CSE 421: Intro Algorithms

2: Analysis

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Elaine presented an introduction to analysis and "big-O" on the whiteboard. Her notes are linked from the 421 web page.

The Powerpoint slides below supplement that (plus a bit of new material, especially "little-o").

Why big-O: measuring algorithm efficiency

What's big-O: definition and related concepts

Reasoning with big-O: examples & applications

- polynomials
- exponentials
- logarithms
- sums

Polynomial Time

Why big-O: measuring algorithm efficiency

Our correct TSP algorithm was incredibly slow

- No matter what computer you have

As a 2nd example, for large problems, mergesort beats insertion sort – n log n *vs* n^2 matters a lot

- Even tho the alg is more complex & inner loop is slower
- No matter what computer you have

We want a general theory of "efficiency" that is

- Simple
- Objective
- Relatively independent of changing technology
- Measures *algorithm*, not code
- But still *predictive* – "theoretically bad" algorithms should be bad in practice and vice versa (usually)

"Runs fast on typical real problem instances"

Pro:

sensible, bottom-line-oriented

Con:

moving target (diff computers, compilers, Moore's law)

highly subjective (how fast is "fast"? What's "typical"?)

"Runs fast on a specific suite of benchmarks"

Pro:

again sensible, bottom-line-oriented

Con:

all the problems above

are benchmarks representative

algorithms can be "tuned" to the well-known benchmarks

generating/maintaining benchmarks is a burden

benchmarking a new algorithm is a lot of work

Instead:

a) Give up on detailed timing, focus on *scaling*

Nanoseconds matter of course, but we often want to push to bigger problems tomorrow than we can solve today, so an algorithm that scales as n^2, say, will very likely beat one that grows as 2^n or n^{10} or even n^3, even if the later uses fewer nanoseconds for today's n.

b) Give up on "typical," focus on *worst case* behavior

Over all inputs of size n, how fast are we on the worst? Removes all debate about "typical" / "average."

Overall, these yield a big win in terms of technology independence, ease of analysis, robustness

