CSE 421
Algorithms

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Lecture 18

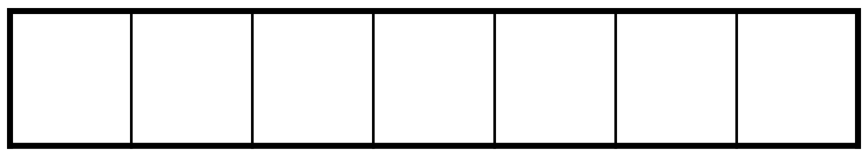
Dynamic Programming

Announcements

- Nov 11, No class (holiday)
- HW 7 is available
  - 5 dynamic programming problems

One dimensional dynamic programming: Interval scheduling

$$Opt[\,j\,] = \max\,(Opt[\,j - 1], w_j + Opt[\,p[\,j\,]\,])$$



Two dimensional dynamic programming

K-segment linear approximation

$Opt_k[\,j\,] = \min_i \{ Opt_{k-1}[\,i\,] + E_{i,j} \}$ for $0 < i < j$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Two dimensional dynamic programming

Subset sum and knapsack

$\text{Opt}[\,j, K] = \max(\text{Opt}[\,j - 1, K], \text{Opt}[\,j - 1, K - w_j] + w_j)$

$\text{Opt}[\,j, K] = \max(\text{Opt}[\,j - 1, K], \text{Opt}[\,j - 1, K - w_j] + v_j)$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Alternate approach for Subset Sum

- Alternate formulation of Subset Sum dynamic programming algorithm
- Sum[i, K] = true if there is a subset of $\{w_1,\ldots w_k\}$ that sums to exactly K, false otherwise
- Sum [i, K] = Sum [i -1, K] **OR** Sum[i - 1, K - w_i]
- Sum [0, 0] = 1; Sum[i, 0] = 0 for i != 0

- To count the number of solutions
  - Count [i, K] = Count [i -1, K] **+** Count[i - 1, K - w_i]
  - Count [0, 0] = 1; Count[i, 0] = 0 for i != 0

- To allow for negative numbers, we need to fill in the array between K_{min} and K_{max}

Dynamic Programming Examples

- Examples
  - Optimal Billboard Placement
    - Text, Solved Exercise, Pg 307
  - Linebreaking with hyphenation
    - Compare with HW problem 6, Pg 317
  - String approximation
    - Text, Solved Exercise, Page 309

Billboard Placement

- Maximize income in placing billboards
  - $b_i = (p_i, v_i)$, v_i: value of placing billboard at position p_i
- Constraint:
  - At most one billboard every five miles
- Example
  - {(6,5), (8,6), (12, 5), (14, 1)}

Design a Dynamic Programming Algorithm for Billboard Placement

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Input $b_1, \ldots, b_n$, where $b_i = (p_i, v_i)$, position and value of billboard i

Solution

```
j = 0;              // j is five miles behind the current position
                     // the last valid location for a billboard, if one placed at P[k]
for k := 1 to n
        while (P[ j ] < P[ k ] – 5)
                j := j + 1;
        j := j – 1;
        Opt[ k]  = Max(Opt[ k-1] , V[ k ] + Opt[ j ]);
```

String approximation

- Given a string S, and a library of strings B = {b_1, …b_m}, construct an approximation of the string S by using copies of strings in B.

B = {abab, bbbaaa, ccbb, ccaacc}

S = abaccbbbaabbccbbccaabab

Formal Model

- Strings from B assigned to non-overlapping positions of S
- Strings from B may be used multiple times
- Cost of δ for unmatched character in S
- Cost of γ for mismatched character in S
  - MisMatch(i, j) – number of mismatched characters of b_j, when aligned starting with position i in s.

Design a Dynamic Programming Algorithm for String Approximation

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Target string S = $s_1s_2\ldots s_n$
Library of strings B = $\{b_{1,}\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Opt[k] = fun(Opt[0],…,Opt[k-1])

- How is the solution determined from sub problems?

Target string $S = s_1 s_2 \ldots s_n$
Library of strings $B = \{b_{1,} \ldots, b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Solution

```
for i := 1 to n
        Opt[k] = Opt[k-1] + δ;
        for j := 1 to |B|
                p = i – len(b_j);
                Opt[k] = min(Opt[k],  Opt[p-1] + γ MisMatch(p, j));
```

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

ocurranec

occurrence

attacggct

tacgacca

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- Opt[j, k] is the length of $LCS(a_1a_2\ldots a_j, b_1b_2\ldots b_k)$

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

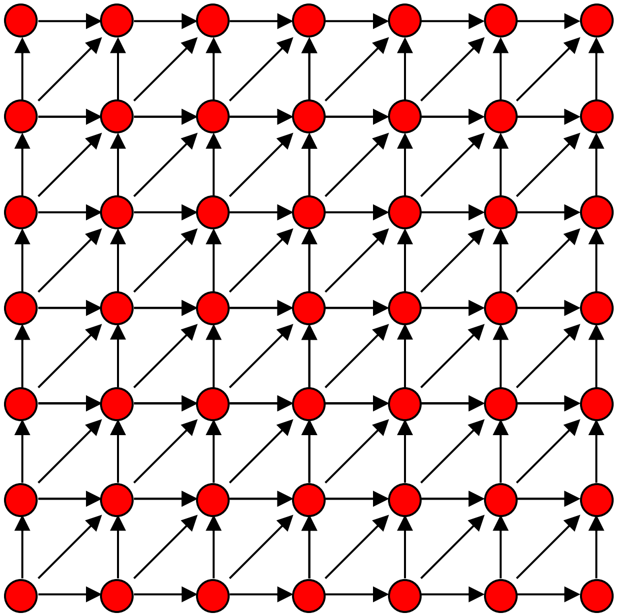
Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

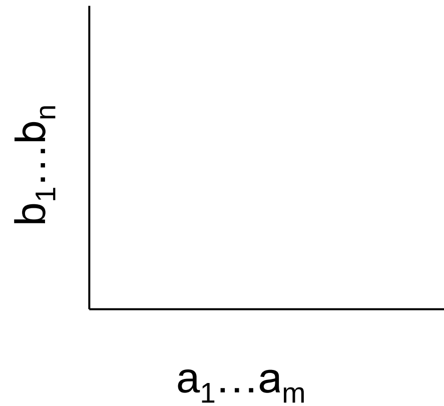
Let a_j = x and b_k = y
Express as minimization

Dynamic Programming Computation



Code to compute Opt[j,k]

Storing the path information




```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else       {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.