CSE 421
Algorithms

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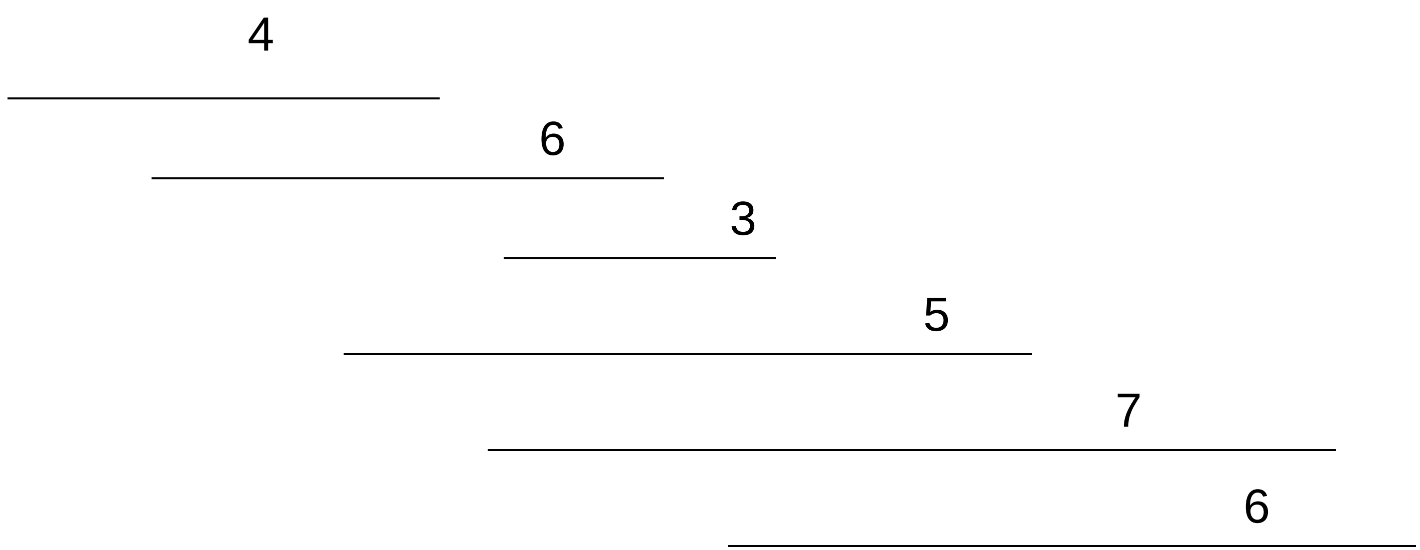
Lecture 16

Dynamic Programming

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Optimality Condition

- Opt[j] is the maximum weight independent set of intervals $I_1, I_2, \ldots, I_j$
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

MaxValue(j) =

  if j = 0 return 0

  else

    return max(MaxValue(j-1),

      w_j + MaxValue(p[j]))

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

```
MaxValue(j) =
    if j = 0 return 0;
    else if M[ j ] != -1 return M[ j ];
    else
        M[ j ] = max(MaxValue(j-1), w_j + MaxValue(p[ j ]));
        return M[ j ];
```