The *time complexity* of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

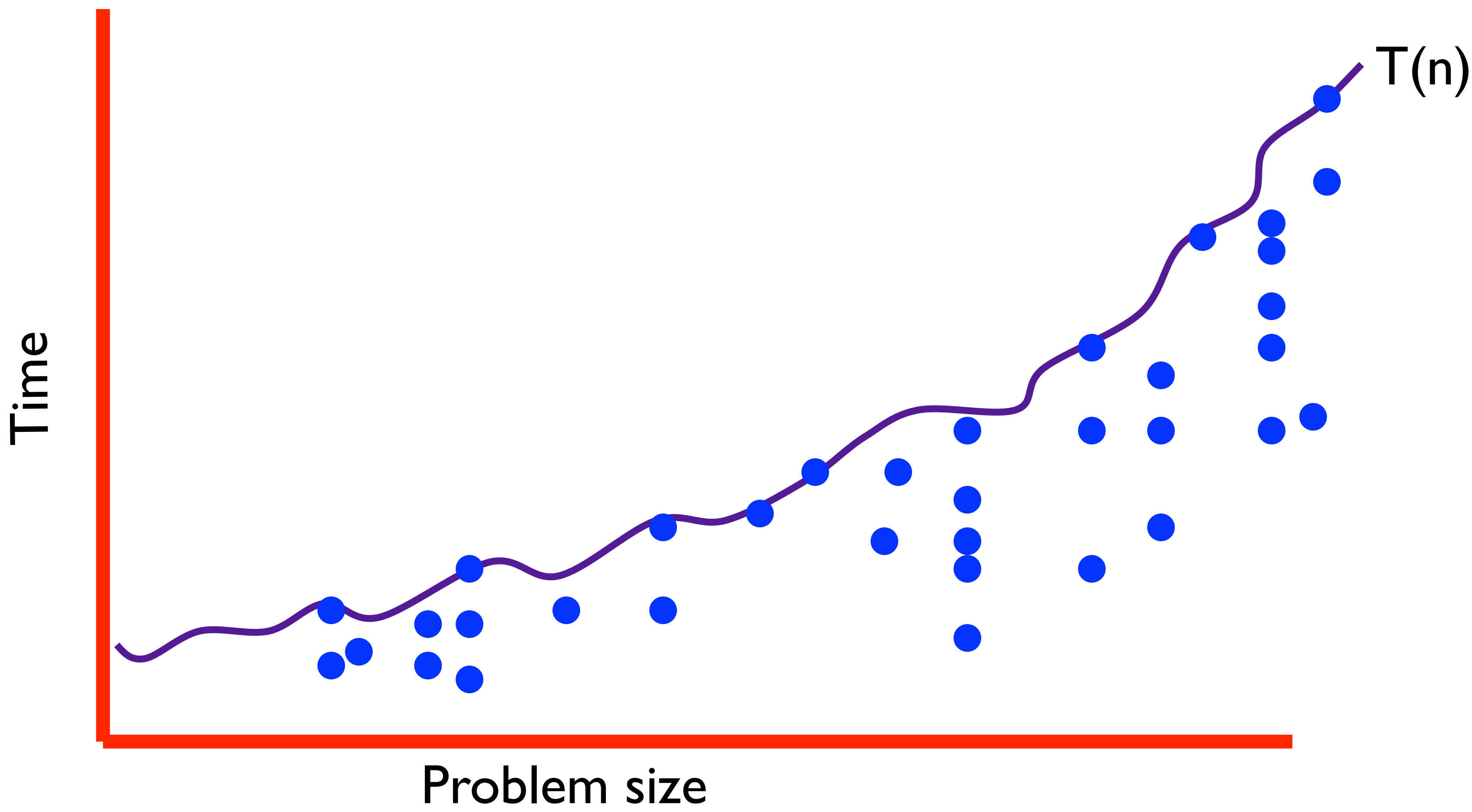
Mathematically,

T: N+ → R

i.e.,T is a function mapping positive integers (problem sizes) to positive real numbers (number of steps).

"Reals" so, e.g., we can say sqrt(n) instead of ⌈sqrt(n)⌉

"Positive" so, e.g., log(n) and 2^n/n aren't problematic


T(n)
Time
Problem size

Appropriate for time-critical applications

- E.g. avionics, nuclear reactors

Unlike Average-Case, no debate over the right definition

- If worst $\gg$ average, then (a) alg is doing something pretty subtle, & (b) are hard instances really that rare?

Analysis often much easier

Result is often representative of "typical" problem instances

Of course there are exceptions…

Asymptotic growth rate, i.e., characterize growth rate of worst-case run time as a function of problem size, up to a constant factor, e.g. $T(n) = O(n^2)$

Why not try to be more precise?

Average-case, e.g., is hard to define, analyze

Technological variations (computer, compiler, OS, …) easily 10x or more

Being more precise is *much* more work

A key question is "*scale up*": if I can afford this today, how much longer will it take when my business is 2x larger? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.) Big-O analysis is adequate to address this.

Big-O: a math notation for an upper bound on the asymptotic growth rate of a function

E.g., if f(n) = value of the n^{th} prime, f(n) = O(n log n)

In CS, commonly used to describe run time of algorithms, usually worst case run time, but could be other run time functions.

E.g., for Quicksort

$T_{best}(n) = O(n)$

$T_{avg}(n) = O(n \log n)$

$T_{worst}(n) = O(n^2)$

What's big-O: definition and related concepts

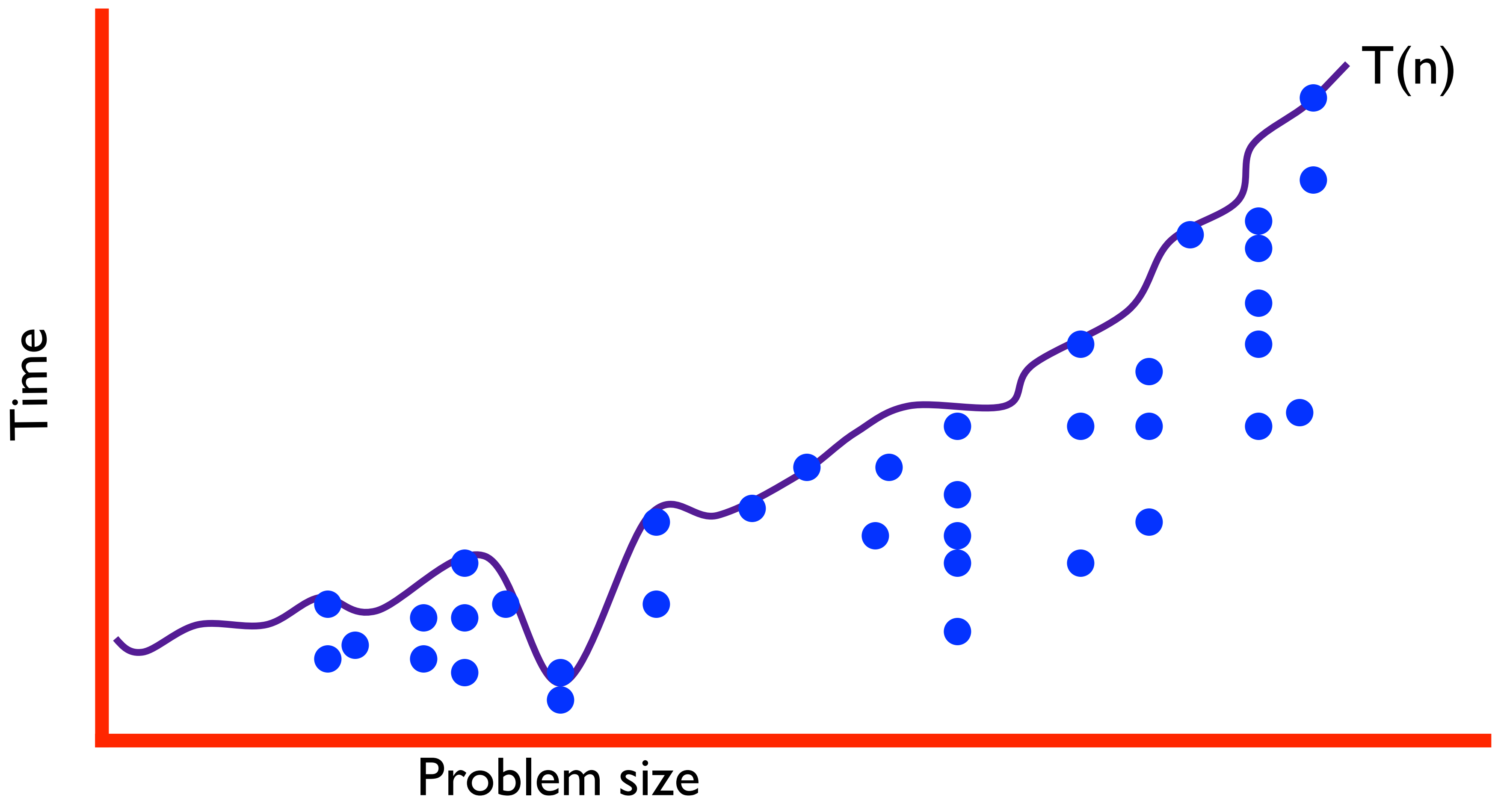
Given two functions f and g: N+ → R

f(n) is O(g(n)) iff there is a constant c > 0 so that f(n) is eventually always ≤ c g(n) — **Upper Bounds**

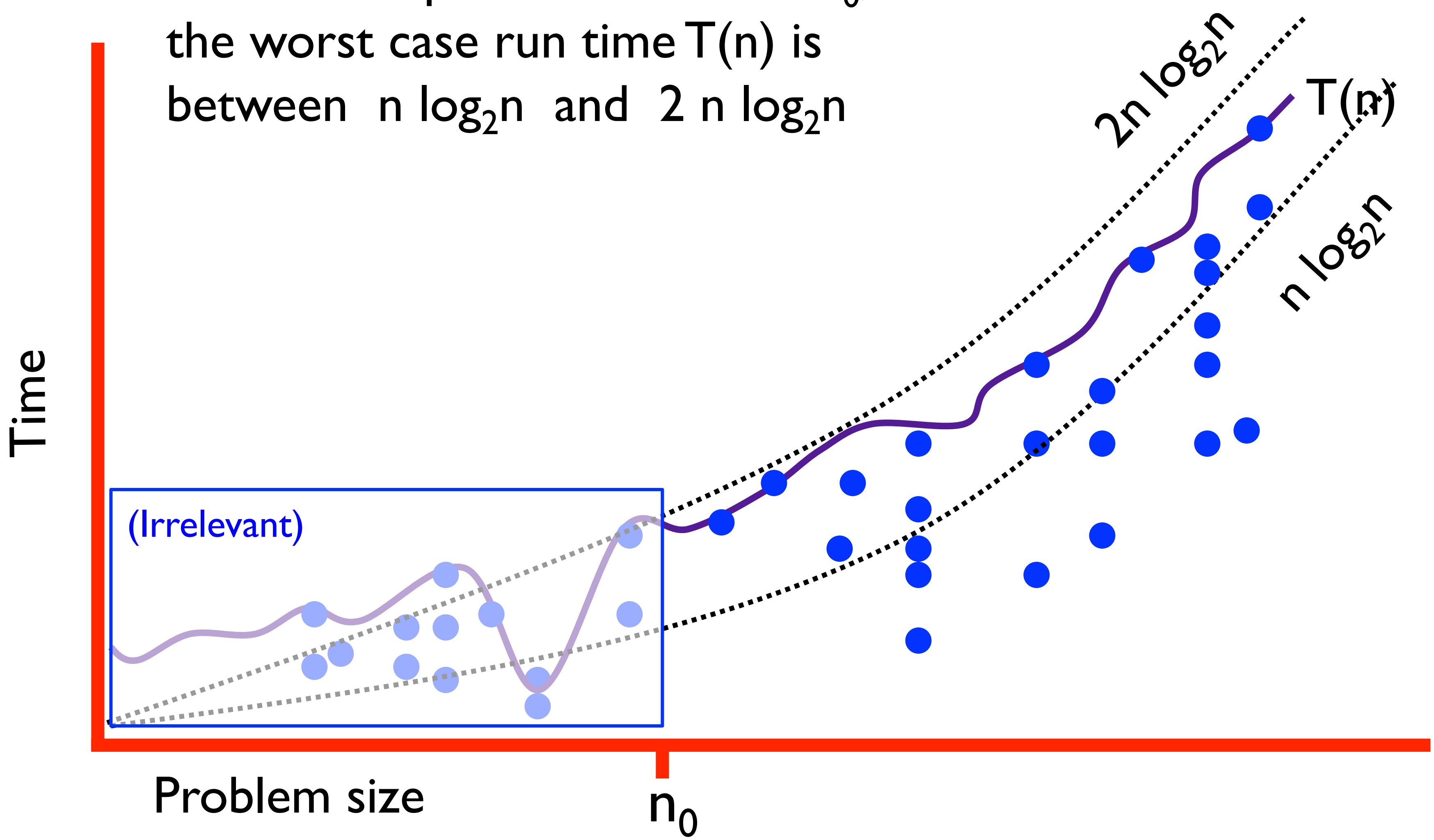
f(n) is Ω(g(n)) iff there is a constant c > 0 so that f(n) is eventually always ≥ c g(n) — **Lower Bounds**

f(n) is Θ(g(n)) iff there is are constants c_1, $c_2 > 0$ so that eventually always $c_1 g(n) \le f(n) \le c_2 g(n)$ — **Both**

