CSE 421
Algorithms

Huffman Codes:
An Optimal Data Compression Method

Compression Example

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits

$2^3 > 6$; 3 bits/char: 300kbits

Why?

Storage, transmission vs 5 Ghz cpu

Compression Example

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100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits

$2^3 > 6$; 3 bits/char: 300kbits

better: →

2.52 bits/char 74%*2 +26%*4: 252kbits

Optimal?

E.g.:		Why not:
a	00	00
b	01	01
d	10	10
c	1100	110
e	1101	1101
f	1110	1110

1101110 = cf or ec?

Data Compression

Binary character code ("code")

- each k-bit *source string* maps to unique *code word* (e.g. k=8)
- "compression" alg: concatenate code words for successive k-bit "characters" of source

Fixed/variable length codes

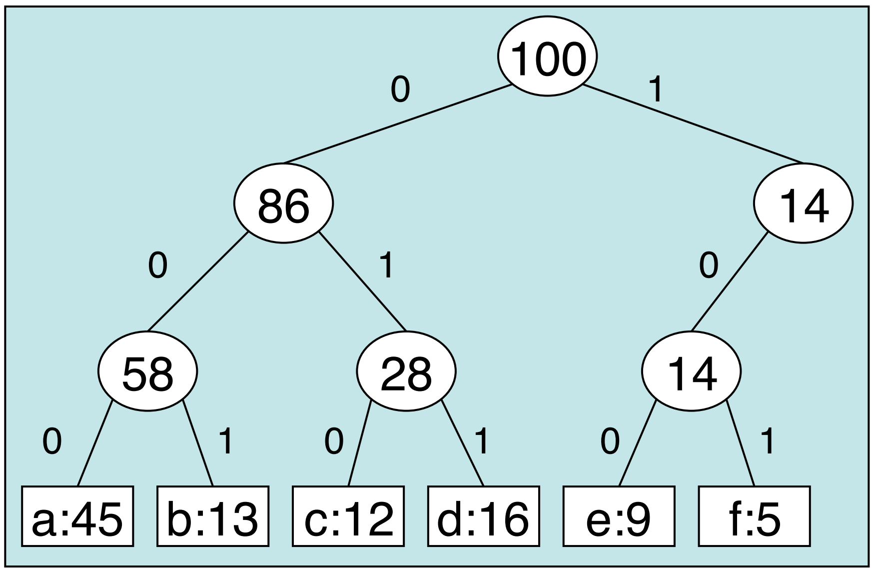
- all code words equal length?

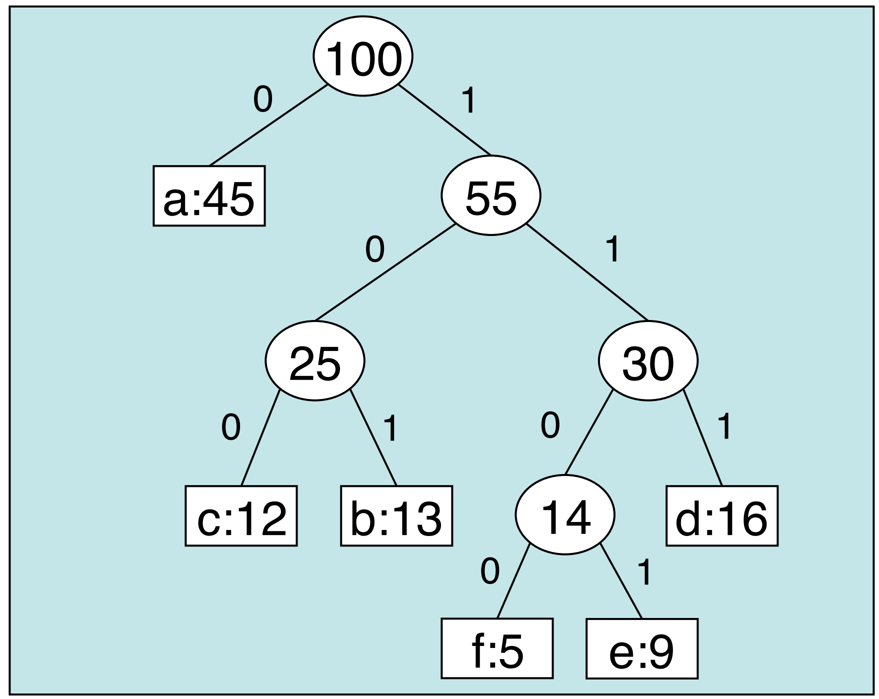
Prefix codes

- no code word is prefix of another (unique decoding)

