CSE 421 Algorithms: Divide and Conquer

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Thanks to Richard Anderson, Paul Beame, Kevin Wayne for some slides

algorithm design paradigms: divide and conquer

Outline:

- General Idea
- Review of Merge Sort
- Why does it work?
  - Importance of balance
  - Importance of super-linear growth
- Some interesting applications
  - Closest points
  - Integer Multiplication
- Finding & Solving Recurrences

Divide & Conquer

Reduce problem to one or more sub-problems of the same type

Typically, each sub-problem is at most a constant fraction of the size of the original problem

Subproblems typically disjoint

Often gives significant, usually polynomial, speedup

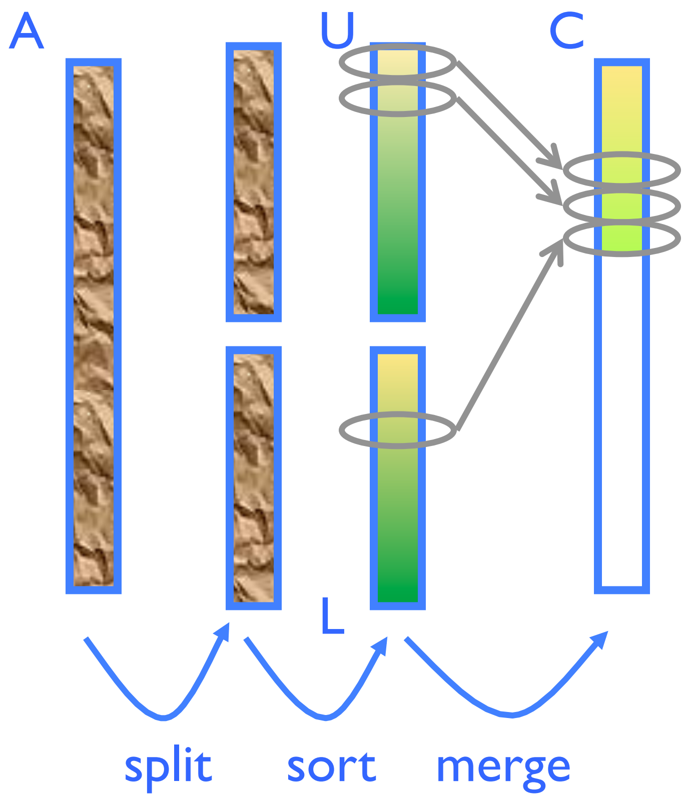
Examples:

Binary Search, Mergesort, Quicksort (roughly), Strassen's Algorithm, integer multiplication, powering, FFT, …