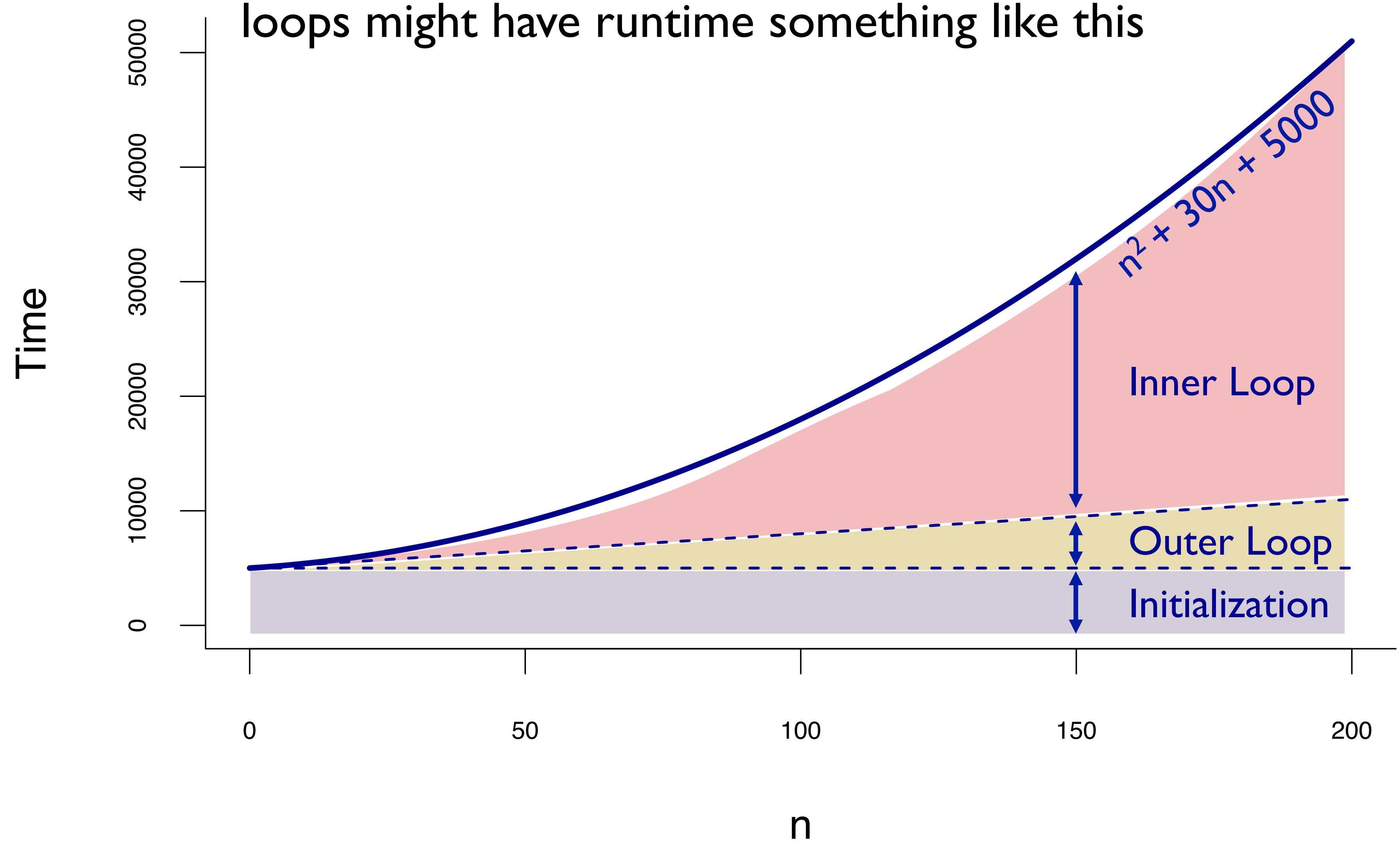
"Eventually always P(n)" means "$\exists n_0$ s.t. $\forall n > n_0$ P(n) is true." I.e., there can be exceptions, but only for finitely many "small" values of n.


