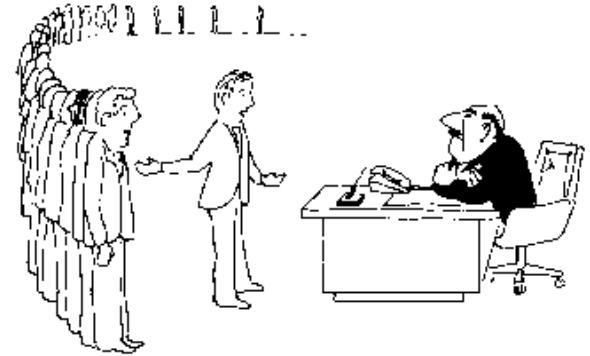


CSE 421

Algorithms

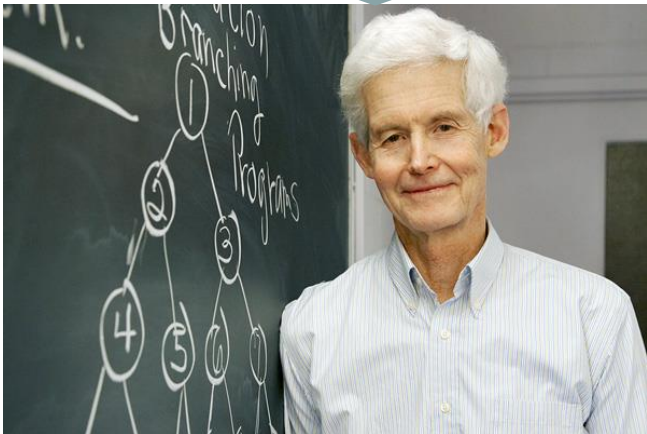
Richard Anderson
Lecture 26
NP-Completeness



I can't find an efficient algorithm, but neither can all these famous people.

NP Completeness: The story so far

Circuit Satisfiability is
NP-Complete



Background

- P: Class of problems that can be solved in polynomial time
- NP: Class of problems that can be solved in non-deterministic polynomial time
- Y is Polynomial Time Reducible to X
 - Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
 - Notation: $Y <_p X$
- Suppose $Y <_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time
- A problem X is NP-complete if
 - X is in NP
 - For every Y in NP, $Y <_p X$
- If X is NP-Complete, Z is in NP and $X <_p Z$
 - Then Z is NP-Complete

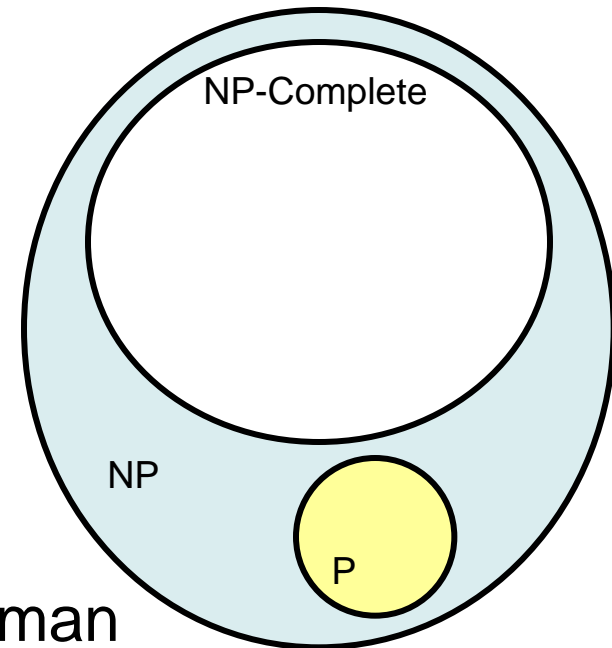
Today

There are a whole bunch of other important problems which are NP-Complete



Populating the NP-Completeness Universe

- Circuit Sat \leq_p 3-SAT
- 3-SAT \leq_p Independent Set
- 3-SAT \leq_p Vertex Cover
- Independent Set \leq_p Clique
- 3-SAT \leq_p Hamiltonian Circuit
- Hamiltonian Circuit \leq_p Traveling Salesman
- 3-SAT \leq_p Integer Linear Programming
- 3-SAT \leq_p Graph Coloring
- 3-SAT \leq_p Subset Sum
- Subset Sum \leq_p Scheduling with Release times and deadlines



Satisfiability

Literal: A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

Clause: A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

SAT: Given CNF formula Φ , does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

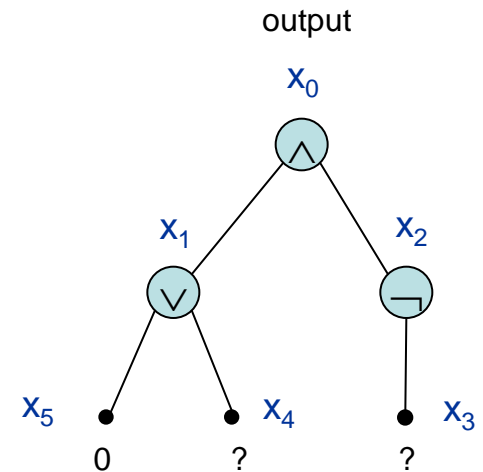
Yes: $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}.$

3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

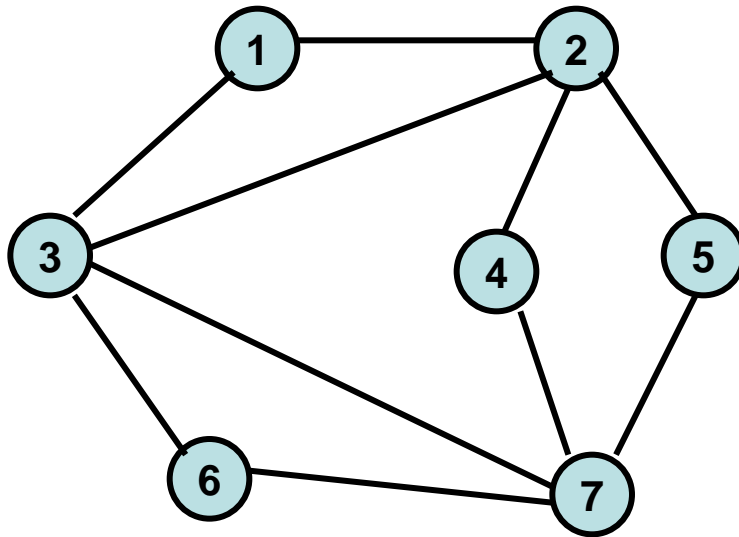
Pf. Suffices to show that $\text{CIRCUIT-SAT} \leq_p \text{3-SAT}$ since 3-SAT is in NP.

- Let K be any circuit.
- Create a 3-SAT variable x_i for each circuit element i .
- Make circuit compute correct values at each node:
 - $x_2 = \neg x_3 \Rightarrow$ add 2 clauses: $x_2 \vee x_3, \overline{x_2} \vee \overline{x_3}$
 - $x_1 = x_4 \vee x_5 \Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}, x_1 \vee \overline{x_5}, x_1 \vee x_4 \vee x_5$
 - $x_0 = x_1 \wedge x_2 \Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1, \overline{x_0} \vee x_2, x_0 \vee \overline{x_1} \vee \overline{x_2}$
- Hard-coded input values and output value.
 - $x_5 = 0 \Rightarrow$ add 1 clause: $\overline{x_5}$
 - $x_0 = 1 \Rightarrow$ add 1 clause: x_0
- Final step: turn clauses of length < 3 into clauses of length exactly 3. ▀



Independent Set

- Independent Set
 - Graph $G = (V, E)$, a subset S of the vertices is independent if there are no edges between vertices in S