Motivating Example: Mergesort

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }

Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
        "C[i] =  smaller of U[a], L[b] and correspondingly a++ or b++,
                 while being careful about running past end of either";
    Return C;
    }
```



Time: $\Theta(n \log n)$

Why does it work? Suppose we've already invented DumbSort, taking time n^2

Try *Just One Level* of divide & conquer:

DumbSort(first n/2 elements)	$O((n/2)^2)$
DumbSort(last n/2 elements)	$O((n/2)^2)$
Merge results	$O(n)$

Time: $\boxed{2\,(n/2)^2 + n = n^2/2 + n \ll n^2}$

D&C in a nutshell

Almost twice as fast!

d&c approach, cont.

Moral 1: “two halves are better than a whole”

Two problems of half size are *better* than one full-size problem, even given O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: “If a little's good, then more's better”

Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing.
Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

In the limit: you’ve just rediscovered mergesort!

Moral 3: unbalanced division good, but less so:

$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

Moral 4: but consistent, completely unbalanced division doesn't help much:

$(1)^2 + (n-1)^2 + n = n^2 - n + 2$

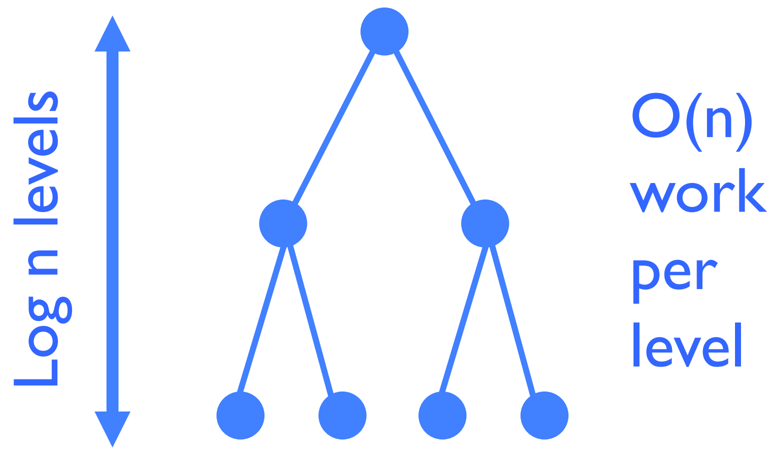
Little improvement here.

Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n) = 2T(n/2)+cn, \quad n \geq 2$

$T(1) = 0$

Solution: $\Theta(n \log n)$
(details later)



Example:
Counting Inversions

Inversion Problem

Let $a_1, \ldots a_n$ be a permutation of $1 \ldots n$

(a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

Problem: given a permutation, count the number of inversions

This can be done easily in $O(n^2)$ time

- Can we do better?

Application

Counting inversions can be use to measure closeness of ranked preferences

People rank 20 movies, based on their rankings you cluster people who like the same types of movies

Can also be used to measure nonlinear correlation

Inversion Problem

Let $a_1, \ldots a_n$ be a permutation of $1 \ldots n$

(a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

Problem: given a permutation, count the number of inversions

This can be done easily in $O(n^2)$ time

Can we do better?

Counting Inversions

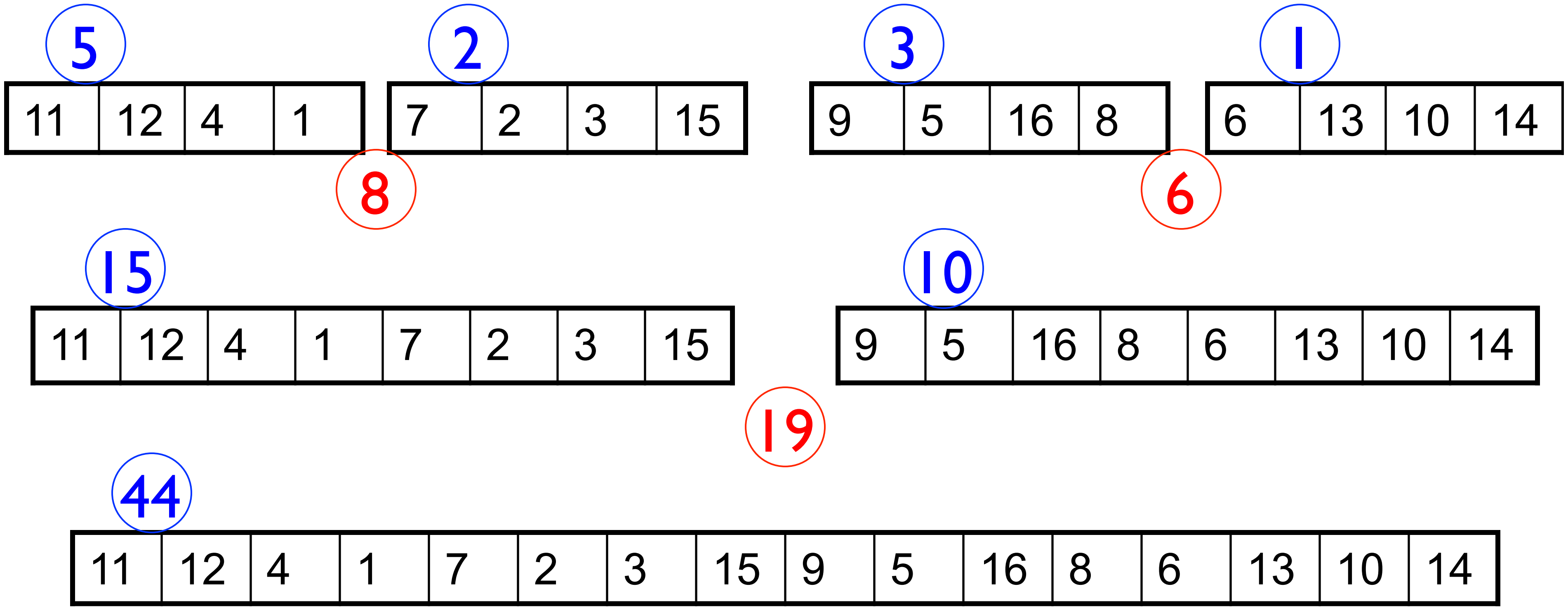
11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on left half

Count inversions on right half

Count the inversions between the halves

Count the Inversions



Can we count inversions between sub-problems in O(n) time?

Yes – Count inversions while merging

1	2	3	4	7	11	12	15

5	6	8	9	10	13	14	16

Standard merge algorithm – add to inversion count when an element is moved from the right array to the solution. (Add how much? Why not left array?)

Counting inversions while merging

1	4	11	12

2	3	7	15

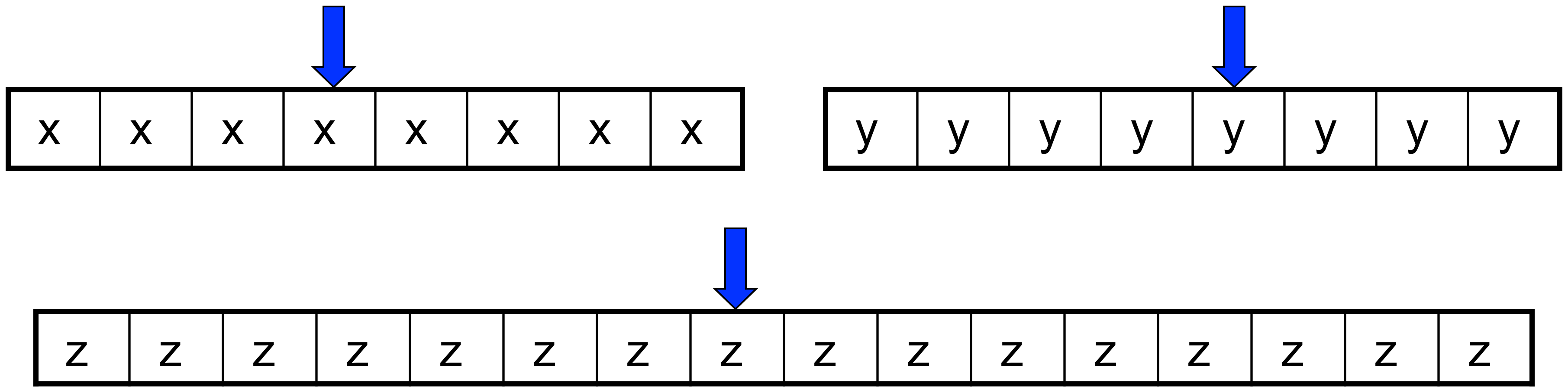
5	8	9	16

6	10	13	14

Indicate the number of inversions for each element detected when merging

Counting inversions between two sorted lists

O(1) per element to count inversions



Algorithm summary

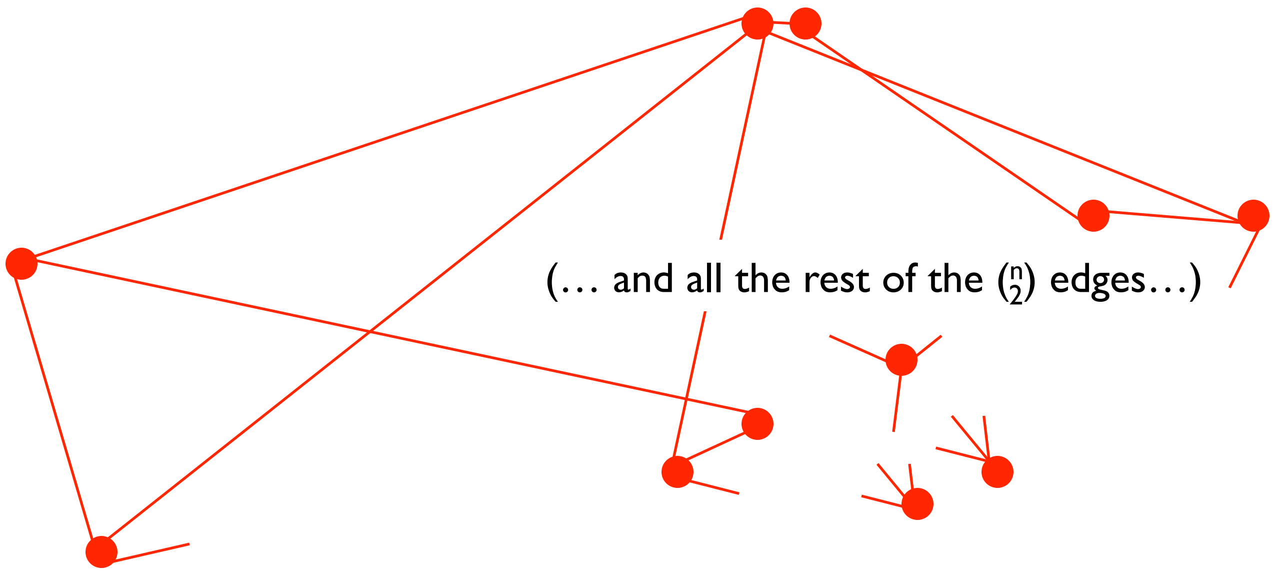
Satisfies the “Standard recurrence”

$T(n) = 2\,T(n/2) + cn$

A Divide & Conquer Example: Closest Pair of Points

closest pair of points: non-geometric version

Given n points and *arbitrary* distances between them, find the closest pair. (E.g., think of distance as airfare – definitely *not* Euclidean distance!)

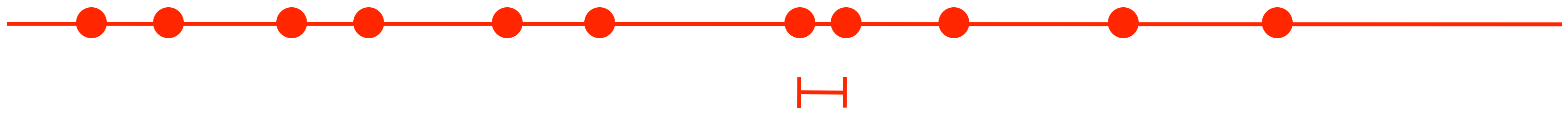


Must look at all n choose 2 pairwise distances, else any one you didn't check might be the shortest.

Also true for Euclidean distance in 1-2 dimensions?

closest pair of points: 1 dimensional version

Given n points on the real line, find the closest pair



Closest pair is *adjacent* in ordered list

Time O(n log n) to sort, if needed

Plus O(n) to scan adjacent pairs

Key point: do *not* need to calc distances between all pairs: exploit geometry + ordering

closest pair of points: 2 dimensional version

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

$\nwarrow$ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

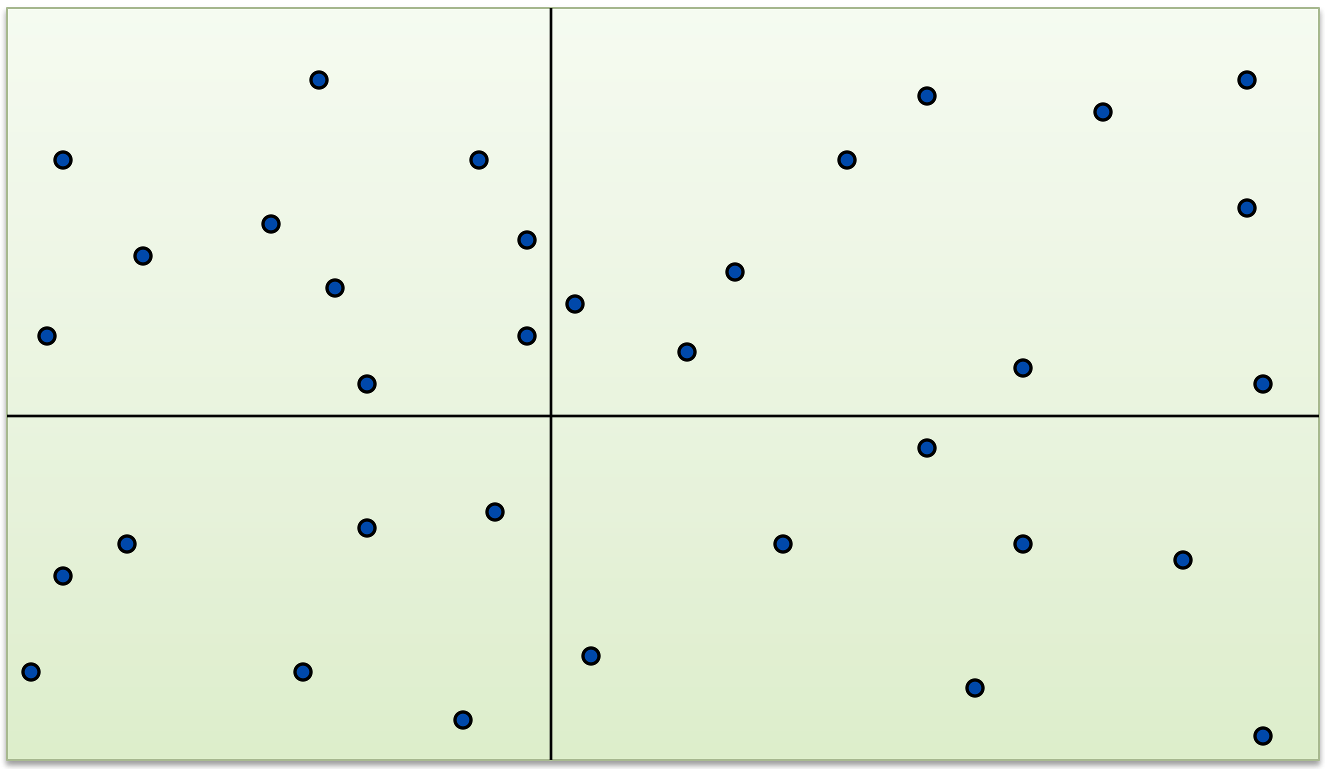
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

$\uparrow$ Just to simplify presentation

closest pair of points. 2d, Euclidean distance: 1st try

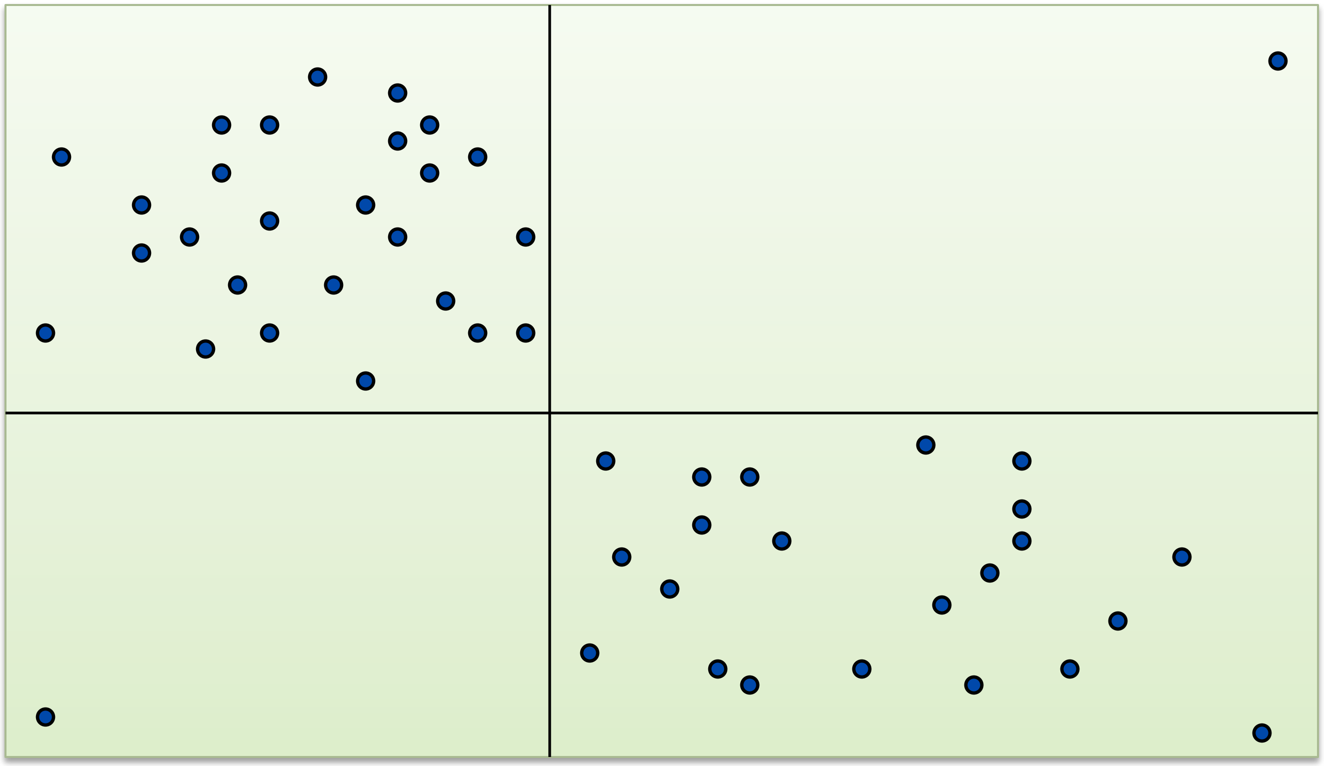
Divide. Sub-divide region into 4 quadrants.



closest pair of points: 1st try

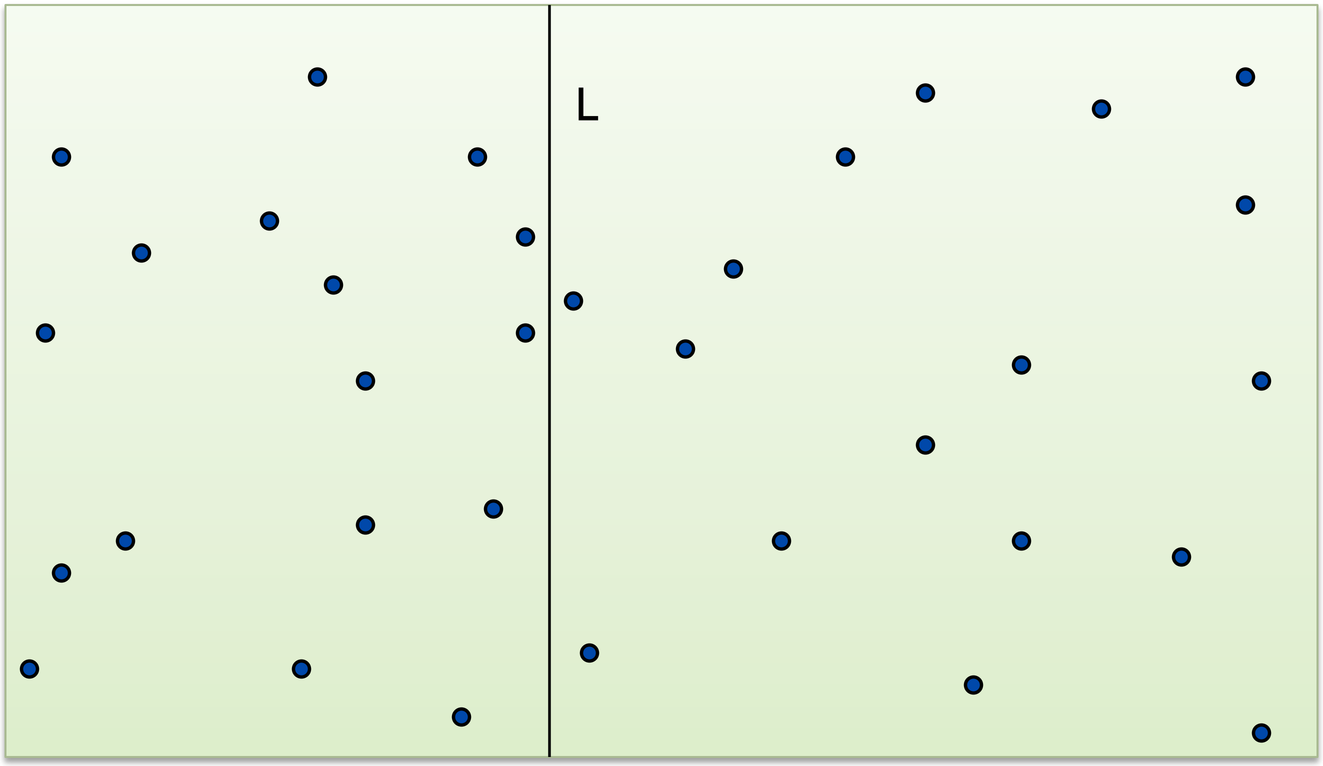
Divide. Sub-divide region into 4 quadrants.

Obstacle. Impossible to ensure n/4 points in each piece, so the “balanced subdivision” goal may be elusive/problematic.



Algorithm.

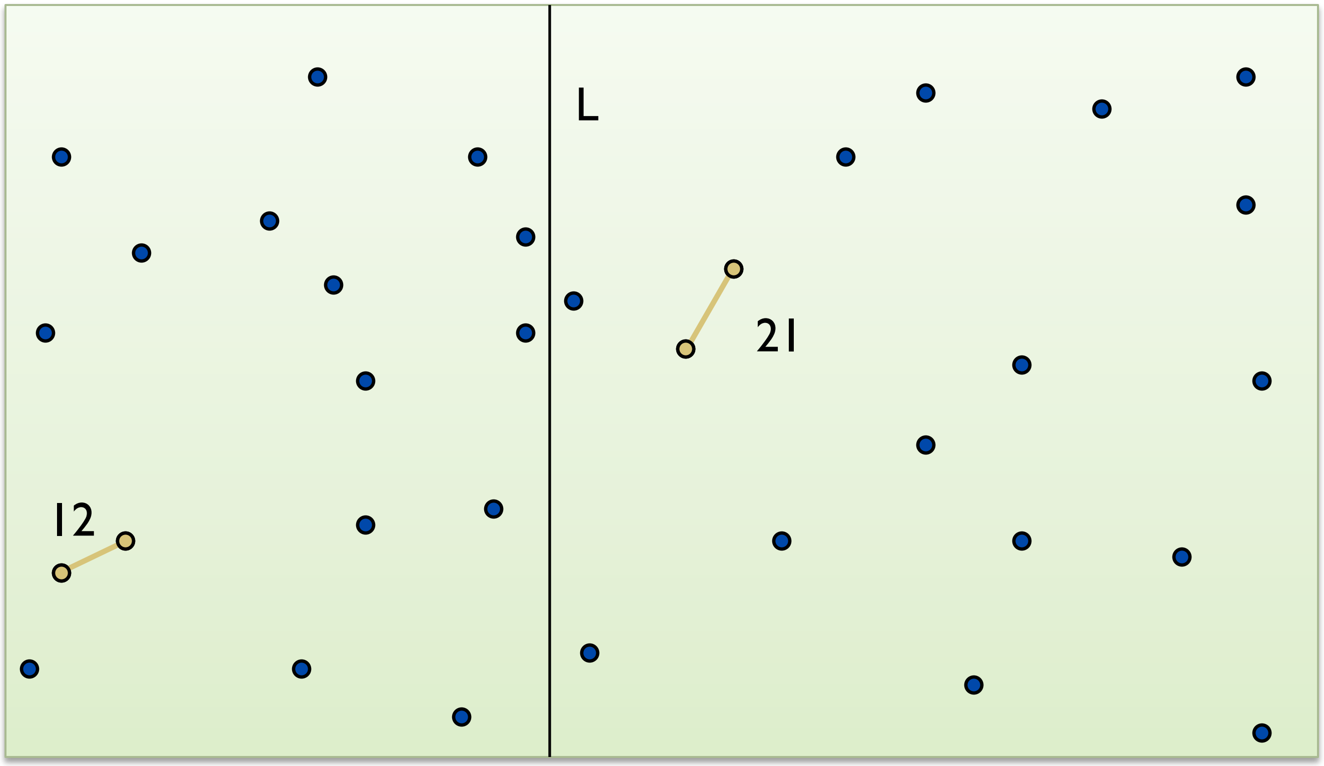
Divide: draw vertical line L with ≈ n/2 points on each side.



Algorithm.

Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.



Algorithm.

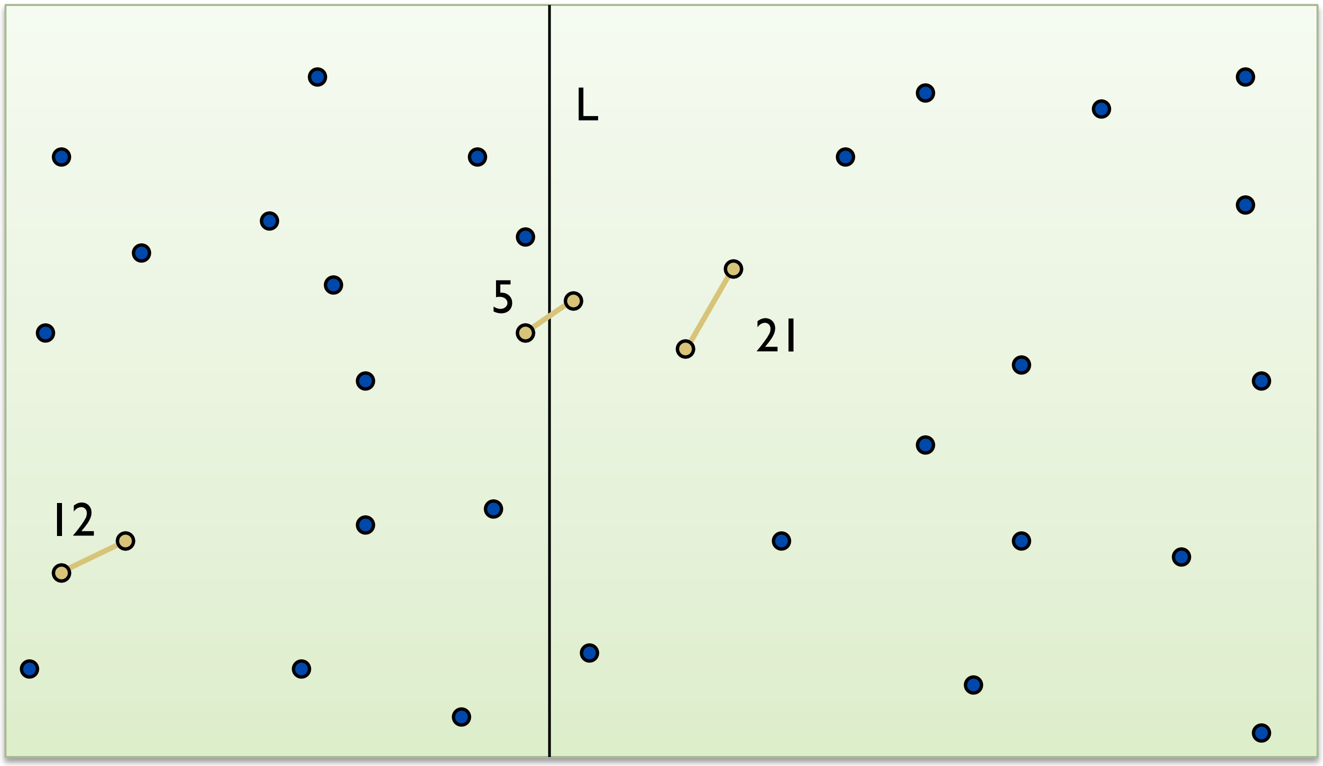
Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.

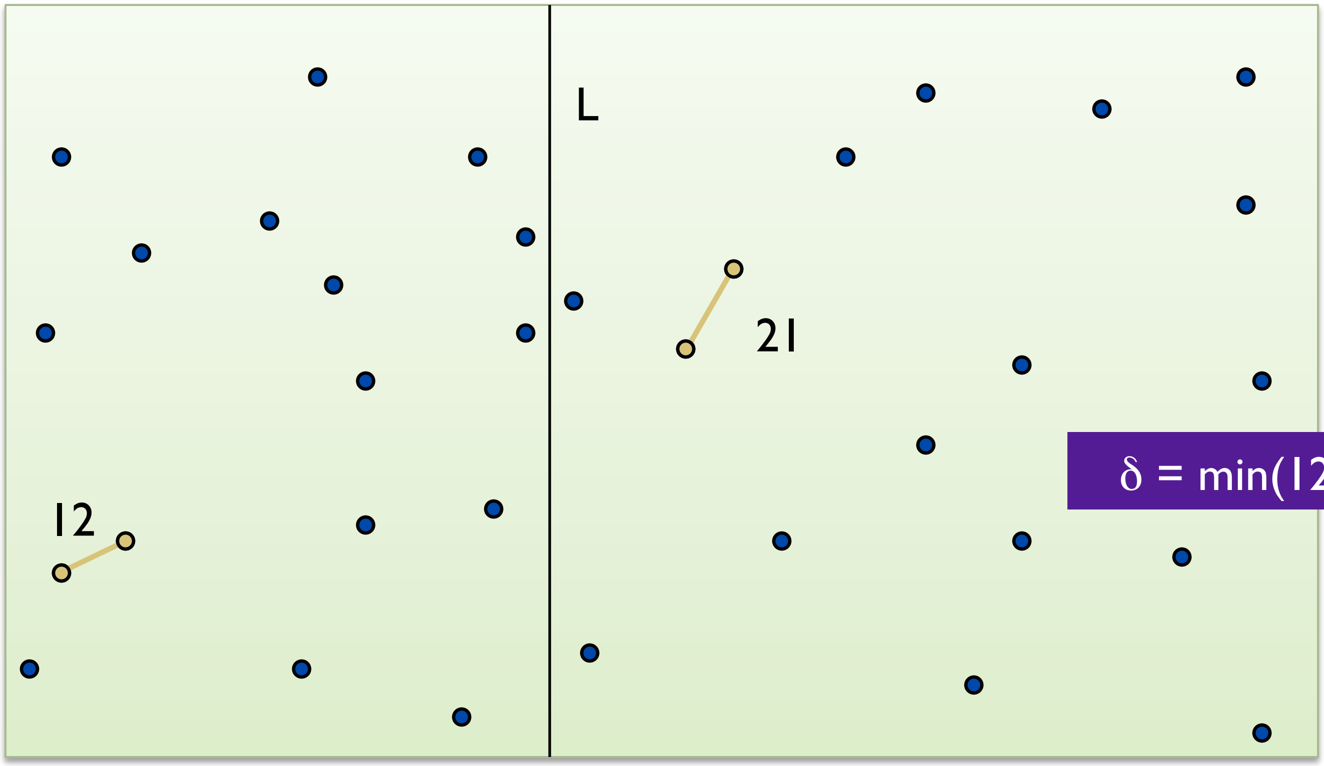
Combine: find closest pair with one point in each side.

Return best of 3 solutions.

seems like $\Theta(n^2)$?



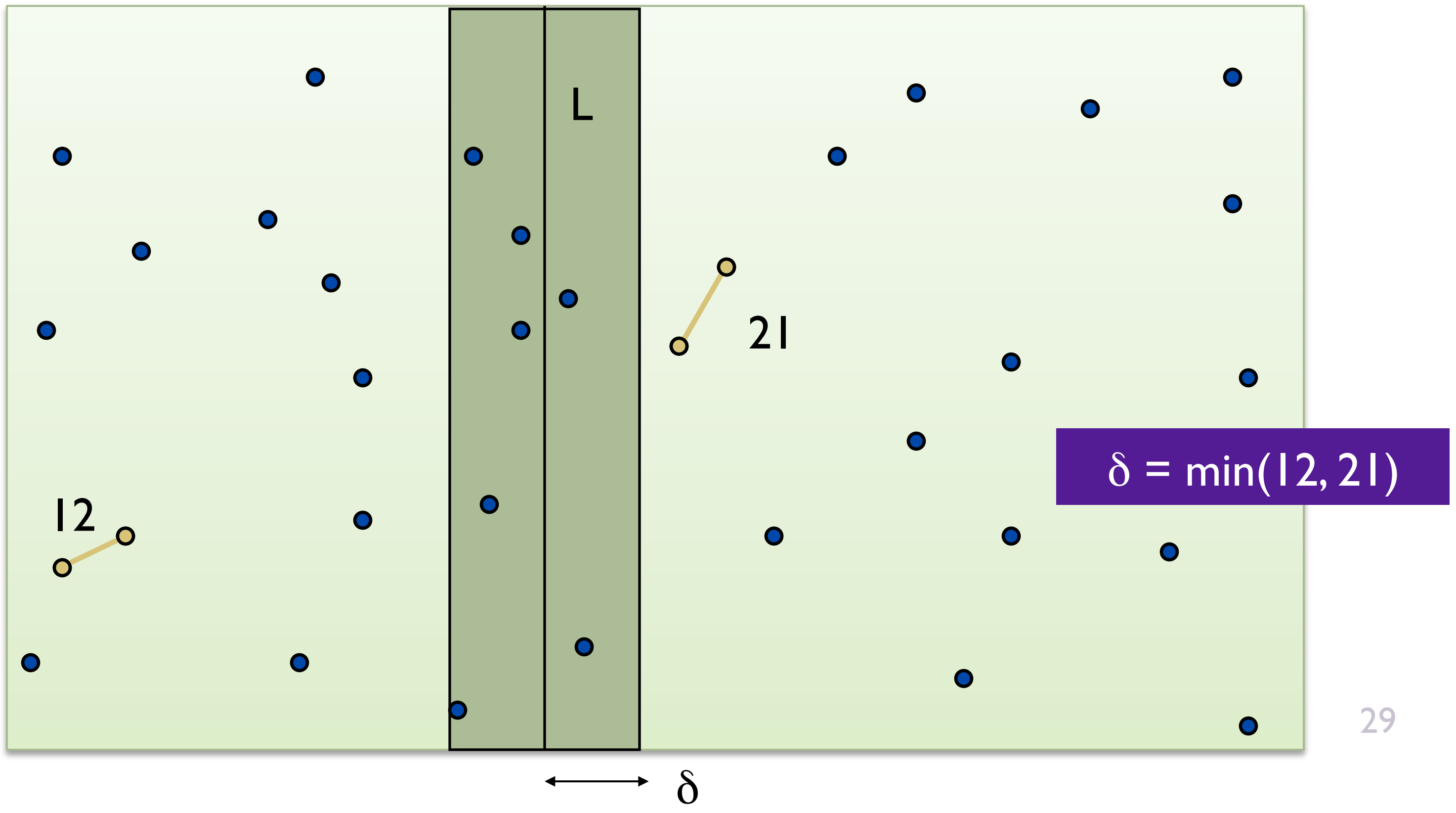
Find closest pair with one point in each side, *assuming* distance < δ.



δ = min(12, 21)

Find closest pair with one point in each side, *assuming distance < δ.*

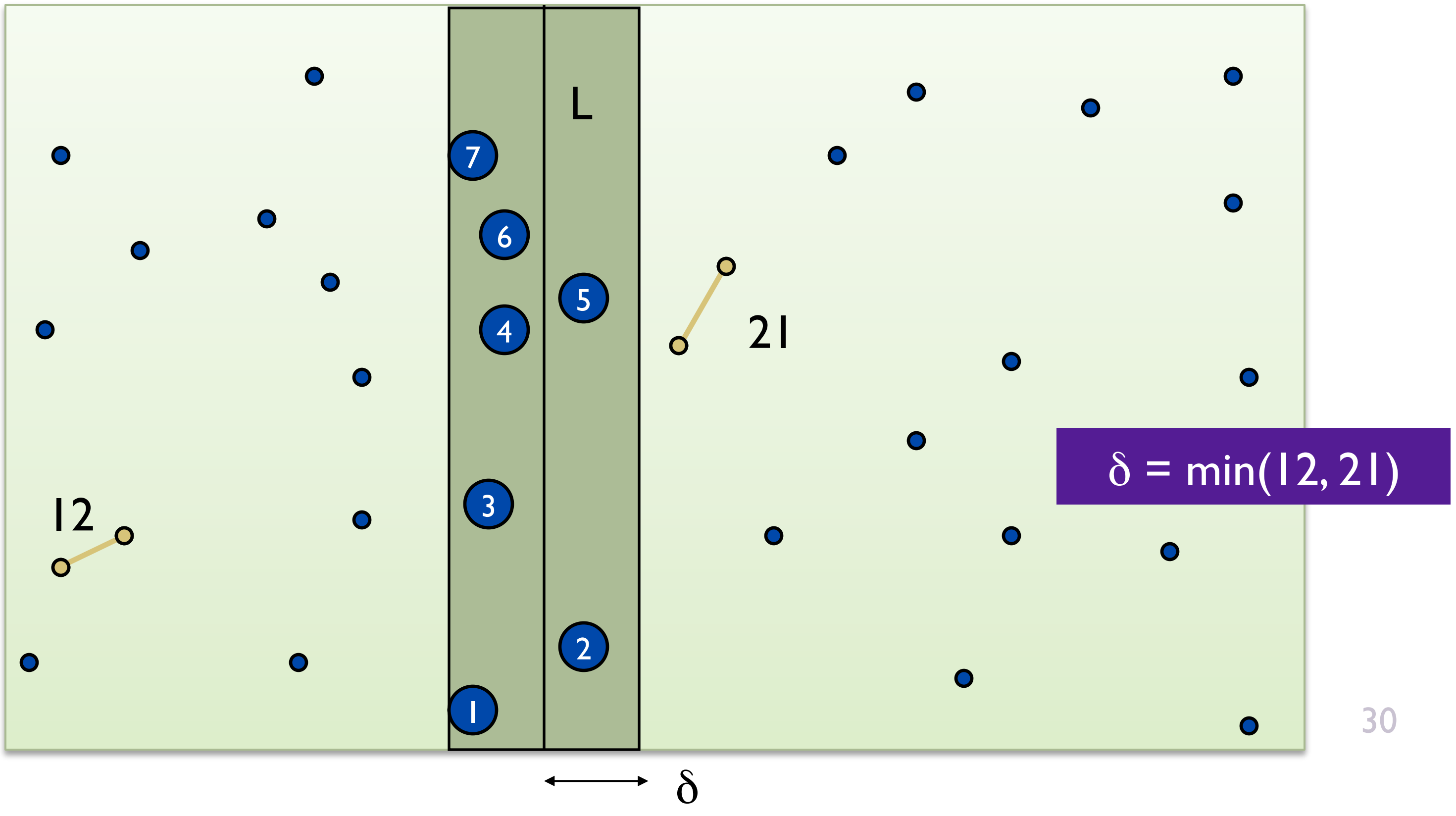
Observation: suffices to consider points within δ of line L.



Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

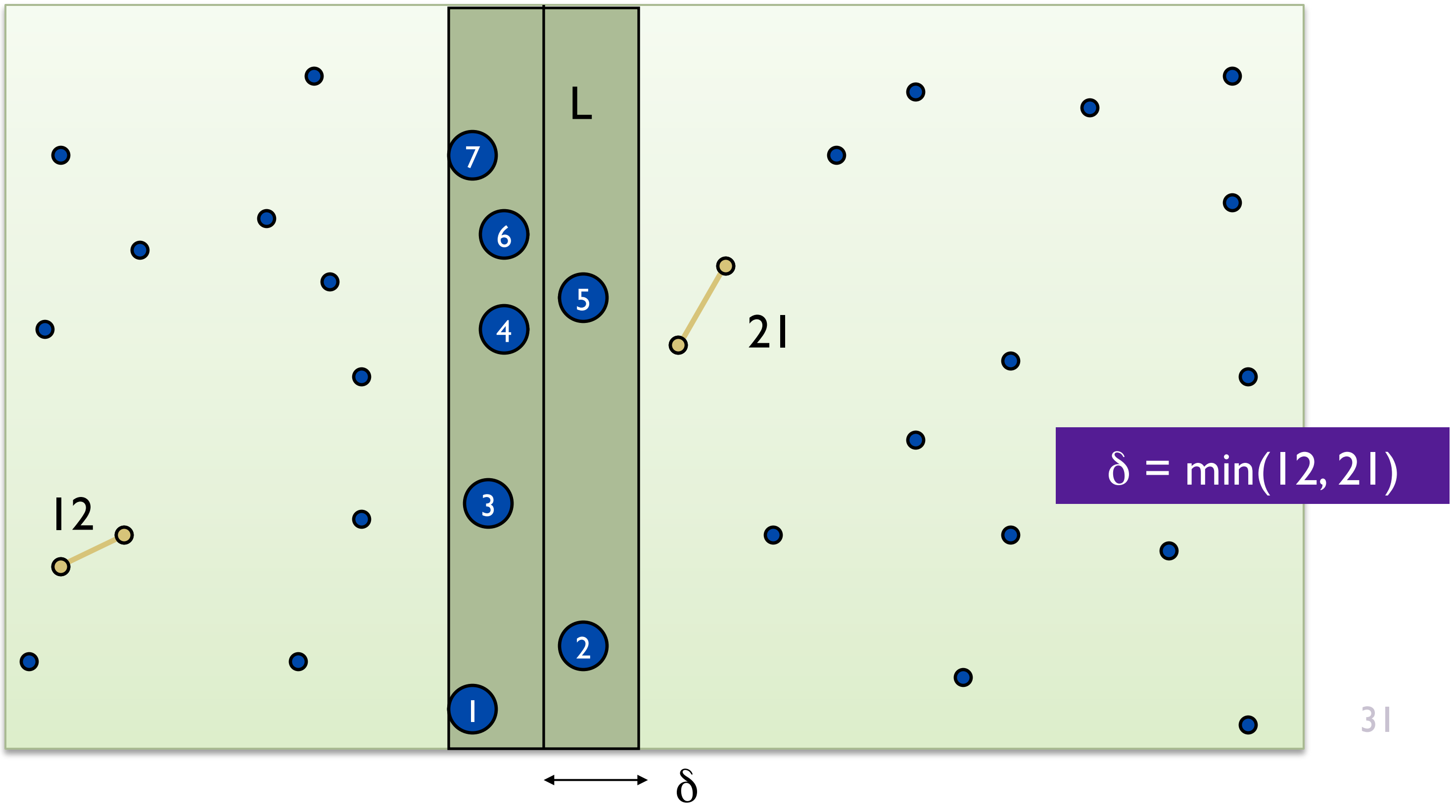
Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate. Only check pts within 8 in sorted list!



closest pair of points

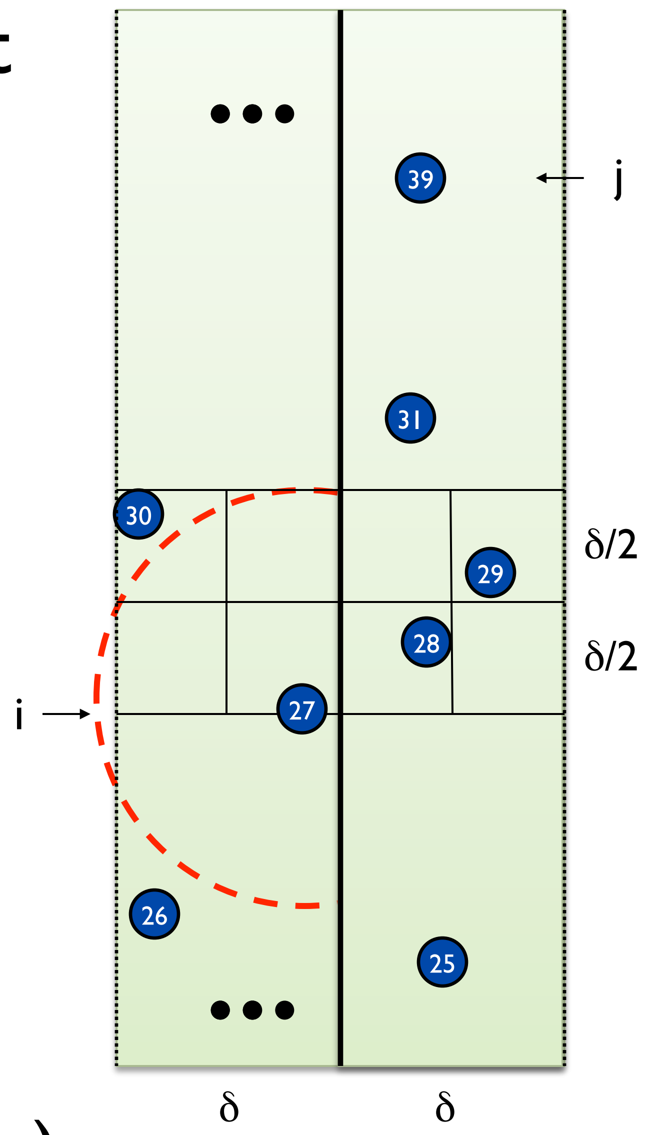
Def. Let s_i have the ith smallest y-coordinate among points in the 2δ-width-strip.

Claim. If $|j - i| \geq 8$, then the distance between s_i and s_j is > δ.

Pf: No two points lie in the same δ/2-by-δ/2 square:

$$\sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \frac{\sqrt{2}}{2}\delta \approx 0.7\delta < \delta$$

so ≤ 8 points within +δ of $y(s_i)$.



```
Closest-Pair($p_1$, …, $p_n$) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + $\delta$
        $\delta$ = min($\delta$, distance between p[i] and p[i+k]);
        k++;

   return $\delta$.
}
```

closest pair of points: analysis

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

closest pair of points: analysis

Analysis, II: Let C(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \left\{ \begin{array}{cc} 0 & n=1 \\ 2C(n/2) + kn \log n & n>1 \end{array} \right\} \Rightarrow C(n) = O(n \log^2 n)$$

for some constant k

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

is it worth the effort?

Code is longer & more complex

O(n log n) vs O(n^2) may hide 10x in constant?

How many points?

n	Speedup: n^2 / (10 n $\log_2$ n)
10	0.3
100	1.5
1,000	10
10,000	75
100,000	602
1,000,000	5,017
10,000,000	43,004

Going From Code to Recurrence

going from code to recurrence

Carefully define what you're counting, and *write it down*!

“Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1”

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)

merge sort

```
MS(A: array[1..n]) returns array[1..n] {
        If(n=1) return A;
        New L:array[1:n/2] = MS(A[1..n/2]);
        New R:array[1:n/2] = MS(A[n/2+1..n]);
        Return(Merge(L,R));
        }
Merge(A,B: array[1..n]) {
        New C: array[1..2n];
        a=1; b=1;
        For i = 1 to 2n {
                C[i] = "smaller of A[a], B[b] and a++ or b++";
        Return C;
        }
```

Base Case

Recursive calls

One Recursive Level

Operations being counted

the recurrence

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)

going from code to recurrence

Carefully define what you're counting, and *write it down*!

"Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

closest pair algorithm

Basic operations: distance calcs

```
Closest-Pair($p_1$, …, $p_n$) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   $\delta_1$ = Closest-Pair(left half)
   $\delta_2$ = Closest-Pair(right half)
   $\delta$  = min($\delta_1$, $\delta_2$)

   Delete all points further than $\delta$ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + $\delta$
         $\delta$ = min($\delta$, distance between p[i] and p[i+k]);
         k++;

   return $\delta$.
}
```

Base Case

0

Recursive calls (2)

2D(n / 2)

One recursive level

Basic operations at this recursive level

7n

closest pair of points: analysis

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

going from code to recurrence

Carefully define what you're counting, and *write it down*!

> "Let D(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

closest pair algorithm

Basic operations: *comparisons*

Base Case

Recursive calls (2)

```
Closest-Pair(p_1, …, p_n) {
  if(n <= 1) return ∞

  Compute separation line L such that half the points
  are on one side and half on the other side.

  δ_1 = Closest-Pair(left half)
  δ_2 = Closest-Pair(right half)
  δ  = min(δ_1, δ_2)

  Delete all points further than δ from separation line L

  Sort remaining points p[1]…p[m] by y-coordinate.

  for i = 1..m
     k = 1
     while i+k <= m && p[i+k].y < p[i].y + δ
       δ = min(δ, distance between p[i] and p[i+k]);
       k++;