Iterative Algorithm

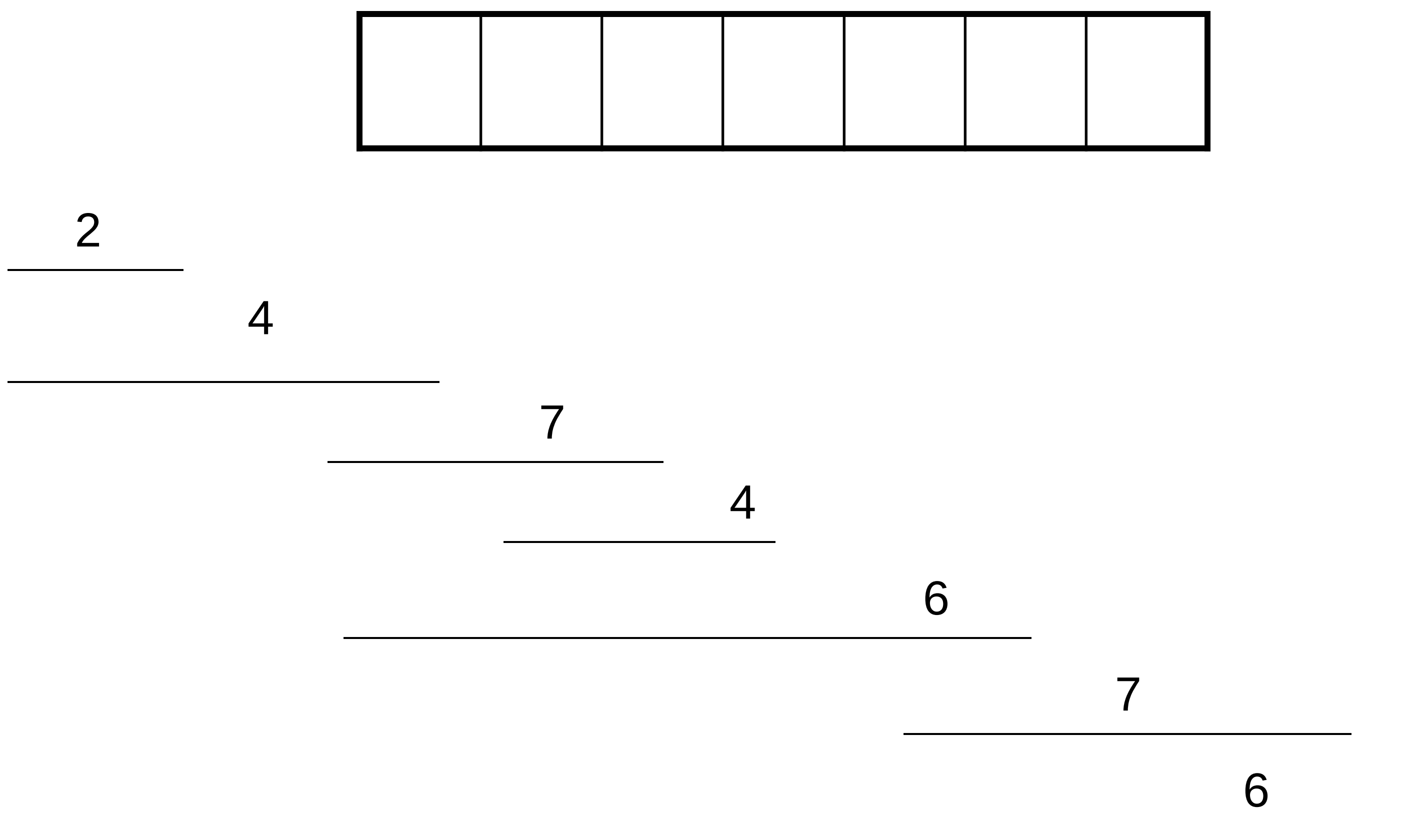
Express the MaxValue algorithm as an iterative algorithm

MaxValue {

}

Fill in the array with the Opt values