Prefix Codes = Trees

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%





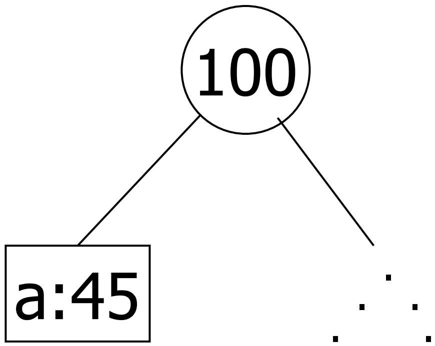
$$\underbrace{1\ 0\ 1}_{f}\ \underbrace{0\ 0\ 0}_{a}\ \underbrace{0\ 0\ 1}_{b}$$

$$\underbrace{1\ 1\ 0\ 0}_{f}\ \underbrace{0}_{a}\ \underbrace{1\ 0\ 1}_{b}$$

Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

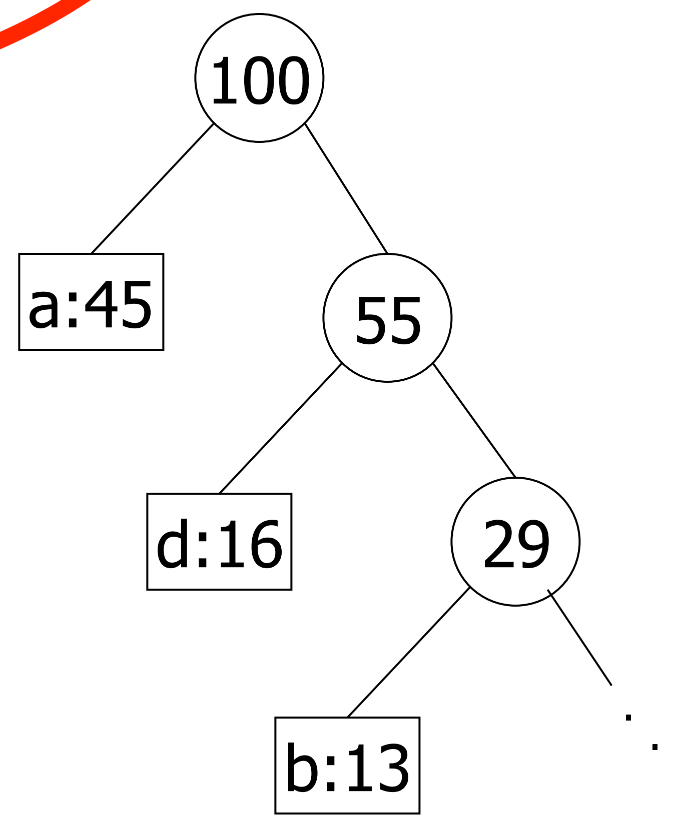
Put <u>most</u> frequent
under root, then recurse …



Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Top down: Put *most* frequent under root, then recurse



Too greedy: unbalanced tree

.45*1 + .16*2 + .13*3 … = 2.34

not too bad, but imagine if all freqs were ~1/6:

(1+2+3+4+5+5)/6=3.33

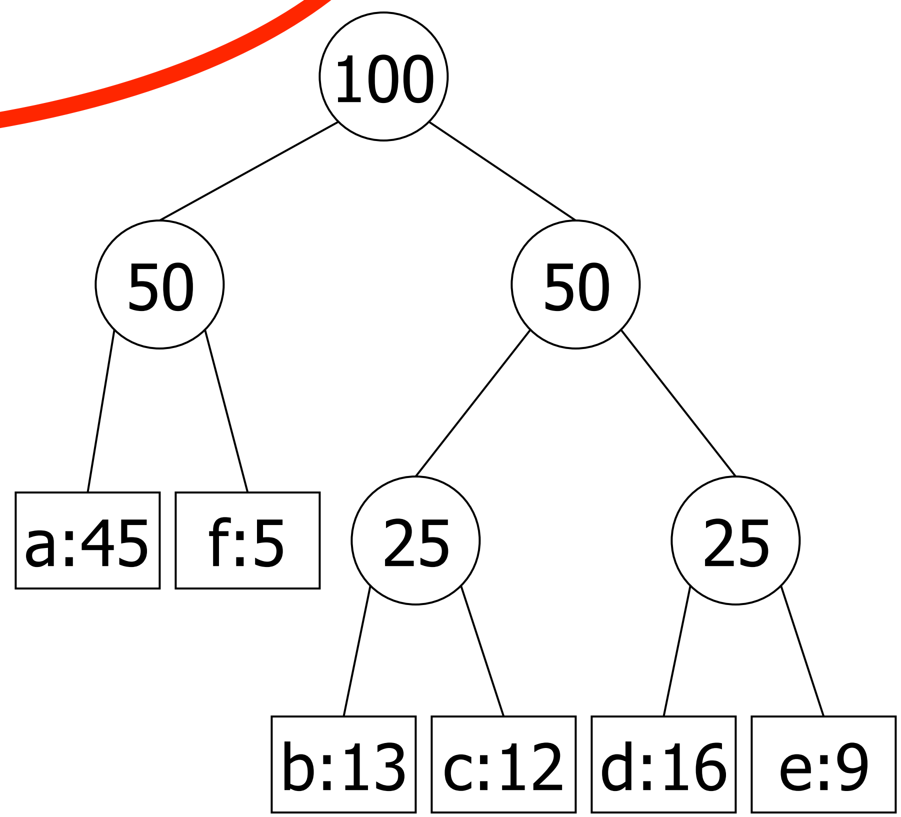
Greedy Idea #2

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Top down: Divide letters into 2 groups, with ~50% weight in each; recurse
(Shannon-Fano code)

Again, not terrible
2*.5+3*.5 = 2.5

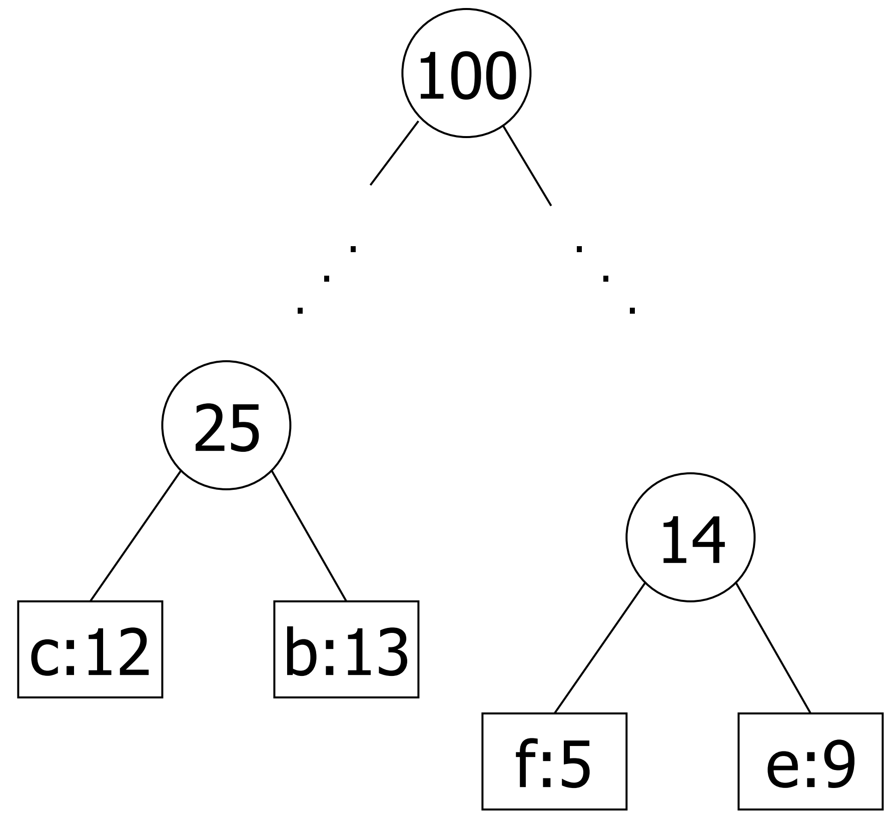
But this tree can easily be improved! (How?)

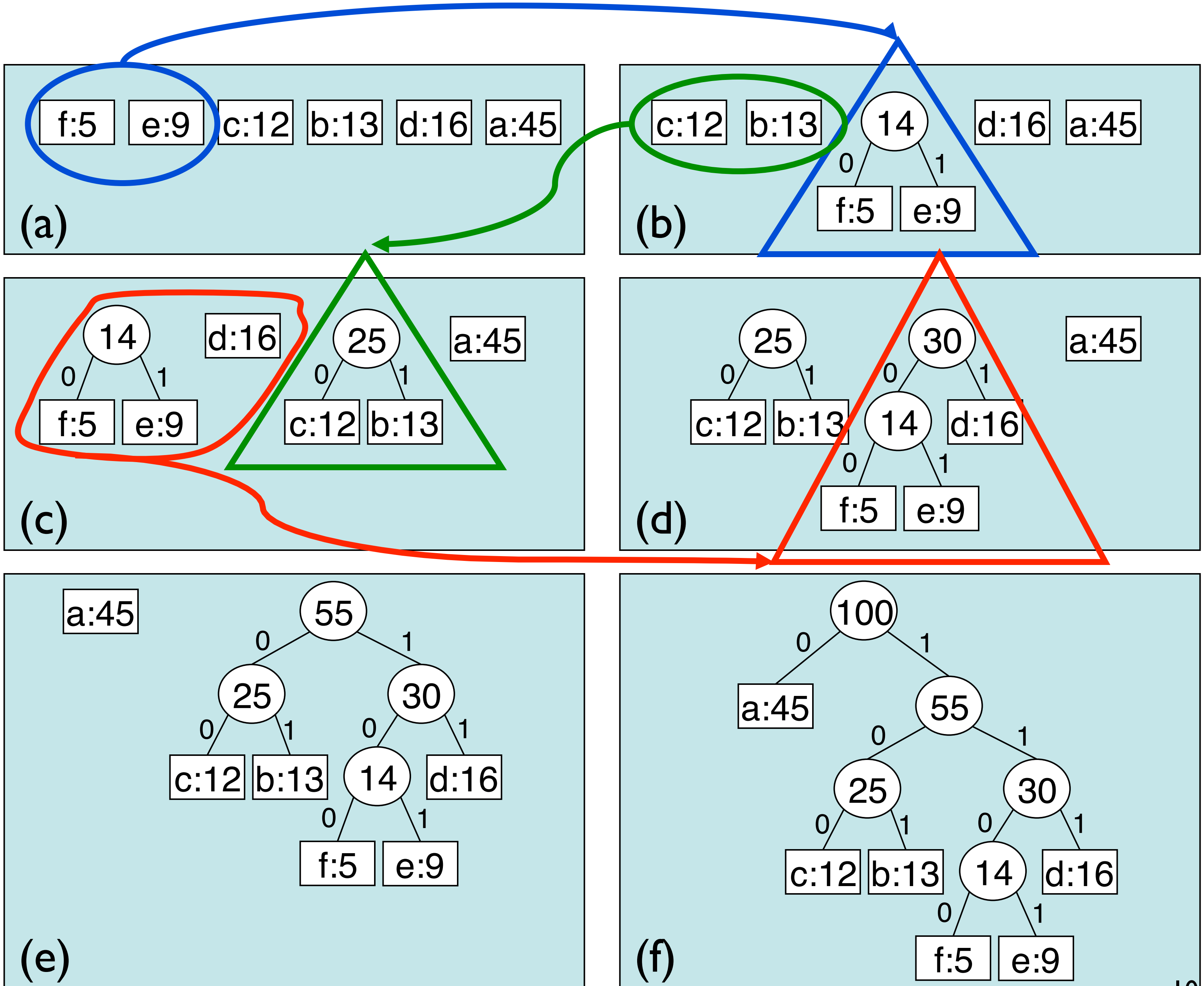


Greedy idea #3

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Bottom up: Group *least* frequent letters near bottom





.45*1 + .41*3 + .14*4 = 2.24 bits per char

Huffman's Algorithm (1952)

Algorithm:

insert node for each letter into priority queue by freq
while queue length > 1 do
    remove smallest 2; call them x, y
    make new node z from them, with f(z) = f(x) + f(y)
    insert z into queue

Analysis: O(n) heap ops: O(n log n)

Goal: Minimize $B(T) = \sum_{c \in C} \mathrm{freq}(c) * \mathrm{depth}(c)$

T = Tree
C = alphabet

Correctness: ???

Correctness Strategy

Optimal solution may not be unique, so cannot prove that greedy gives the *only* possible answer.

Instead, show greedy's solution is as good as any.

How: an exchange argument

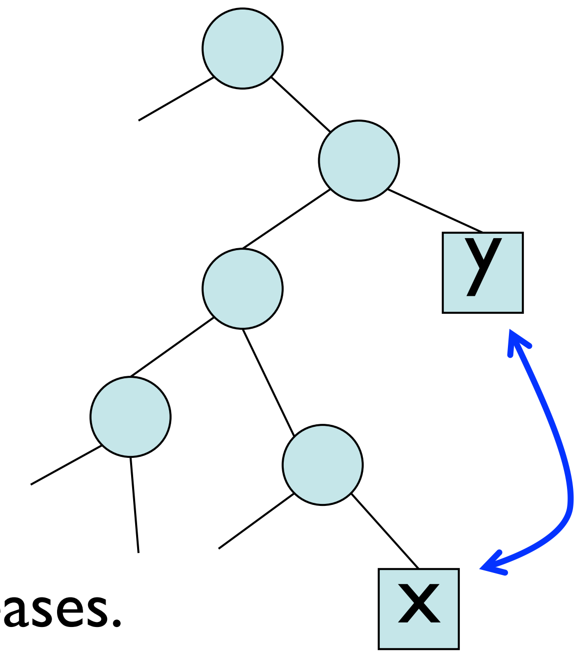
- Identify *inversions:* node-pairs whose swap improves tree
- To compare trees T (arbitrary) to H (Huffman): run Huff alg, tracking subtrees in common to T & H; discrepancies flag inversions; swapping them incrementally xforms T to H

