CSE 421
Algorithms

Richard Anderson

Lecture 19

Memory Efficient Dynamic Programming

Announcements

- Guest lecturers
  - Wednesday, Nov 16, Shortest Paths
  - Friday, Nov 18, Network Flow
  - Monday, Nov 21, Network Flow

Longest Common Subsequence

- $C=c_1\ldots c_g$ is a subsequence of $A=a_1\ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

LCS(BARTHOLEMEWSIMPSON, KRUSTYTHECLOWN)
= RTHOWN

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

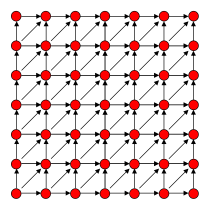
- Opt[j, k] is the length of LCS($a_1a_2\ldots a_j$, $b_1b_2\ldots b_k$)

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If a_j != b_k, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Dynamic Programming Computation

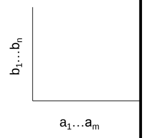


Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
       Opt[ i, j ] := Opt[ i-1, j ];
    else
       Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
    for j := 1 to n
        if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
        else if Opt[i-1, j] >= Opt[i, j-1]
            {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
        else      {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



Reconstructing Path from Distances

```
LCS Arguments
 21103132110203321212000032130 2
 100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

How good is this algorithm?

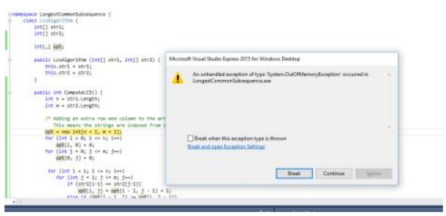
- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Implementation 1

```
public int ComputeLCS() {
          int n = str1.Length;
          int m = str2.Length;

          int[,] opt = new int[n + 1, m + 1];
          for (int i = 0; i <= n; i++)
              opt[i, 0] = 0;
          for (int j = 0; j <= m; j++)
              opt[0, j] = 0;

           for (int i = 1; i <= n; i++)
              for (int j = 1; j <= m; j++)
                  if (str1[i-1] == str2[j-1])
                      opt[i, j] = opt[i - 1, j - 1] + 1;
                  else if (opt[i - 1, j] >= opt[i, j - 1])
                      opt[i, j] = opt[i - 1, j];
                  else
                      opt[i, j] = opt[i, j - 1];

          return opt[n,m];
      }
```

N = 17000

Runtime should be about 5 seconds*



* Personal PC, 3 years old

Manufacturer:	Dell
Model:	Optiplex 990
Processor:	Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM):	8.00 GB (7.88 GB usable)
System type:	64-bit Operating System, x64-based processor

Implementation 2

```
public int SpaceEfficientLCS() {
        int n = str1.Length;
        int m = str2.Length;
        int[] prevRow = new int[m + 1];
        int[] currRow = new int[m + 1];

        for (int j = 0; j <= m; j++)
            prevRow[j] = 0;

        for (int i = 1; i <= n; i++) {
            currRow[0] = 0;
            for (int j = 1; j <= m; j++) {
                if (str1[i - 1] == str2[j - 1])
                    currRow[j] = prevRow[j - 1] + 1;
                else if (prevRow[j] >= currRow[j - 1])
                    currRow[j] = prevRow[j];
                else
                    currRow[j] = currRow[j - 1];
            }
            for (int j = 1; j <= m; j++)
                prevRow[j] = currRow[j];
        }

        return currRow[m];
    }
```

N = 300000

N: 10000 Base 2 Length: 8096 Gamma: 0.8096 Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231 Gamma: 0.81155 Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317 Gamma: 0.8105667 Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510 Gamma: 0.81275 Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563 Gamma: 0.81126 Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700 Gamma: 0.8116667 Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824 Gamma: 0.8117715 Runtime:00:01:33.36

N: 300000 Base 2 Length: 243605 Gamma: 0.8120167 Runtime:00:28:07.32

Observations about the Algorithm

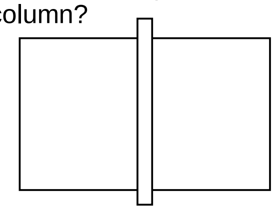
- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

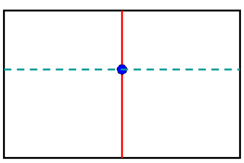
- From the left, compute $LCS(a_1…a_{m/2},b_1…b_j)$
- From the right, compute $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$
- Add values for corresponding j's
- Note – this is space efficient

Divide and Conquer

- $A = a_1,…,a_m$ $B = b_1,…,b_n$
- Find j such that
  - $LCS(a_1…a_{m/2}, b_1…b_j)$ and
  - $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$ yield optimal solution
- Recurse

Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$



Prove by induction that
T(m,n) <= 2cmn

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes