent Set is NP-complete
  - Class of all the hardest problems that have the property above

Competitive Facility Location

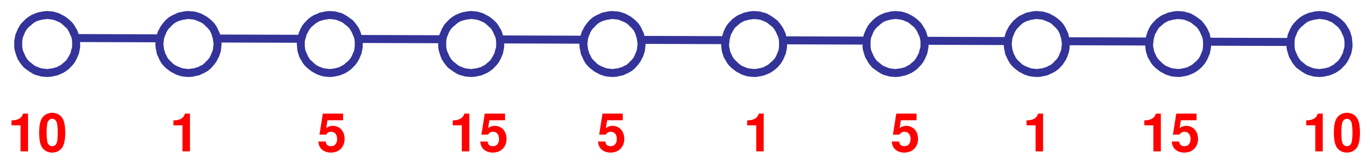
- Two players competing for market share in a geographic area
  - e.g. McDonald's, Burger King
- Rules:
  - Region is divided into **n** zones, **1,…,n**
  - Each zone **i** has a value $\mathbf{b_i}$
    - Revenue derived from opening franchise in that zone
  - No adjacent zones may contain a franchise
    - i.e., zoning regulations limit density
  - Players alternate opening franchises
- Find: Given a target total value **B** is there a strategy for the second player that always achieves ≥ **B**?

Competitive Facility Location

- Model geography by
  - A graph $G=(V,E)$ where
    - V is the set $\{1,\ldots,n\}$ of zones
    - E is the set of pairs (i,j) such that i and j are adjacent zones

- Observe:
  - The set of zones with franchises will form an independent set in G

Competitive Facility Location



Target **B = 20** achievable ?

What about **B = 25** ?

Competitive Facility Location

- Checking that a strategy is good seems hard
  - You'd have to worry about all possible responses at each round!
    - a giant search tree of possibilities

- Problem is PSPACE-complete
  - Likely strictly harder than NP-complete problems
  - PSPACE-complete problems include
    - Game-playing problems such as **$n \times n$** chess and checkers
    - Logic problems such as whether quantified boolean expressions are always true
    - Verification problems for finite automata

Five Representative Problems

- Variations on a theme: independent set.
- Interval scheduling: **O**(**n** log **n)** greedy algorithm.
- Weighted interval scheduling: **O**(**n** log **n)** dynamic programming algorithm.
- Bipartite matching: **O**(**n**^{k}) max-flow based algorithm.
- Independent set: NP-complete.
- Competitive facility location: PSPACE-complete.