3 Satisfiability Reduces to Independent Set

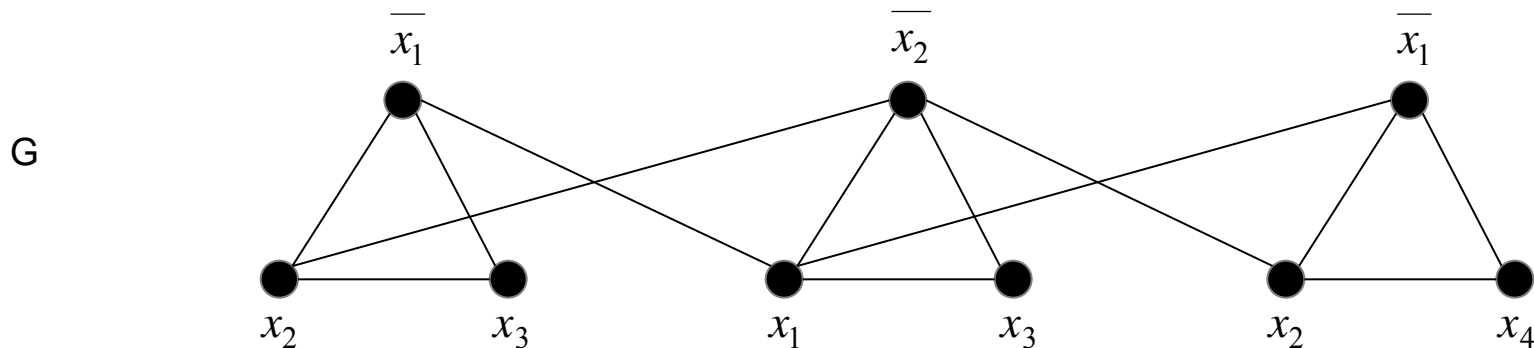


Claim. $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$



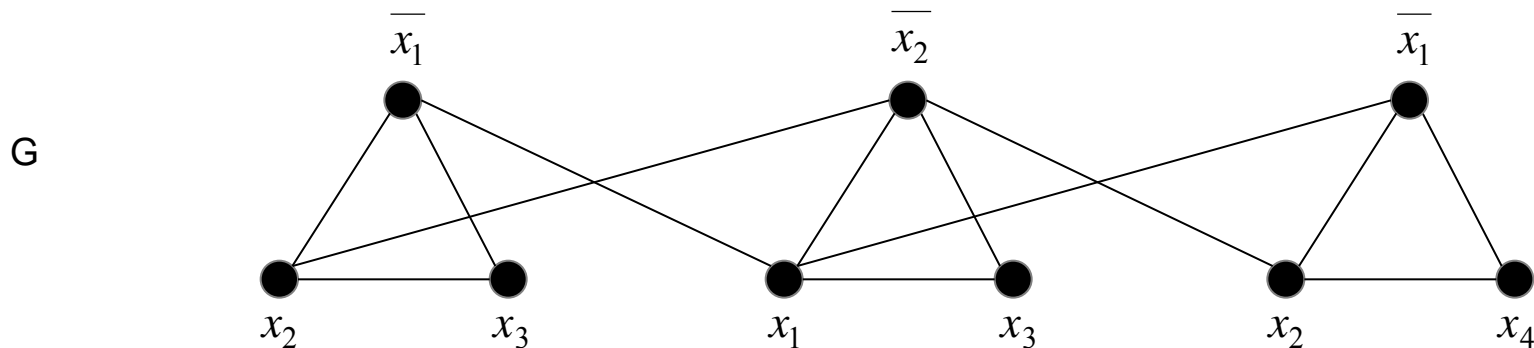
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Pf. \Rightarrow Let S be independent set of size k .

- S must contain exactly one vertex in each triangle.
- Set these literals to true. \leftarrow and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf \Leftarrow Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k . \blacksquare

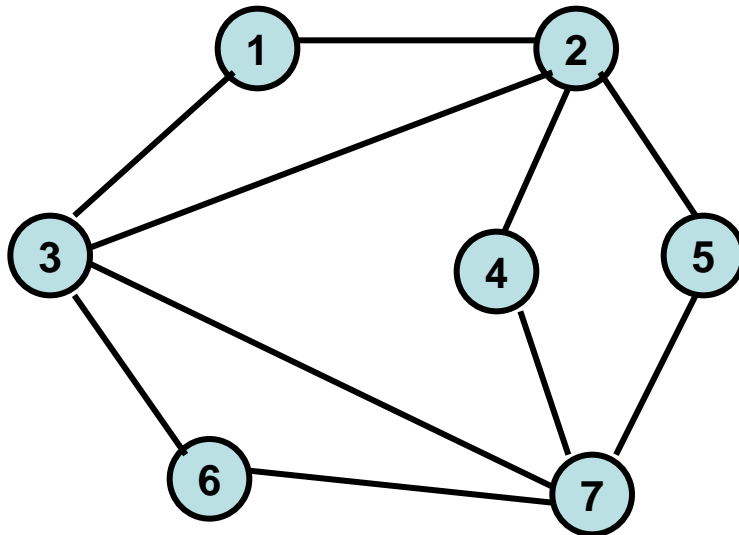


$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

Vertex Cover

- Vertex Cover
 - Graph $G = (V, E)$, a subset S of the vertices is a vertex cover if every edge in E has at least one endpoint in S

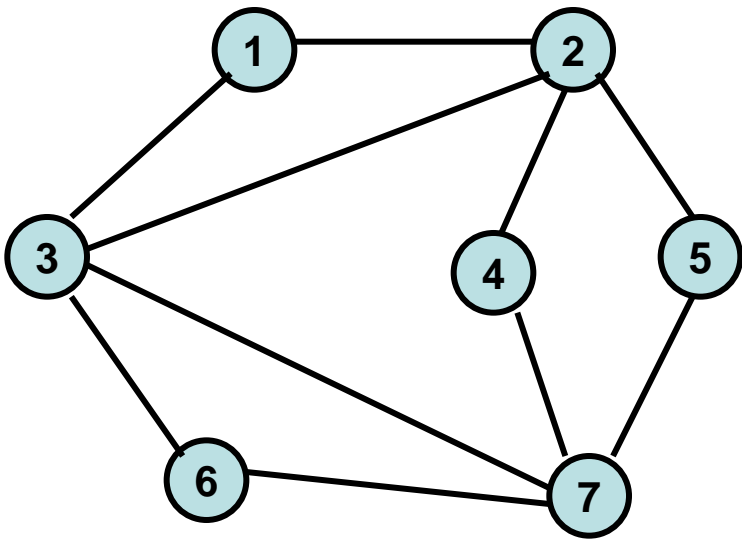


$$IS \leq_P VC$$

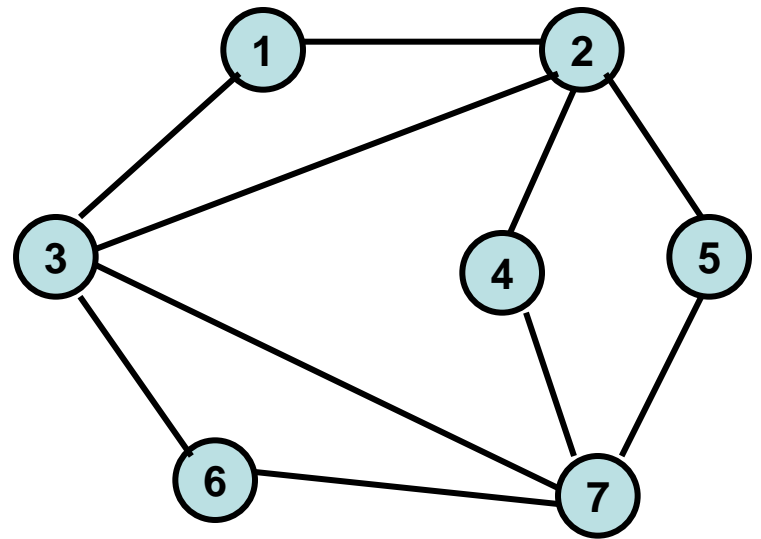
- Lemma: A set S is independent iff $V-S$ is a vertex cover
- To reduce IS to VC , we show that we can determine if a graph has an independent set of size K by testing for a Vertex cover of size $n - K$