  return δ.
}
```

Basic operations at this recursive level

0

$k_1 n \log n$

$2C(n / 2)$

1

$k_2 n$

$k_3 n \log n$

$8n$

$7n$

One recursive level

closest pair of points: analysis

Analysis, II: Let C(n) be the number of comparisons of coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \left\{ \begin{array}{cc} 0 & n=1 \\ 2C(n/2) + k_4 n \log n + 1 & n>1 \end{array} \right\} \Rightarrow C(n) = O(n \log^2 n)$$

for some $k_4 \le k_1 + k_2 + k_3 + 15$

Q. Can we achieve time O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

Integer Multiplication

integer arithmetic

Add. Given two n-bit integers a and b, compute a + b.

Add

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

O(n) bit operations.

Add. Given two n-bit integers a and b, compute a + b.

O(n) bit operations.

Add

Carries

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Multiply. Given two n-bit integers a and b, compute a × b.
The "grade school" method:

$\Theta(n^2)$ bit operations.

Multiply

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

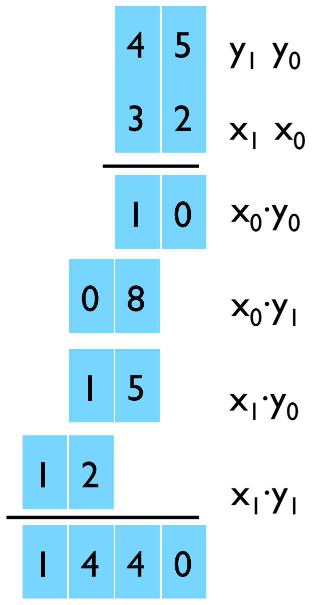
divide & conquer multiplication: warmup

To multiply two 2-digit integers:

Multiply four 1-digit integers.

Add, shift some 2-digit integers to obtain result.

$$\begin{aligned}
x &= 10\cdot x_1 + x_0\\
y &= 10\cdot y_1 + y_0\\
xy &= (10\cdot x_1 + x_0)(10\cdot y_1 + y_0)\\
&= 100\cdot x_1y_1 + 10\cdot(x_1y_0 + x_0y_1) + x_0y_0
\end{aligned}$$



Same idea works for *long* integers – can split them into 4 half-sized ints ("10" becomes "10^k", k = length/2)

divide & conquer multiplication: warmup

To multiply two n-bit integers:

Multiply four ½n-bit integers.

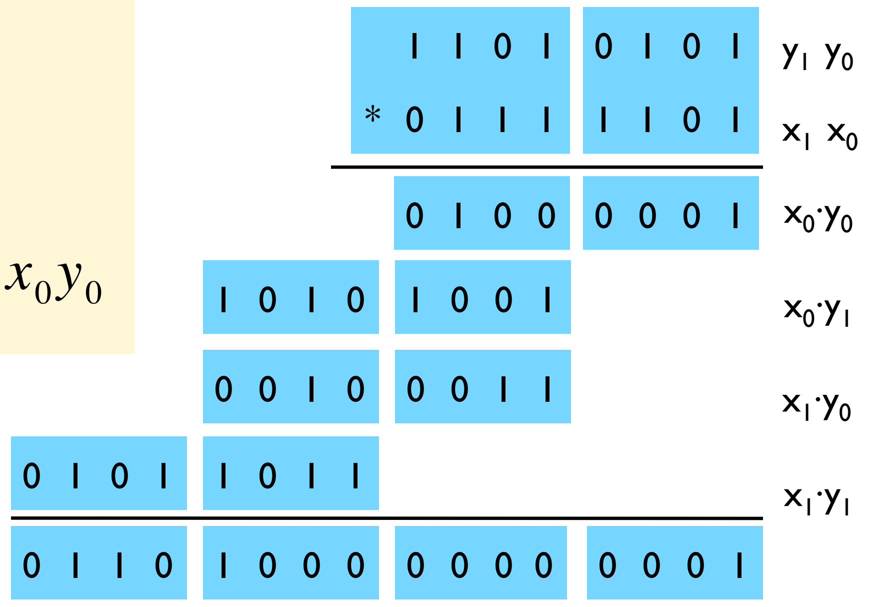
Shift/add four n-bit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)
$$

↑

assumes n is a power of 2



1 1 0 1 0 1 0 1 y_1 y_0
* 0 1 1 1 1 1 0 1 x_1 x_0
0 1 0 0 0 0 0 1 $x_0 \cdot y_0$
1 0 1 0 1 0 0 1 $x_0 \cdot y_1$
0 0 1 0 0 0 1 1 $x_1 \cdot y_0$
0 1 0 1 1 0 1 1 $x_1 \cdot y_1$
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1

key trick: 2 multiplies for the price of 1:

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \boxed{\left(x_1 y_0 + x_0 y_1\right)} + x_0 y_0
\end{aligned}
$$

Well, ok, 4 for 3 is more accurate…

$$
\begin{aligned}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\
&= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
\left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0
\end{aligned}
$$

Karatsuba multiplication

To multiply two n-bit integers:

- Add two pairs of ½n bit integers.
- Multiply *three* pairs of ½n-bit integers.
- Add, subtract, and shift n-bit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1 \right) + x_0 y_0 \\
&= 2^n \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \left(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \right) + \underset{C}{x_0 y_0}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in O($n^{1.585}$) bit operations.

$$
\mathrm{T}(n) \le \underbrace{T\left(\lfloor n/2 \rfloor \right) + T\left(\lceil n/2 \rceil \right) + T\left(1 + \lceil n/2 \rceil \right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

$$
\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)
$$

$$
\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})
$$

Karatsuba multiplication

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$\mathrm{T}(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1 + \lceil n/2 \rceil\big)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

$$\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)$$

$$\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

Best: Solve the exact recurrence

2nd best: Solve the sloppy version

Almost always gives the right asymptotics

Alternatively, you can often change the algorithm to simplify the recurrence so that you can solve it. E.g., "sloppy" = exact if

Get rid of $\lceil \bullet \rceil$ and $\lfloor \bullet \rfloor$ by "padding" n to $2^{\lceil \log_2 n \rceil}$

Get rid of $\lceil 1 + n/2 \rceil$ by peeling off one bit: T(n/2)+O(n)

Karatsuba multiplication

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in O($n^{1.585}$) bit operations.

$$\mathrm{T}(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1+\lceil n/2 \rceil\big)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

$$\textit{Sloppy version}:\ T(n) \le 3T(n/2) + O(n)$$

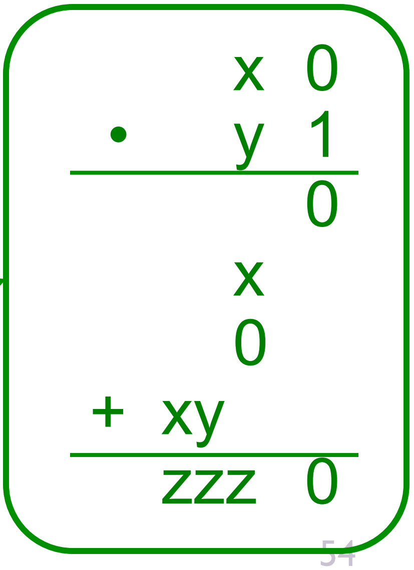
$$\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

Alternatively, you can often change the algorithm to simplify the recurrence so that you can solve it. E.g., "sloppy" becomes exact if

Kill $\lceil\bullet\rceil$ & $\lfloor\bullet\rfloor$: pad n to $2^{\lceil \log_2 n \rceil}$

Kill $\lceil 1 + n/2 \rceil$: peel off one bit: T(n/2)+O(n)

e.g.



multiplication – the bottom line

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems $\rightarrow \Theta(n^{1.46\ldots})$

Best known: $\Theta(n \log n \log\log n)$

- "Fast Fourier Transform"
- but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

High precision arithmetic *IS* important for crypto, among other uses

Recurrences

Above: Where they come from, how to find them

Next: how to solve them

Mergesort: (recursively) sort 2 half-lists, then merge results.

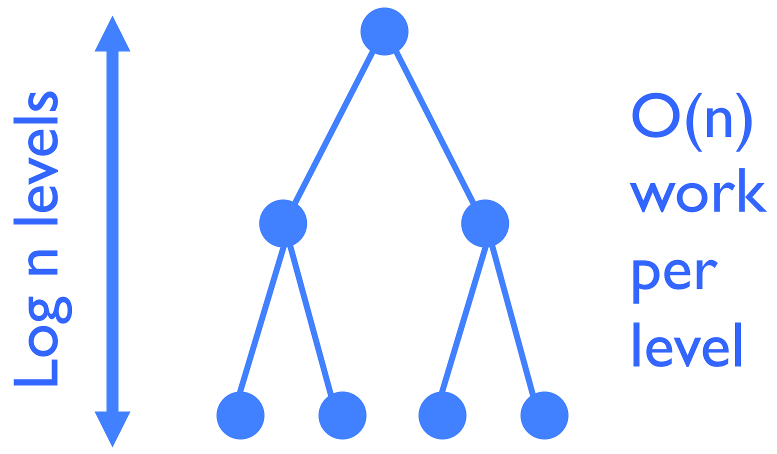
$T(n) = 2T(n/2)+cn, \quad n \geq 2$

$T(1) = 0$

Solution: $\Theta(n \log n)$

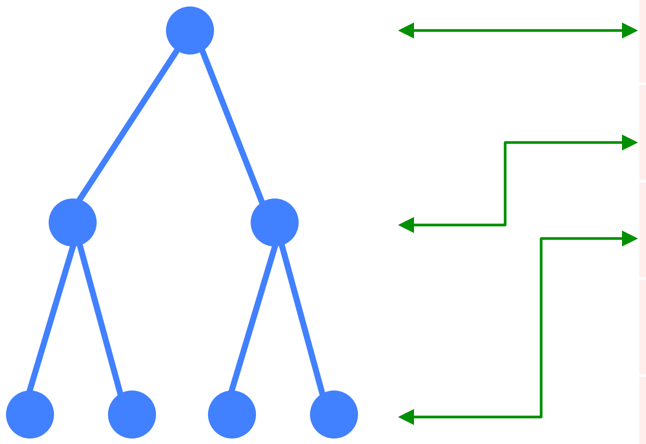
(~~details~~ later)

now!



Solve: $T(1) = c$

$T(n) = 2\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 2^0$	n	cn
1	$2 = 2^1$	$n/2$	$2cn/2$
2	$4 = 2^2$	$n/4$	$4cn/4$
…	…	…	…
i	2^i	$n/2^i$	$2^i\ c\ n/2^i$
…	…	…	…
$k-1$	2^{k-1}	$n/2^{k-1}$	$2^{k-1}\ c\ n/2^{k-1}$
k	2^k	$n/2^k = 1$	$2^k\,T(1)$

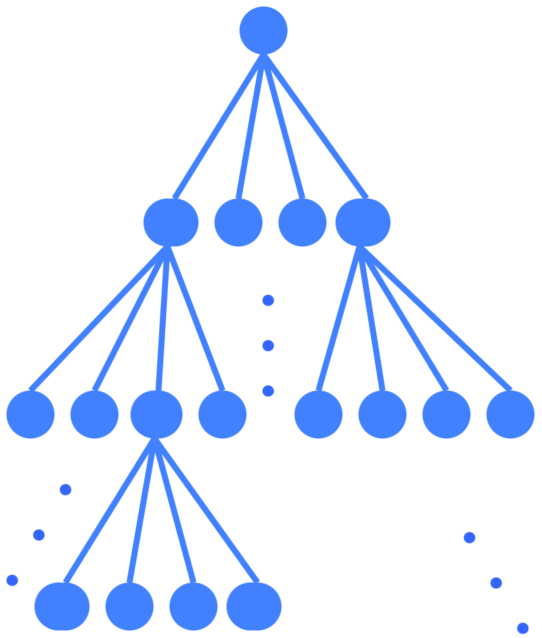
$n = 2^k$; $k = \log_2 n$

Total Work: $c\ n\ (1+\log_2 n)$

(add last col)

Solve: $T(1) = c$

$T(n) = 4\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 4^0$	n	cn
1	$4 = 4^1$	$n/2$	$4cn/2$
2	$16 = 4^2$	$n/4$	$16cn/4$
…	…	…	…
i	4^i	$n/2^i$	$4^i\ c\ n/2^i$
…	…	…	…
$k-1$	4^{k-1}	$n/2^{k-1}$	$4^{k-1}\ c\ n/2^{k-1}$
k	4^k	$n/2^k = 1$	$4^k\,T(1)$

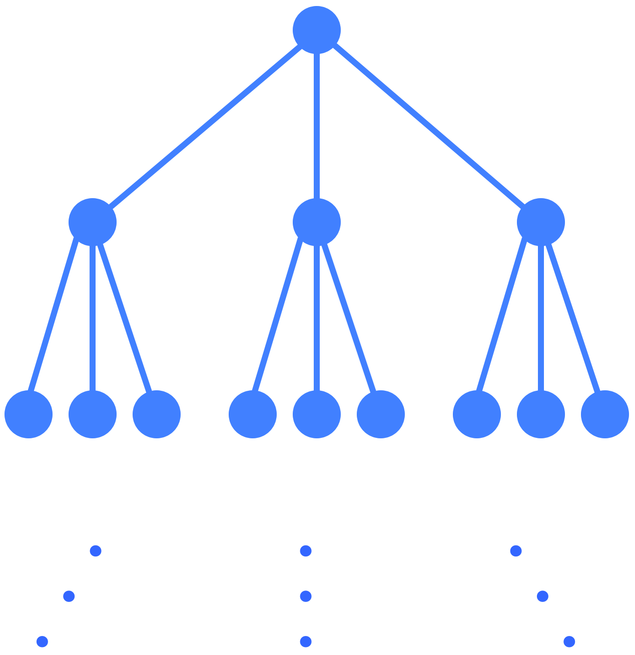
$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 4^i cn/2^i = O(n^2)$

$4^k = (2^2)^k = (2^k)^2 = n^2$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$



Level	Num	Size	Work
0	$1 = 3^0$	n	cn
1	$3 = 3^1$	$n/2$	$3cn/2$
2	$9 = 3^2$	$n/4$	$9cn/4$
…	…	…	…
i	3^i	$n/2^i$	$3^i\,c\,n/2^i$
…	…	…	…
$k-1$	3^{k-1}	$n/2^{k-1}$	$3^{k-1}\,c\,n/2^{k-1}$
k	3^k	$n/2^k = 1$	$3^k\,T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

a useful identity

Theorem: for $x \neq 1$,

$$1 + x + x^2 + x^3 + \ldots + x^k = (x^{k+1}-1)/(x-1)$$

proof:

$$\begin{aligned} y &= 1 + x + x^2 + x^3 + \ldots + x^k \\ xy &= \phantom{1 + {}} x + x^2 + x^3 + \ldots + x^k + x^{k+1} \\ xy-y &= x^{k+1} - 1 \\ y(x-1) &= x^{k+1} - 1 \\ y &= (x^{k+1}-1)/(x-1) \end{aligned}$$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$ (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$$

$$= cn \sum_{i=0}^{k} 3^i / 2^i$$

$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$

$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\boxed{\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)}$$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$ (cont.)

$$cn\frac{\left(\frac{3}{2}\right)^{k+1}-1}{\left(\frac{3}{2}\right)-1} = 2cn\left(\left(\frac{3}{2}\right)^{k+1}-1\right)$$

$$< 2cn\left(\frac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\frac{3}{2}\right)^{k}$$

$$= 3cn\frac{3^k}{2^k}$$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$ (cont.)

$$3cn\frac{3^k}{2^k} = 3cn\frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn\frac{3^{\log_2 n}}{n}$$

$$= 3c3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$\boxed{= O\left(n^{1.585\ldots}\right)}$$

$$a^{\log_b n}$$

$$= \left(b^{\log_b a}\right)^{\log_b n}$$

$$= \left(b^{\log_b n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

divide and conquer – master recurrence

$T(n) = aT(n/b)+cn^k$ for $n > b$ then

$a > b^k \Rightarrow T(n) = \Theta(n^{\log_b a})$ [many subprobs → leaves dominate]

$a < b^k \Rightarrow T(n) = \Theta(n^k)$ [few subprobs → top level dominates]

$a = b^k \Rightarrow T(n) = \Theta(n^k \log n)$ [balanced → all log n levels contribute]

Fine print:

$T(1) = d$; $a \geq 1$; $b > 1$; $c, d, k \geq 0$; $n = b^t$ for some $t > 0$; a, b, k, t integers. True even if it is $\lceil n/b \rceil$ instead of n/b.

master recurrence: proof sketch

Expand recurrence as in earlier examples, to get

$$T(n) = n^h (d + c S)$$

where $h = \log_b(a)$ (and n^h = number of tree leaves) and $S = \sum_{j=1}^{\log_b n} x^j$, where $x = b^k/a$.

If $c = 0$ the sum S is irrelevant, and $T(n) = O(n^h)$: all work happens in the base cases, of which there are n^h, one for each leaf in the recursion tree.

If $c > 0$, then the sum matters, and splits into 3 cases (like previous slide):

if $x < 1$, then $S < x/(1-x) = O(1)$.	[S is the first log n terms of the infinite series with that sum.]
if $x = 1$, then $S = \log_b(n) = O(\log n)$.	[All terms in the sum are 1 and there are that many terms.]
if $x > 1$, then $S = x \bullet (x^{1+\log_b(n)}-1)/(x-1)$.	[And after some algebra, $n^h * S = O(n^k)$.]

Another Example: Exponentiation

another d&c example: fast exponentiation

Power(a,n)

Input: integer n and number a

Output: a^n

Obvious algorithm

n-1 multiplications

Observation:

if n is even, $n = 2m$, then $a^n = a^m \bullet a^m$

divide & conquer algorithm

