CSE 421
Algorithms

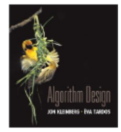
Richard Anderson

Autumn 2016

Lecture 2

Announcements

- Homework 1, due Wednesday Oct 5
  - in class, paper turn in
  - pay attention to making explanations clear and understandable
- Reading
  - Chapter 1, Sections 2.1, 2.2



Office Hours

- Richard Anderson
  - Monday, 2:30 pm - 3:30 pm, CSE 582
  - Wednesday, 2:30 pm - 3:30 pm, CSE 582
- Deepali Aneja
  - Monday, 5:30 pm - 6:30 pm, CSE 220
- Max Horton
  - Monday, 4:30 pm – 5:30 pm, CSE 220
  - Tuesday, 2:00 pm – 3:00 pm, CSE 218
- Ben Jones
  - Tuesday, 1:00 pm – 2:00 pm, CSE 218
  - Friday, 2:30 pm – 3:30 pm, CSE 220

Stable Matching: Formal Problem

- Input
  - Preference lists for $m_1, m_2, \ldots, m_n$
  - Preference lists for $w_1, w_2, \ldots, w_n$
- Output
  - Perfect matching M satisfying stability property (e.g., no instabilities) :

For all m', m'', w', w''
If (m', w') $\in$ M and (m'', w'') $\in$ M then
(m' prefers w' to w'') or (w'' prefers m'' to m')

Idea for an Algorithm

m proposes to w

If w is unmatched, w accepts

If w is matched to m_2

If w prefers m to m_2 w accepts m, dumping m_2

If w prefers m_2 to m, w rejects m

Unmatched m proposes to the highest w on its preference list that it has not already proposed to

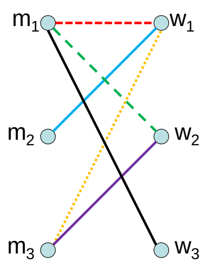
Algorithm

```
Initially all m in M and w in W are free
While there is a free m
    w highest on m's list that m has not proposed to
    if w is free, then match (m, w)
    else
        suppose (m2, w) is matched
        if w prefers m to m2
            unmatch (m2, w)
            match (m, w)
```

Example

m_1: w_1 w_2 w_3
m_2: w_1 w_3 w_2
m_3: w_1 w_2 w_3

w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_3 m_1 m_2



Order: m_1, m_2, m_3, m_1, m_3, m_1

Does this work?

- Does it terminate?
- Is the result a stable matching?

- Begin by identifying invariants and measures of progress
  - m's proposals get worse (have higher m-rank)
  - Once w is matched, w stays matched
  - w's partners get better (have lower w-rank)

Claim: If an m reaches the end of its list, then all the w's are matched

Claim: The algorithm stops in at most n^2 steps

When the algorithms halts, every w is matched

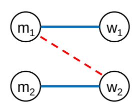
Hence, the algorithm finds a perfect matching

The resulting matching is stable

Suppose

$(m_1, w_1) \in M$, $(m_2, w_2) \in M$

m_1 prefers w_2 to w_1



How could this happen?

Result

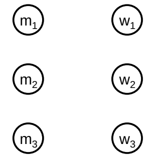
- Simple, $O(n^2)$ algorithm to compute a stable matching
- Corollary
  - A stable matching always exists

A closer look

Stable matchings are not necessarily fair

m_1: w_1 w_2 w_3
m_2: w_2 w_3 w_1
m_3: w_3 w_1 w_2

w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_1 m_2 m_3



How many stable matchings can you find?

Algorithm under specified

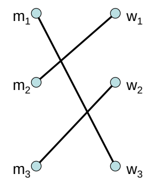
- Many different ways of picking m's to propose
- Surprising result
  - All orderings of picking free m's give the same result

- Proving this type of result
  - Reordering argument
  - Prove algorithm is computing something mores specific
    - Show property of the solution – so it computes a specific stable matching

M-rank and W-rank of matching

- m-rank: position of matching w in preference list
- M-rank: sum of m-ranks
- w-rank: position of matching m in preference list
- W-rank: sum of w-ranks

m_1: w_1 w_2 w_3
m_2: w_1 w_3 w_2
m_3: w_1 w_2 w_3
w_1: m_2 m_3 m_1
w_2: m_3 m_1 m_2
w_3: m_3 m_1 m_2



What is the M-rank?

What is the W-rank?

Suppose there are n m's, and n w's

- What is the minimum possible M-rank?
- What is the maximum possible M-rank?
- Suppose each m is matched with a random w, what is the expected M-rank?

Random Preferences

Suppose that the preferences are completely random

m_1: w_8 w_3 w_1 w_5 w_9 w_2 w_4 w_6 w_7 w_{10}
m_2: w_7 w_{10} w_1 w_9 w_3 w_4 w_8 w_2 w_5 w_6
…
w_1: m_1 m_4 m_9 m_5 m_{10} m_3 m_2 m_6 m_8 m_7
w_2: m_5 m_8 m_1 m_3 m_2 m_7 m_9 m_{10} m_4 m_6
…

If there are n m's and n w's, what is the expected value of the M-rank and the W-rank when the proposal algorithm computes a stable matching?

Best choices for one side may be bad for the other

Design a configuration for problem of size 4:

- M proposal algorithm:
  - All m's get first choice, all w's get last choice
- W proposal algorithm:
  - All w's get first choice, all m's get last choice

m_1:

m_2:

m_3:

m_4:

w_1:

w_2:

w_3:

w_4:

But there is a stable second choice

Design a configuration for problem of size 4:

- M proposal algorithm:
  - All m's get first choice, all w's get last choice
- W proposal algorithm:
  - All w's get first choice, all m's get last choice
- There is a stable matching where everyone gets their second choice

m_1:

m_2:

m_3:

m_4:

w_1:

w_2:

w_3:

w_4:

What is the run time of the Stable Matching Algorithm?

```
Initially all m in M and w in W are free
While there is a free m                    Executed at most n^2 times
        w highest on m's list that m has not proposed to
        if w is free, then match (m, w)
        else
                suppose (m_2, w) is matched
                if w prefers m to m_2
                        unmatch (m_2, w)
                        match (m, w)
```

O(1) time per iteration

- Find free m
- Find next available w
- If w is matched, determine m_2
- Test if w prefer m to m_2
- Update matching

What does it mean for an algorithm to be efficient?

Key ideas

- Formalizing real world problem
  - Model: graph and preference lists
  - Mechanism: stability condition
- Specification of algorithm with a natural operation
  - Proposal
- Establishing termination of process through invariants and progress measure
- Under specification of algorithm
- Establishing uniqueness of solution