Defn: A pair of leaves x,y is an inversion if

depth(x) ≥ depth(y)

and

freq(x) ≥ freq(y)



Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

$$\underbrace{(d(x)*f(x) + d(y)*f(y))}_{\text{before}} - \underbrace{(d(x)*f(y) + d(y)*f(x))}_{\text{after}} =$$

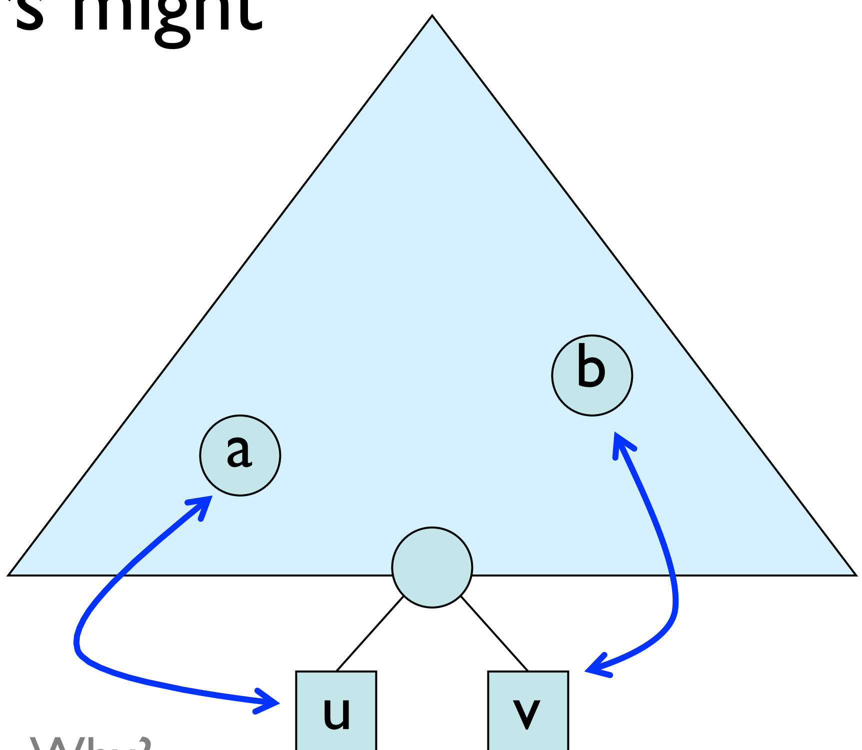
$$(d(x) - d(y)) * (f(x) - f(y)) \geq 0$$

I.e., non-negative cost savings.

Lemma 1:
"Greedy Choice Property"

The 2 least frequent letters might as well be siblings

- Let a be least freq, b 2[nd]
- Let u, v be siblings at max depth, $f(u) \le f(v)$ (why must they exist?)
- Then (a,u) and (b,v) are inversions. Swap them.



NB: $f(a) \le f(u) \le f(v) < f(b)$ impossible. Why?

Why Important? Algorithm is not wrong to join them.

Lemma 2

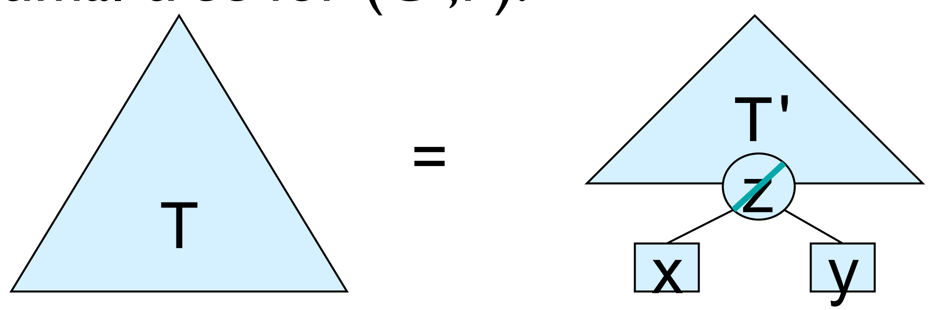
Let (C, f) be a problem instance: C an n-letter alphabet with letter frequencies f(c) for c in C.

For any x, y in C, z not in C, let C' be the (n-1) letter alphabet C - {x,y} ∪ {z} and for all c in C' define

$$f'(c) = \begin{cases} f(c), & \text{if } c \neq x,y,z \\ f(x)+f(y), & \text{if } c = z \end{cases}$$

Let T' be an optimal tree for (C',f').