```
Power(a,n)
   if $n = 0$ then return(1)
   if $n = 1$ then return(a)
   $x \leftarrow$ Power(a,$\lfloor n/2 \rfloor$)
   $x \leftarrow x \bullet x$
   if n is odd then
        $x \leftarrow a \bullet x$
   return(x)
```

Let M(n) be number of multiplies

Worst-case recurrence:

$$M(n) = \begin{cases} 0 & n \le 1 \\ M\left(\lfloor n/2 \rfloor\right) + 2 & n > 1 \end{cases}$$

By master theorem

M(n) = O(log n)   (a=1, b=2, k=0)

More precise analysis:

M(n) = $\lfloor \log_2 n \rfloor$ + (# of 1's in n's binary representation) - 1

Time is O(M(n)) if numbers < word size, else also depends on length, multiply algorithm

a practical application - RSA

Instead of a^n want a^n mod N

- $a^{i+j} \bmod N = ((a^i \bmod N) \cdot (a^j \bmod N)) \bmod N$
- same algorithm applies with each $x \cdot y$ replaced by
  - $((x \bmod N) \cdot (y \bmod N)) \bmod N$

In RSA cryptosystem (widely used for security)

- need a^n mod N where a, n, N each typically have 1024 bits
- Power: at most 2048 multiplies of 1024 bit numbers
  - relatively easy for modern machines
- Naive algorithm: 2^{1024} multiplies

d & c summary

Utility:

- Correctness often easy; often faster

Idea:

- “Two halves are better than a whole”
  - if the base algorithm has super-linear complexity.
- “If a little's good, then more's better”
  - repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

- Binary Search, Merge Sort, (Quicksort), Closest Points, Integer Multiply, Exponentiation,…