$$IS \leq_P VC$$

Find a maximum independent set S

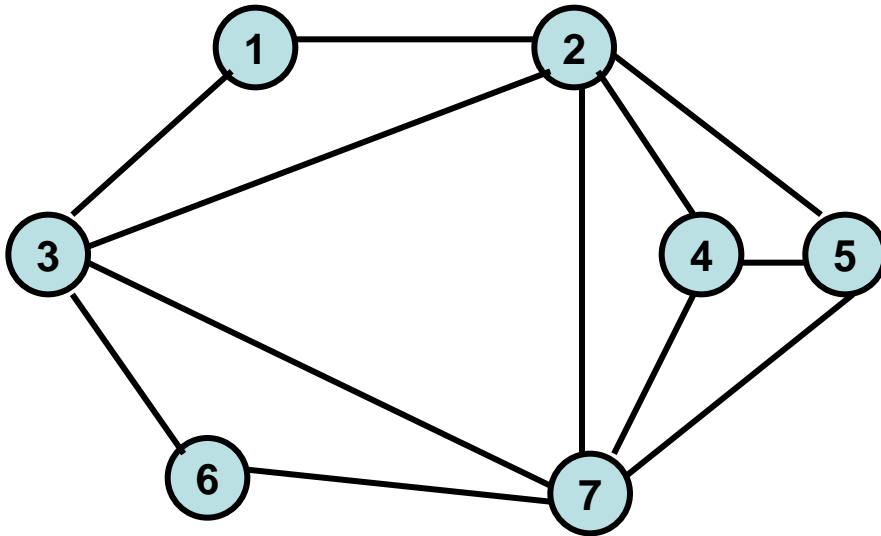


Show that $V-S$ is a vertex cover



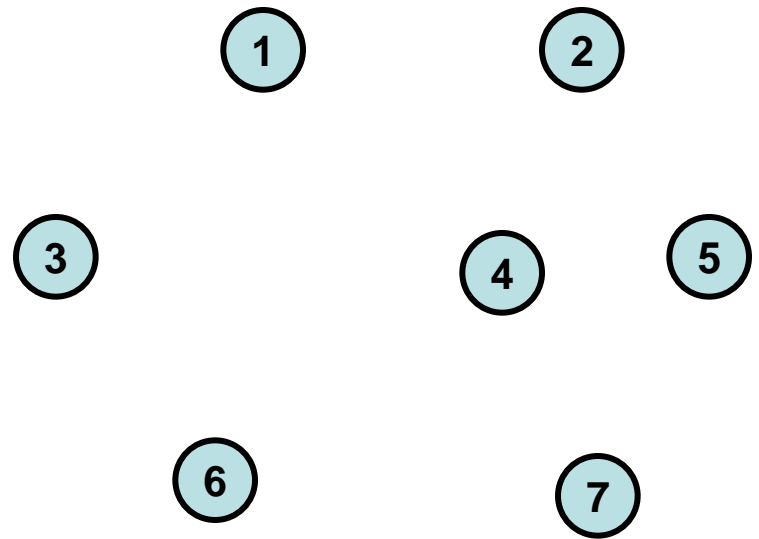
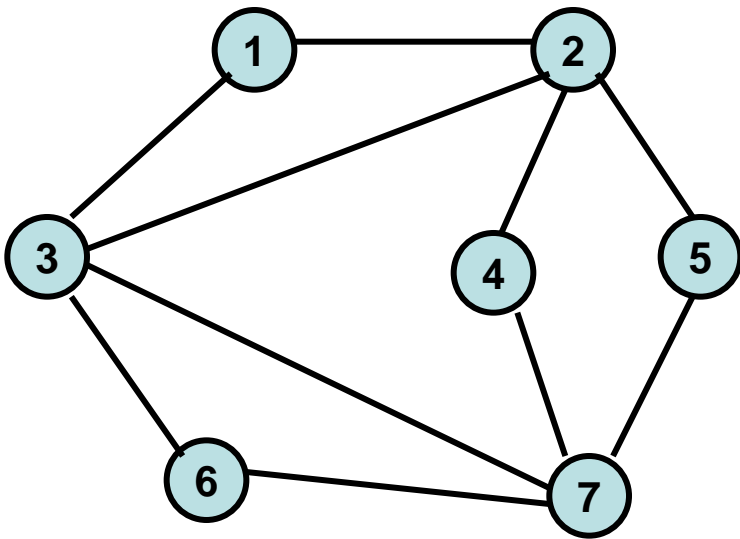
Clique