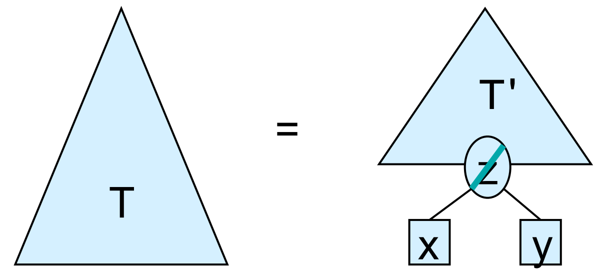
Then



is optimal for (C,f) among all trees having x,y as siblings

Why Important? Algorithm is not wrong to treat x:y as z.

To show: T' opt for C' $\Rightarrow$ T opt for C/x/y



Proof:

$$B(T) = \sum_{c \in C} d_T(c) \cdot f(c)$$

$$B(T) - B(T') = d_T(x) \cdot (f(x) + f(y)) - d_{T'}(z) \cdot f'(z)$$
$$= (d_{T'}(z) + 1) \cdot f'(z) - d_{T'}(z) \cdot f'(z)$$
$$= f'(z)$$

Suppose $\hat{T}$ (having x & y as siblings) is better than T, i.e.

$B(\hat{T}) < B(T)$. Collapse x & y to z, forming $\hat{T}'$; as above:

$$B(\hat{T}) - B(\hat{T}') = f'(z)$$

Then:

$$B(\hat{T}') = B(\hat{T}) - f'(z) < B(T) - f'(z) = B(T')$$

Contradicting optimality of T'

Theorem:
Huffman gives optimal codes

Proof: induction on |C|

Basis: n=1,2 – immediate

Induction: n>2

- Let x,y be least frequent
- Form C′, f′, & z, as above
- By induction, T′ is opt for (C′,f′)
- By lemma 2, T′ →T is opt for (C,f) among trees with x,y as siblings
- By lemma 1, some opt tree has x, y as siblings
- Therefore, T is optimal.