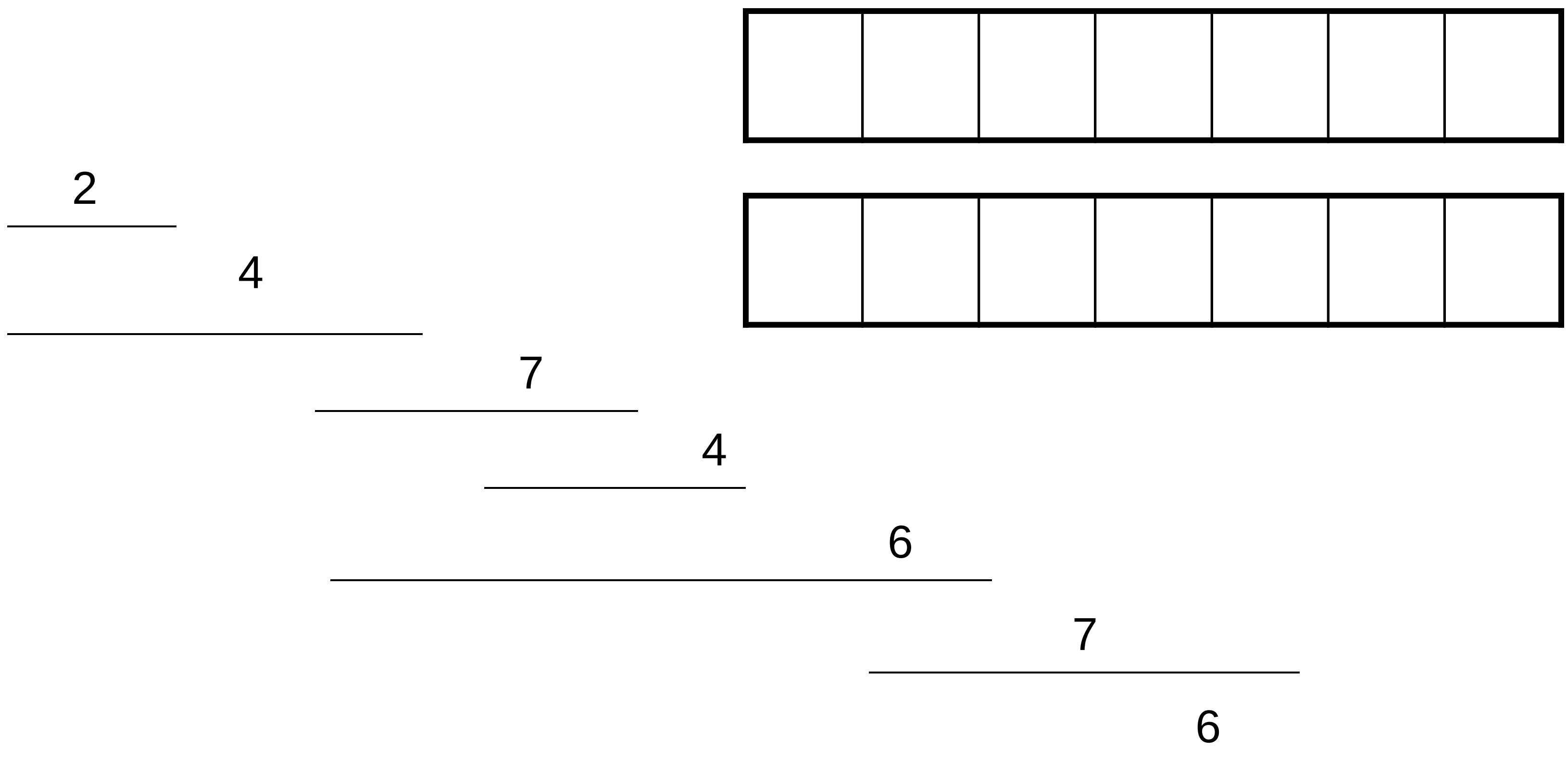
$$\mathrm{Opt}[\,j\,] = \max\left(\mathrm{Opt}[\,j-1],\ w_j + \mathrm{Opt}[\,p[\,j\,]\,]\right)$$