- Clique
 - Graph $G = (V, E)$, a subset S of the vertices is a clique if there is an edge between every pair of vertices in S



Complement of a Graph

- Defn: $G'=(V,E')$ is the complement of $G=(V,E)$ if (u,v) is in E' iff (u,v) is not in E

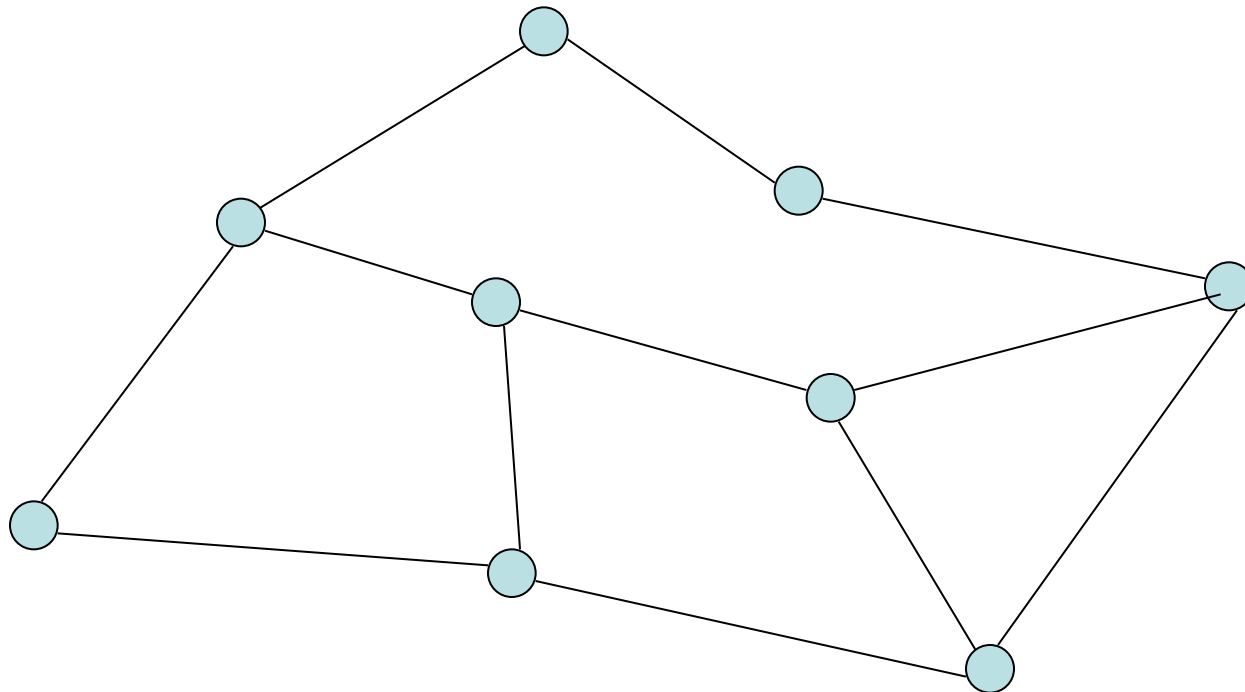


IS \leq_P Clique

- Lemma: S is Independent in G iff S is a Clique in the complement of G
- To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size K iff the original graph has an independent set of size K

Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph

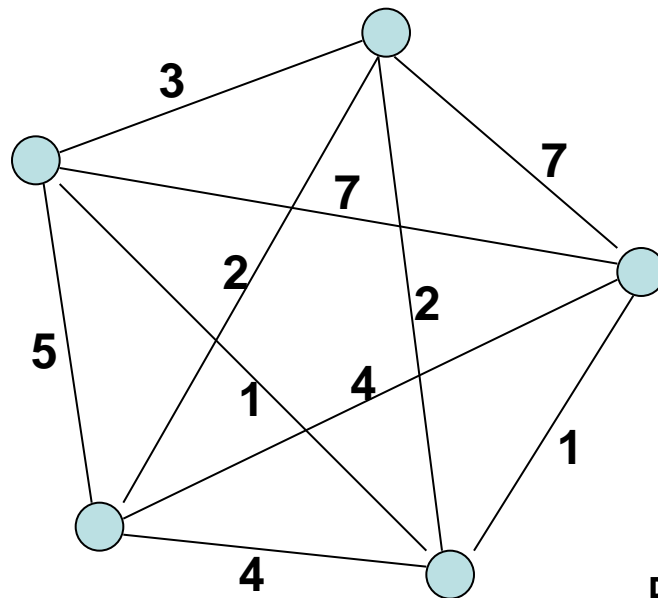


Thm: Hamiltonian Circuit is NP Complete

- Reduction from 3-SAT

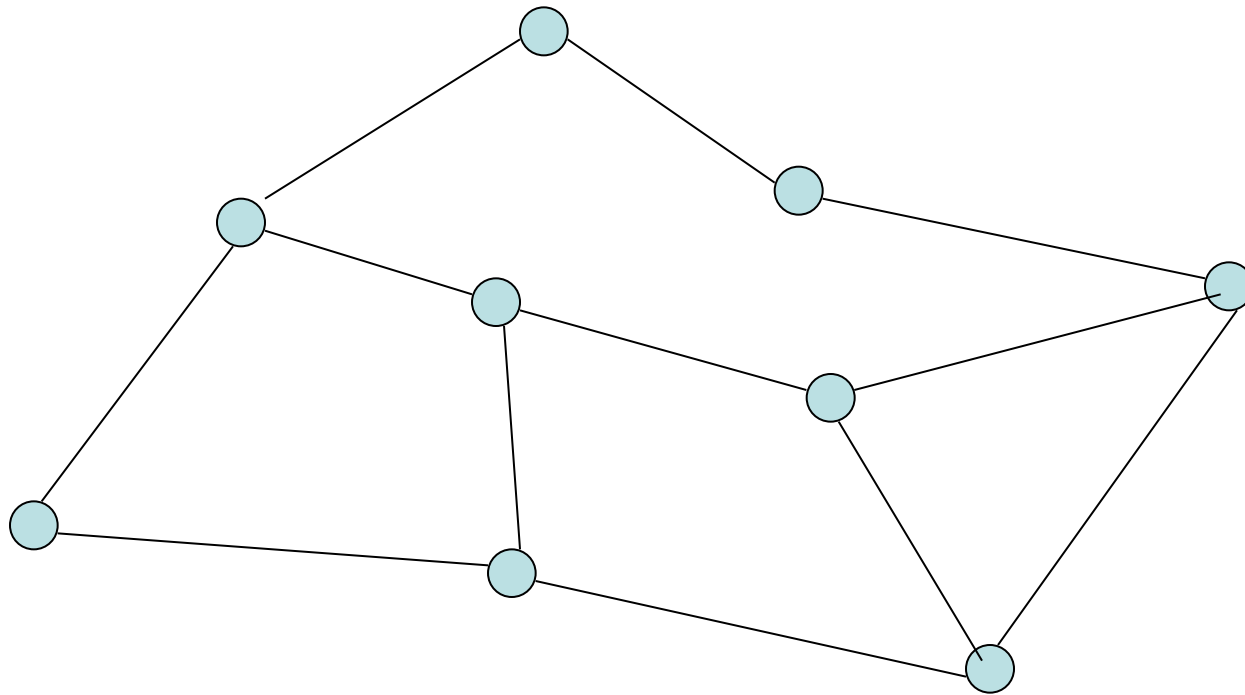
Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)



Find the minimum cost tour

Thm: $HC \leq_p TSP$



Graph Coloring

- NP-Complete
 - Graph K-coloring
 - Graph 3-coloring
- Polynomial
 - Graph 2-Coloring