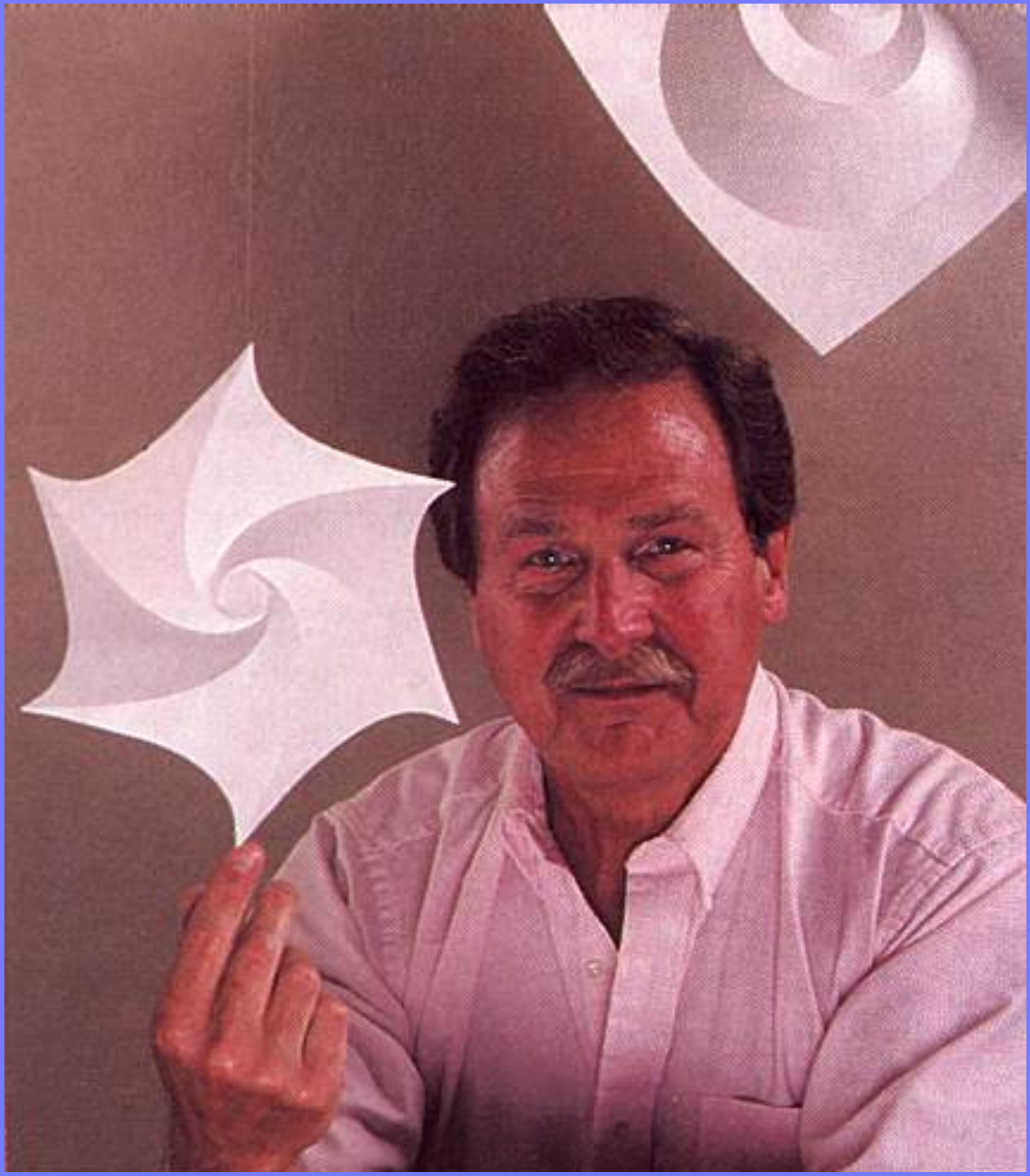
Data Compression

Huffman is optimal.

BUT still might do better!

- Huffman encodes fixed length blocks. What if we vary them?
- Huffman uses one encoding throughout a file. What if characteristics change?
- What if data has structure? E.g. raster images, video,…
- Huffman is lossless. Necessary?

LZW, MPEG, …



David A. Huffman, 1925-1999

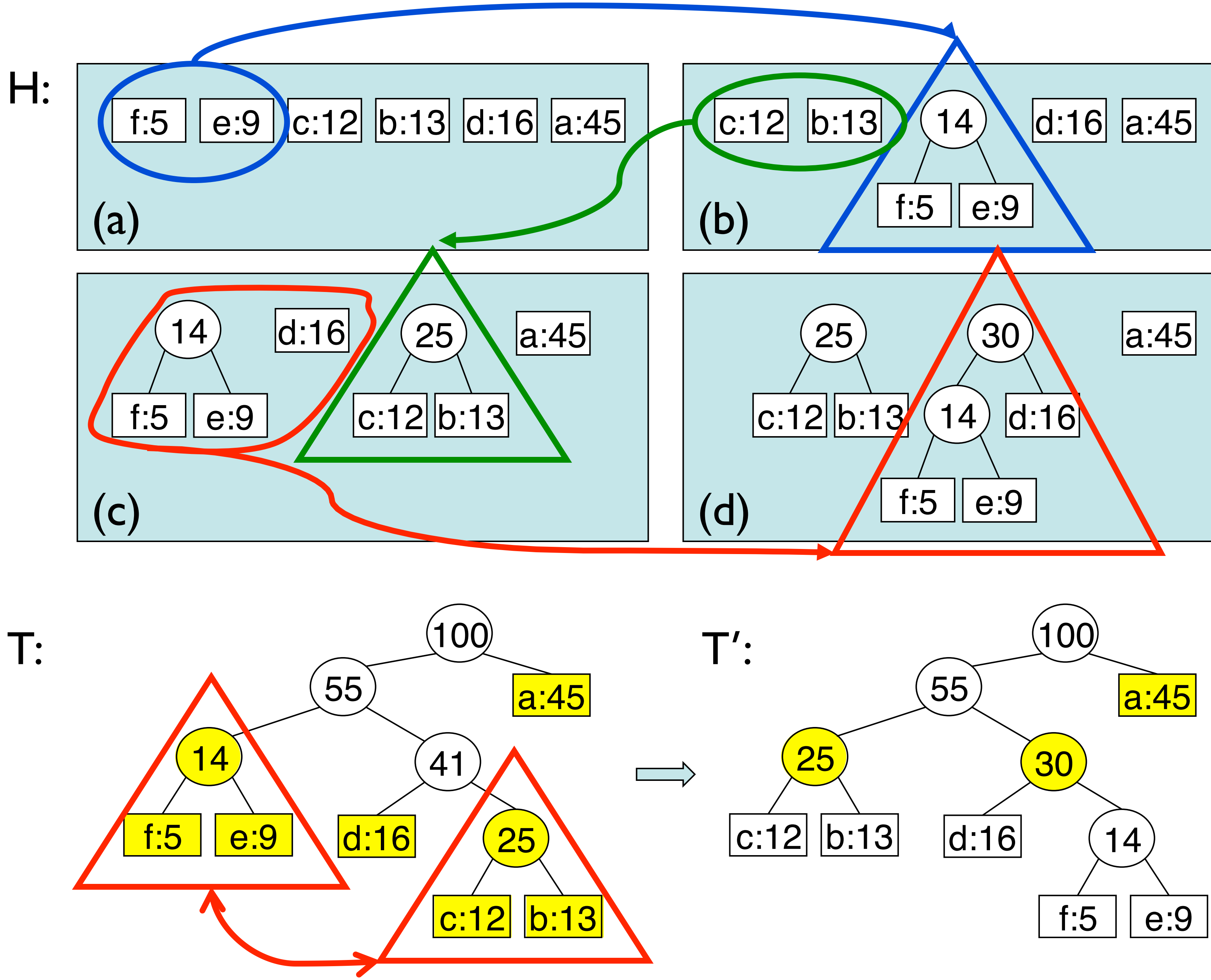
Students: I did NOT present the following slides in 2015 Winter, and you are not required to study them, but if you are interested, they are an alternate correctness proof for the Huffman algorithm, more closely following the usual "exchange argument" template: find a series of exchanges that can turn an arbitrary solution into the greedy solution without increasing cost at any step.