Computing the solution

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])

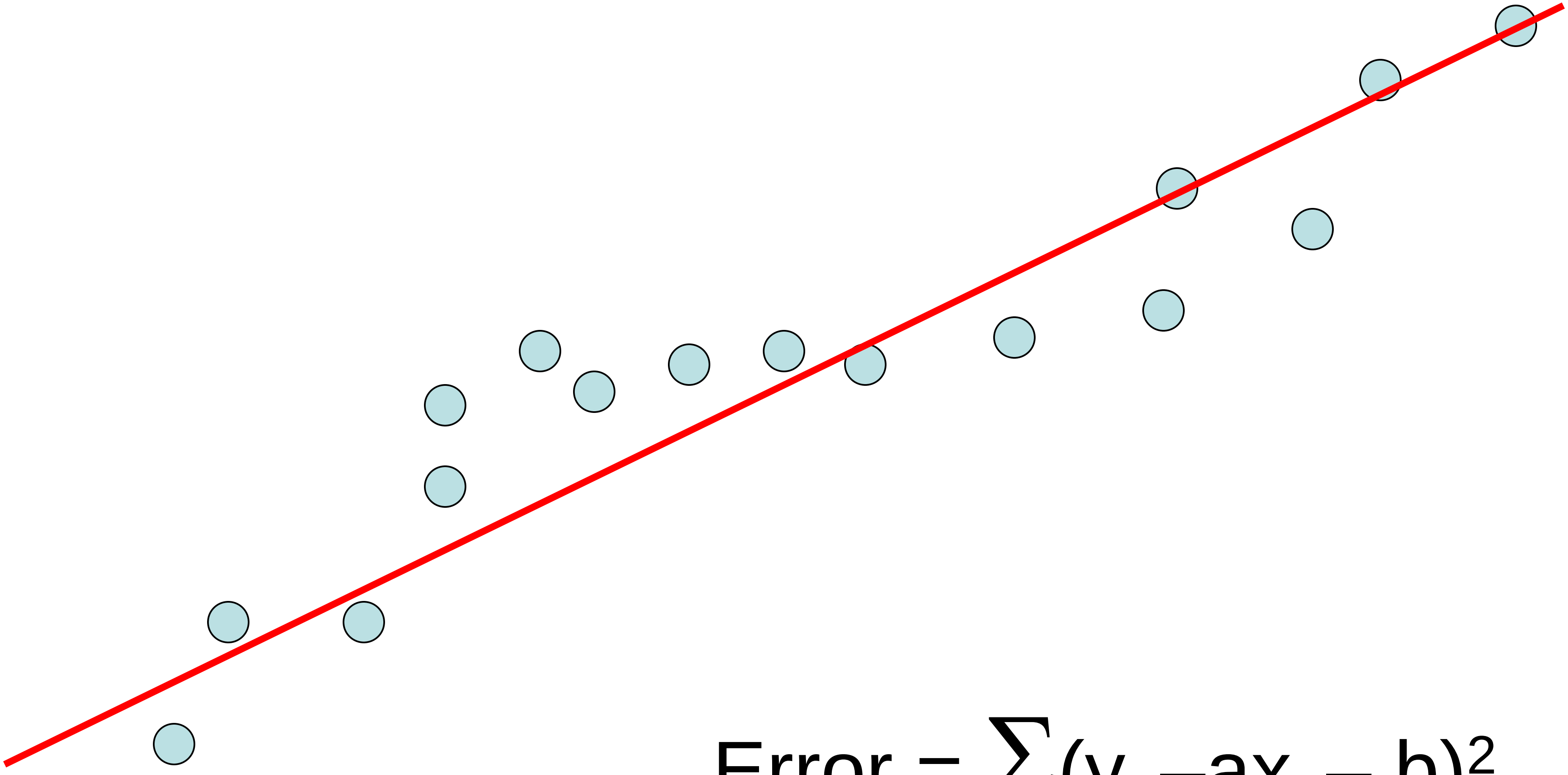
Record which case is used in Opt computation