T(n)
Time
Problem size

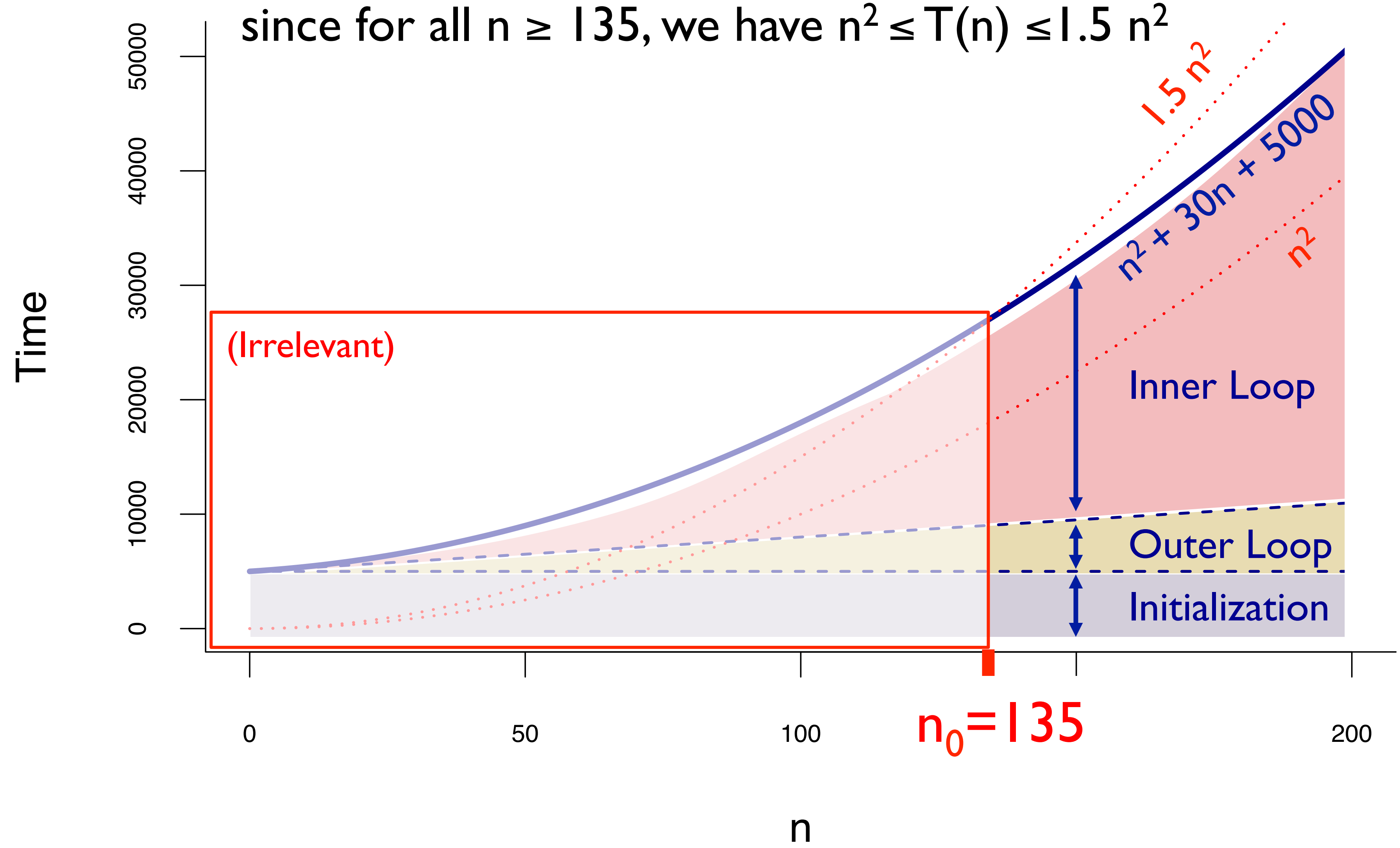
Example: $T(n) = \Theta(n \log_2 n)$
since for all problem sizes $n > n_0$,
the worst case run time $T(n)$ is
between $n \log_2 n$ and $2 n \log_2 n$



A typical program with initialization and two nested loops might have runtime something like this



If $T(n) = n^2 + 30n + 5000$, then $T(n) = \Theta(n^2)$,
since for all $n \geq 135$, we have $n^2 \leq T(n) \leq 1.5\, n^2$