General Inversions

Exercise: generalize the inversion definition/proof on slide 13 to an arbitrary pair of nodes, where the frequency of an internal node is defined to be the sum of the frequencies of the leaves in that subtree.

Possibly useful: show that the contribution to the total cost due to all leaves in one subtree is the sum of the freqs of the internal nodes in that subtree plus d*f, where d = depth of subtree's root, f=its freq

The following slide is heavily animated, which doesn't show too well in print. The point is to illustrate the Lemma on the next slide. Idea is to run Huffman alg on the example above and compare successive subtrees it builds to subtrees in an arbitrary tree T. While they agree, repeat; when they first differ (in this case, when Huffman builds node 30), identify an inversion in T whose removal would allow them to agree for at least one more step, i.e., T' is more like H than T, but costs no more.



In short, where T first differs from H flags an inversion in T

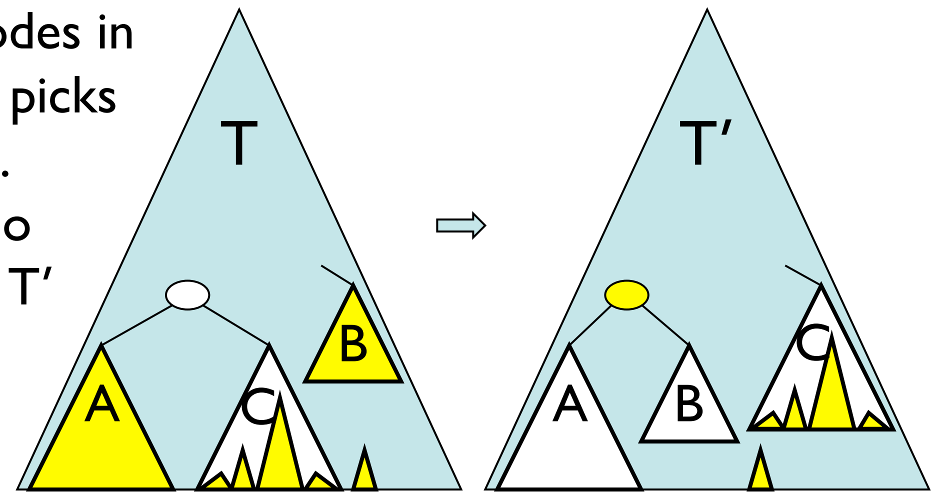
Lemma: Any prefix code tree T can be converted to a Huffman tree H via inversion-exchanges

Pf: Run Huffman alg; "color" nodes to track matching subtrees between T, H. Inductively: yellow nodes in T match subtrees in H that are in Huffman's heap at that stage in the alg. Initially: leaves yellow, rest white.

At each step, Huffman alg. extracts A, B (the 2 smallest items) from its heap.

Case 1: A, B match siblings in T. Then their newly created parent node in H corresponds to their parent in T; paint it yellow, A & B revert to white.

Case 2: A, B not sibs in T. WLOG, in T, depth(A) ≥ depth(B) & A is C's sib. Note B can't overlap C (B = C ⇒ case 1; B subtree of C contradicts depth). In T, the freq of C's root ≥ freqs of all yellow nodes in it (≠ ∅ since leaves are yellow). Huff's picks (A & B) were min, so freq(C) ≥ freq(B). ∴ B:C is an inversion–B is no deeper/no more frequent than C. Swapping gives T′ more like H; repeating ≤ n times converts T to H.



Theorem: Huffman is optimal

Pf: Apply the above lemma to any optimal tree T. The lemma only exchanges inversions, which never increase cost, so, cost(H) $\leq$ cost(T).