Dynamic Programming

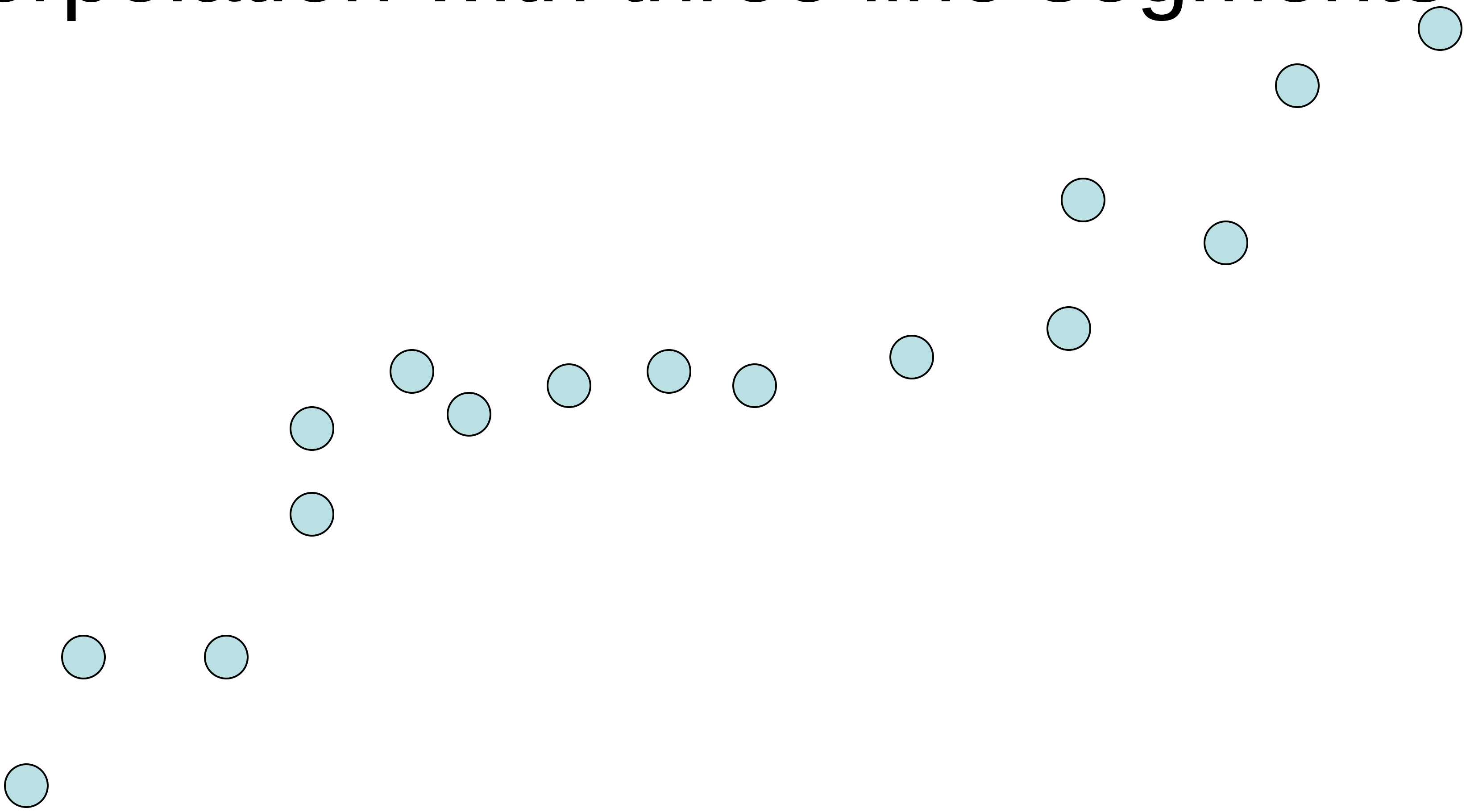
- The most important algorithmic technique covered in CSE 421
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Optimal linear interpolation

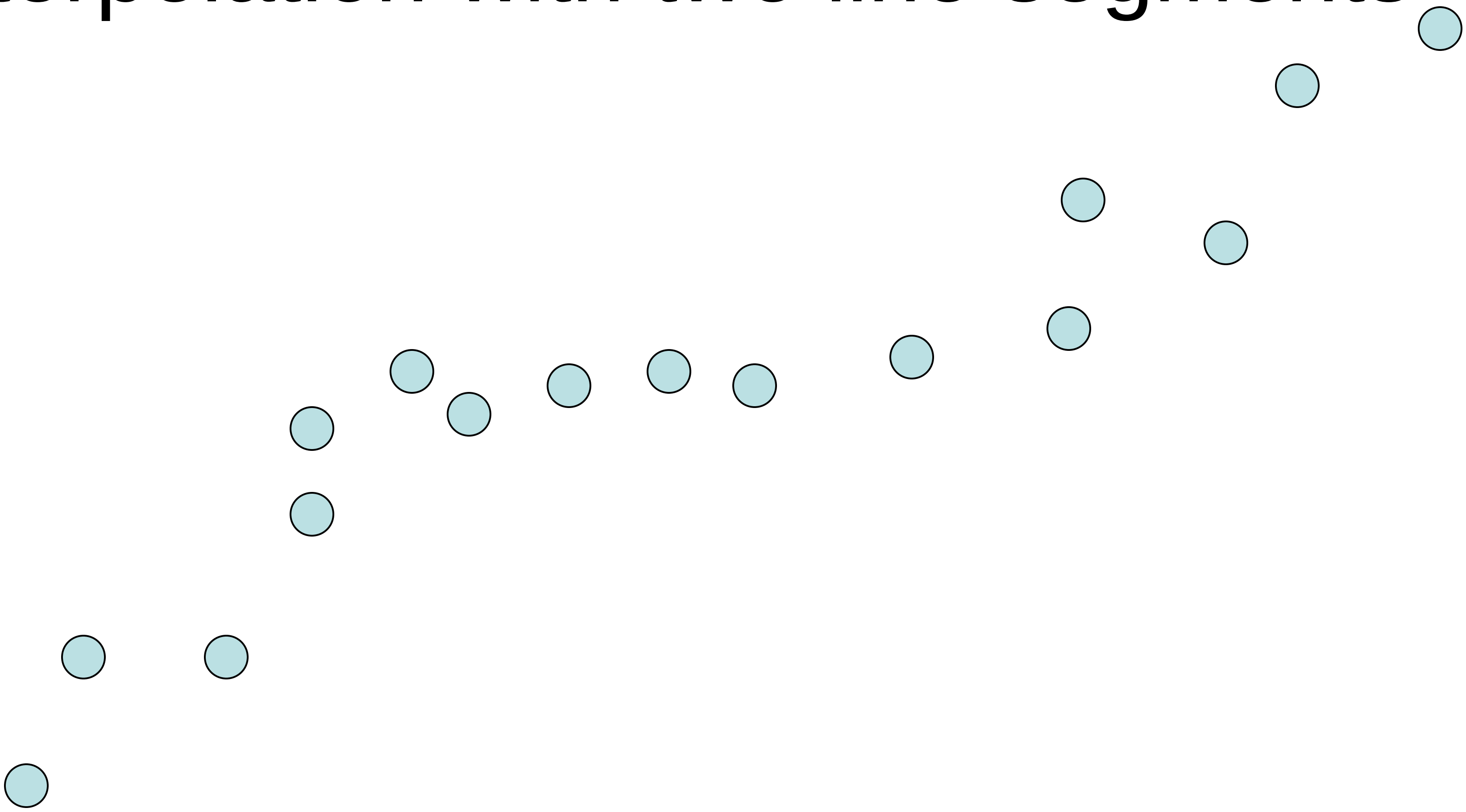


$$\text{Error} = \sum (y_i - ax_i - b)^2$$

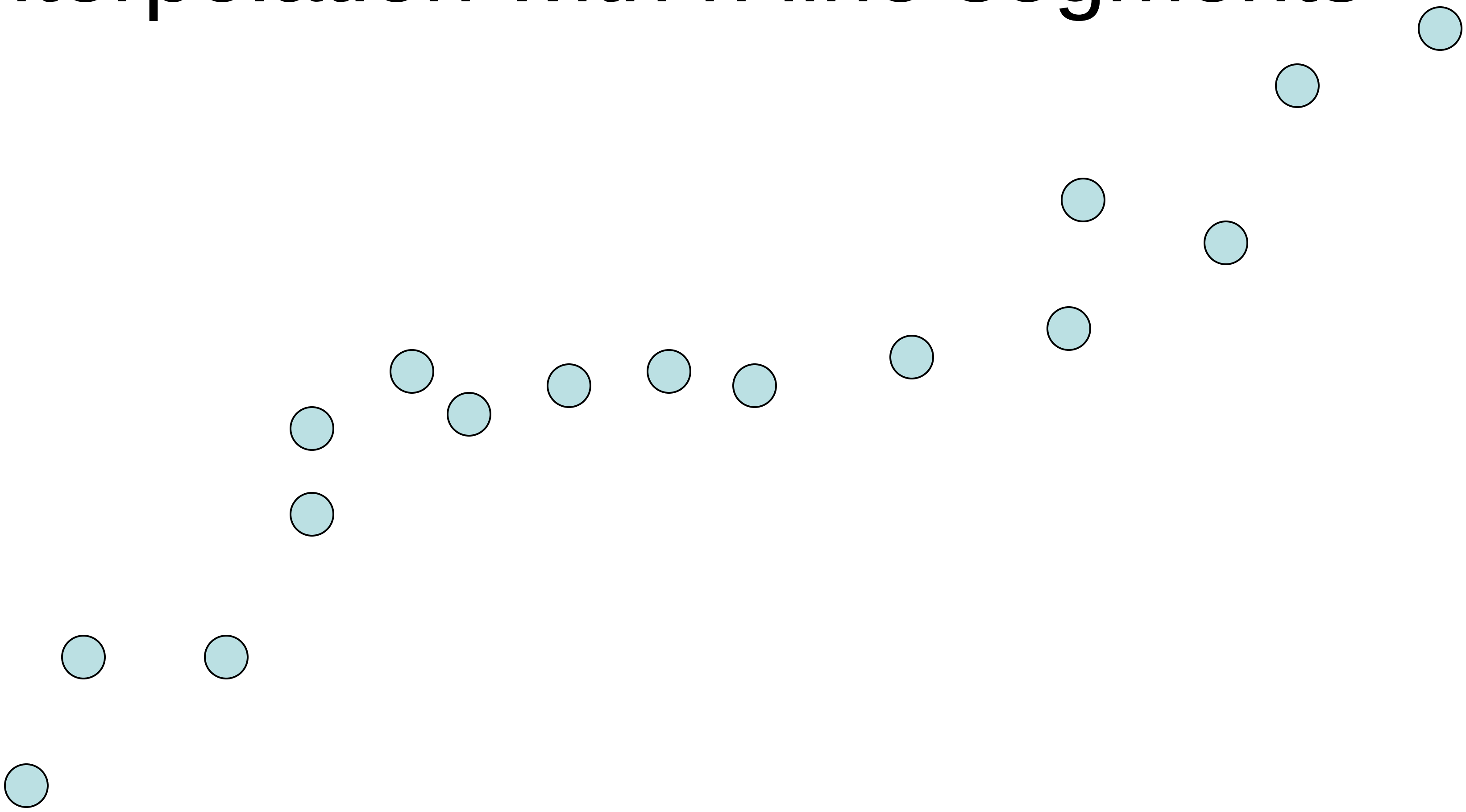
What is the optimal linear interpolation with three line segments



What is the optimal linear interpolation with two line segments

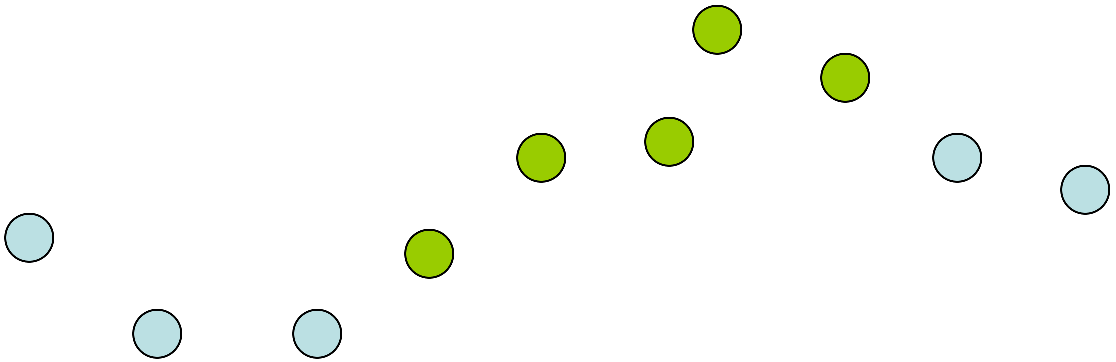


What is the optimal linear interpolation with n line segments



Notation

- Points $p_1, p_2, \ldots, p_n$ ordered by x-coordinate ($p_i = (x_i, y_i)$)
- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$



Optimal interpolation with two segments

- Give an equation for the optimal interpolation of $p_1,\ldots,p_n$ with two line segments

- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$

Optimal interpolation with k segments

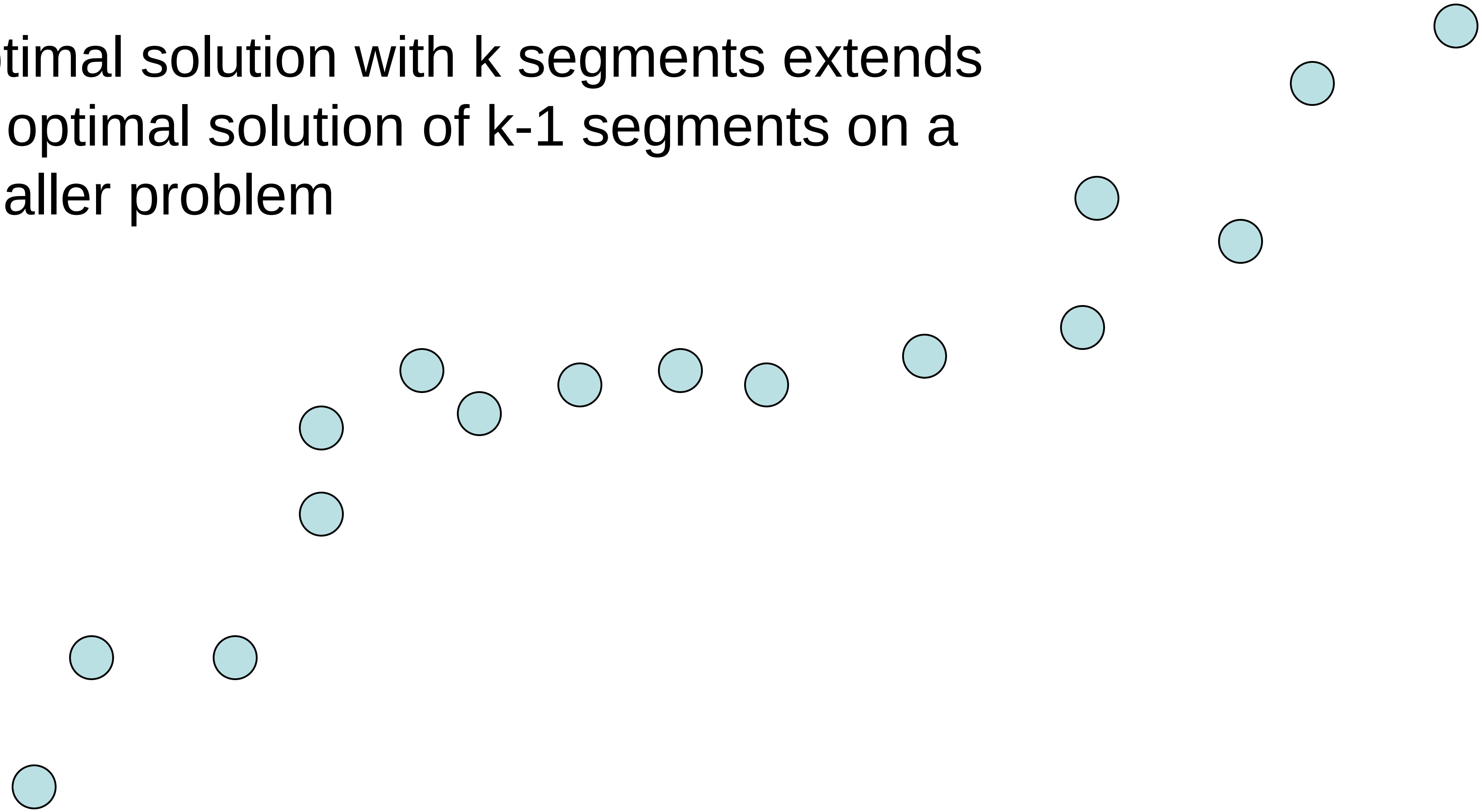
- Optimal segmentation with three segments
  - $\text{Min}_{i,j}\{E_{1,i} + E_{i,j} + E_{j,n}\}$
  - $O(n^2)$ combinations considered
- Generalization to k segments leads to considering $O(n^{k-1})$ combinations

$Opt_k[\ j\ ]$: Minimum error approximating $p_1 \ldots p_j$ with k segments

How do you express $Opt_k[\ j\ ]$ in terms of $Opt_{k-1}[1],\ldots,Opt_{k-1}[\ j\ ]$?

Optimal sub-solution property

Optimal solution with k segments extends an optimal solution of k-1 segments on a smaller problem



Optimal multi-segment interpolation

Compute Opt[k, j] for 0 < k < j < n

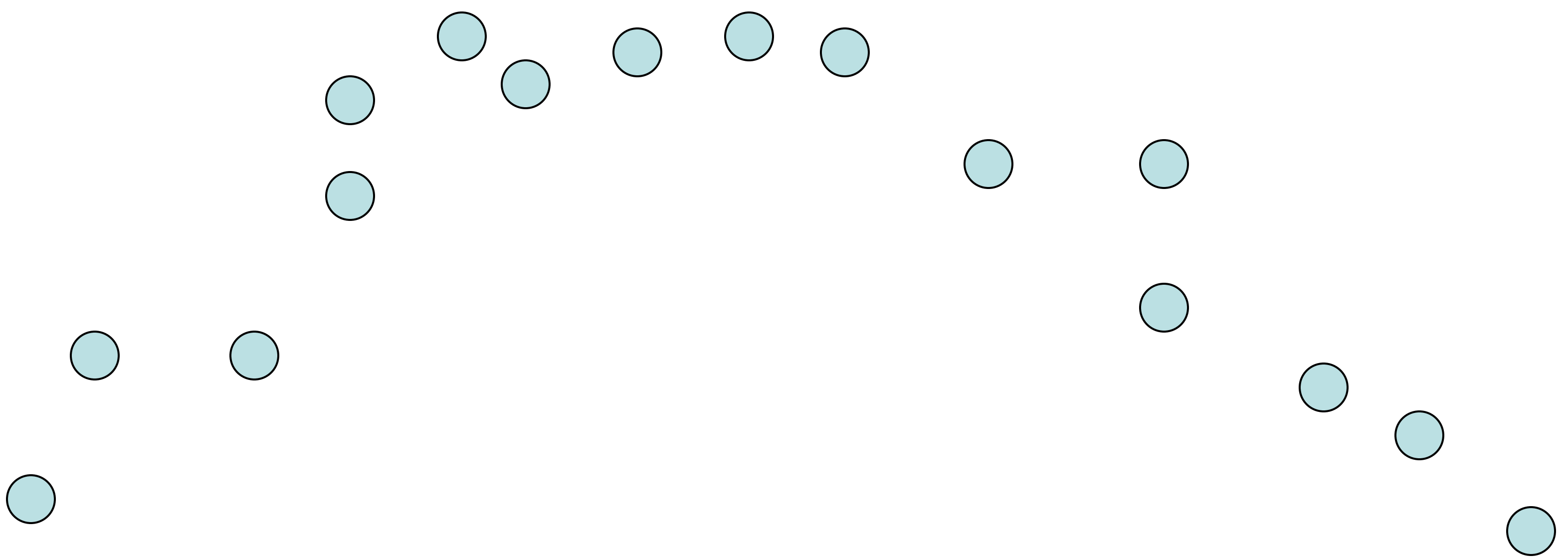
```
for j := 1 to n
    Opt[ 1, j] = E_{1,j};
for k := 2 to n-1
    for j := 2 to n
        t := E_{1,j}
        for i := 1 to j -1
            t = min (t, Opt[k-1, i ] + E_{i,j})
        Opt[k, j] = t
```

Determining the solution

- When Opt[k,j] is computed, record the value of i that minimized the sum
- Store this value in a auxiliary array
- Use to reconstruct solution

Variable number of segments

- Segments not specified in advance
- Penalty function associated with segments
- Cost = Interpolation error + C x #Segments



Penalty cost measure

- $Opt[\ j\ ] = \min(E_{1,j}, \min_i(Opt[\ i\ ] + E_{i,j} + P))$