Reasoning with big-O: examples & applications

polynomials

exponentials

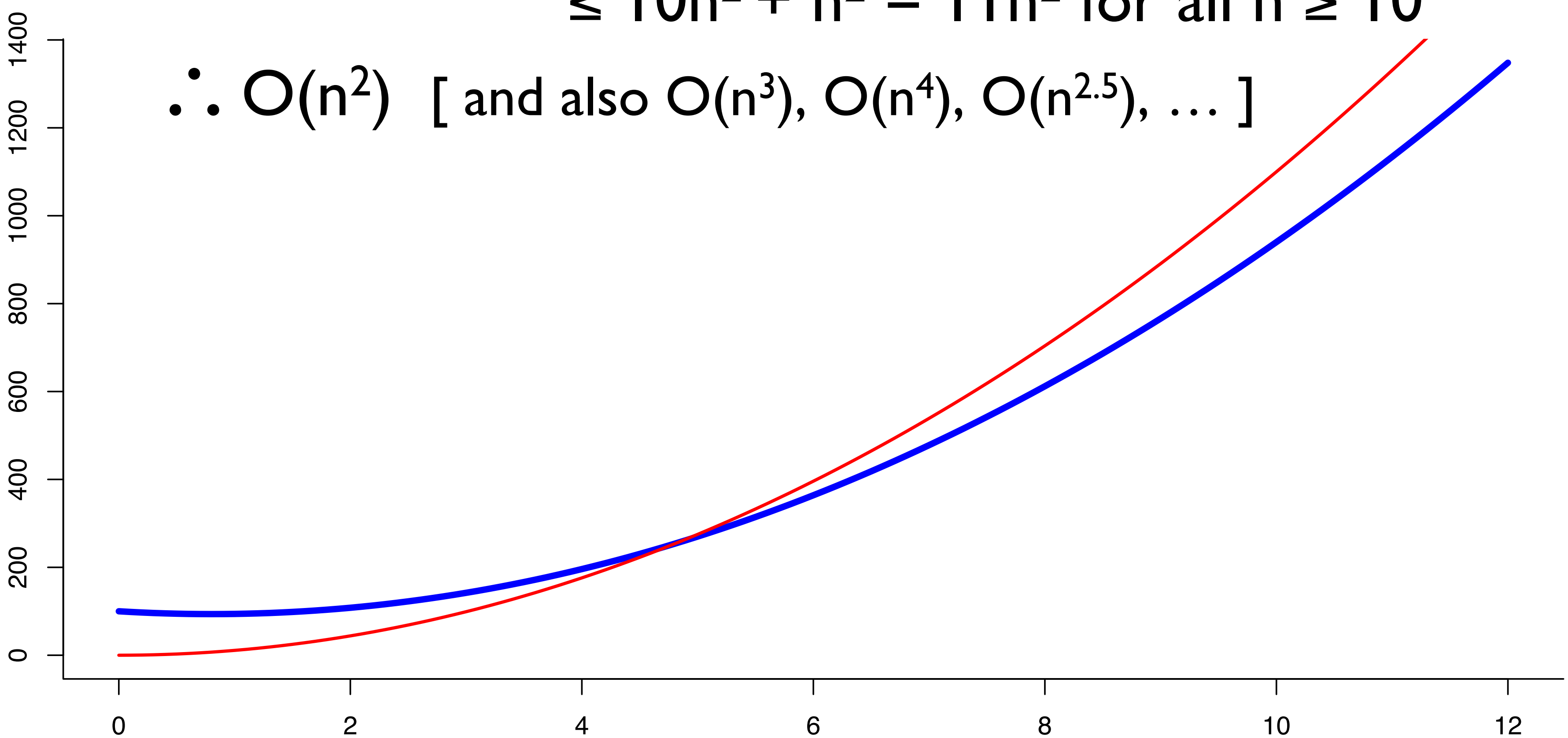
logarithms

sums

Show $10n^2-16n+100$ is $O(n^2)$:

$$
\begin{aligned}
10n^2-16n+100 &\le 10n^2 + 100 \\
&= 10n^2 + 10^2 \\
&\le 10n^2 + n^2 = 11n^2 \text{ for all } n \ge 10
\end{aligned}
$$

$\therefore O(n^2)$ [and also $O(n^3)$, $O(n^4)$, $O(n^{2.5})$, …]



Show $10n^2$-$16n+100$ is $\Omega(n^2)$:

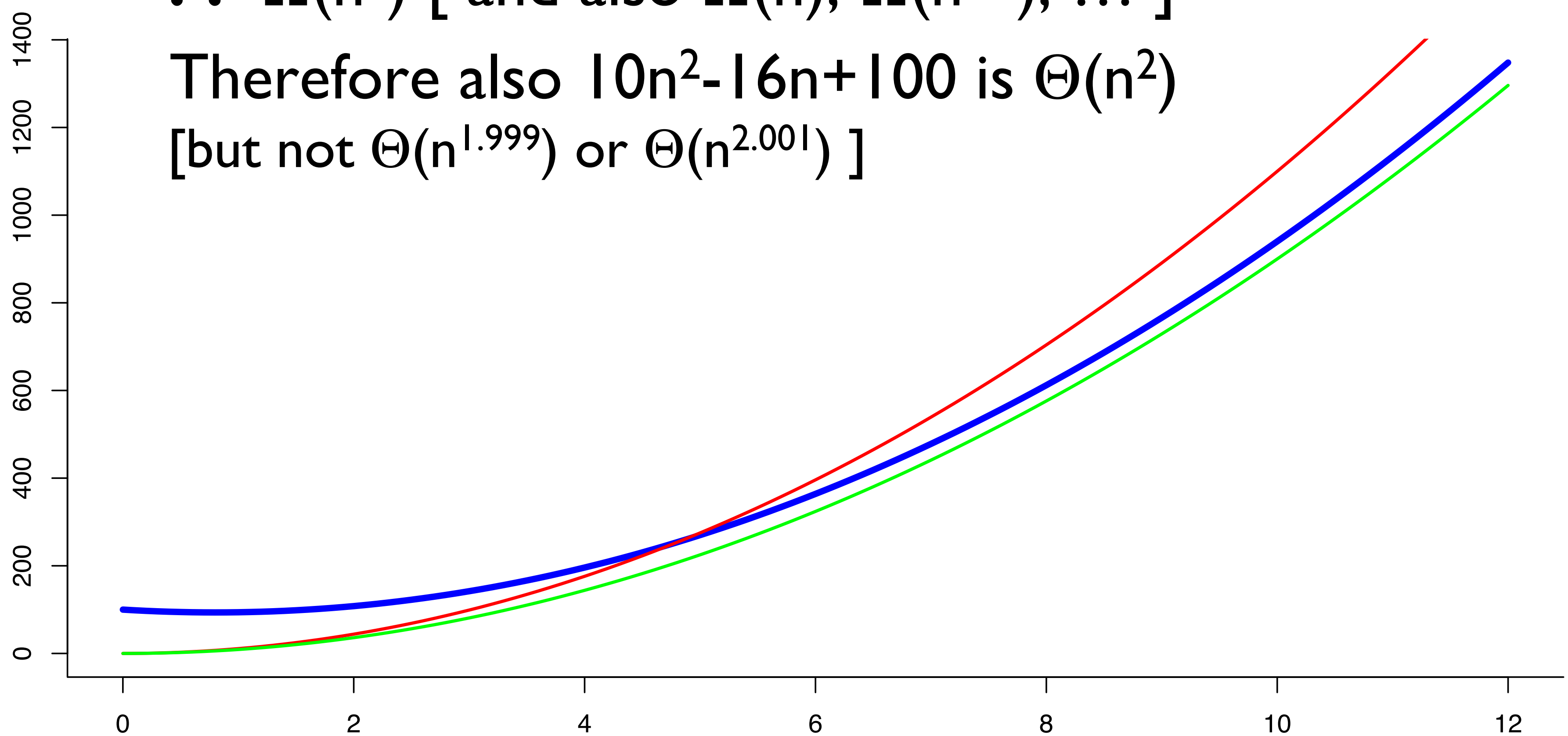
$10n^2 - 16n + 100 \geq 10n^2 - 16n$

$\geq 10n^2 - n^2 = 9n^2$ for all $n \geq 16$

$\therefore$ $\Omega(n^2)$ [and also $\Omega(n)$, $\Omega(n^{1.5})$, …]

Therefore also $10n^2$-$16n+100$ is $\Theta(n^2)$

[but not $\Theta(n^{1.999})$ or $\Theta(n^{2.001})$]



Polynomials:

$p(n) = a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Proof:

$$\begin{aligned} p(n) &= a_0 + a_1 n + \ldots + a_d n^d \\ &\le |a_0| + |a_1| n + \ldots + a_d n^d \\ &\le |a_0| n^d + |a_1| n^d + \ldots + a_d n^d \qquad \text{(for } n \ge 1\text{)} \\ &= c\, n^d, \text{ where } c = (|a_0| + |a_1| + \ldots + |a_{d-1}| + a_d) \end{aligned}$$

$\therefore p(n) = O(n^d)$

Exercise: show that $p(n) = \Omega(n^d)$

Hint: this direction is trickier; focus on the "worst case" where all coefficients except a_d are negative.

another example of working with O-Ω-Θ notation

Example: For any a, and any $b > 0$, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \le (2n)^b$ for $n \ge |a|$

$= 2^b n^b$

$= cn^b$ for $c = 2^b$

so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if $a < 0$)

$= 2^{-b} n^b$

$= c'n$ for $c' = 2^{-b}$

so $(n+a)^b$ is $\Omega\ (n^b)$

Example: $\sum_{1 \leq i \leq n} i = \Theta(n^2)$

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E.g.:  for i = 1..n {
         for j=1 to i {
            . . .
       }}
```

Proof:

(a) An upper bound: each term is ≤ the max term

$$\sum_{1 \leq i \leq n} i \leq \sum_{1 \leq i \leq n} n = n^2 = O(n^2)$$

(b) A lower bound: each term is ≥ the min term

$$\sum_{1 \leq i \leq n} i \geq \sum_{1 \leq i \leq n} 1 = n = \Omega(n)$$

This is valid, but a weak bound.

Better: pick a large subset of large terms

$$\sum_{1 \leq i \leq n} i \geq \sum_{n/2 \leq i \leq n} n/2 \geq \lfloor n/2 \rfloor^2 = \Omega(n^2)$$

Transitivity.

If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.

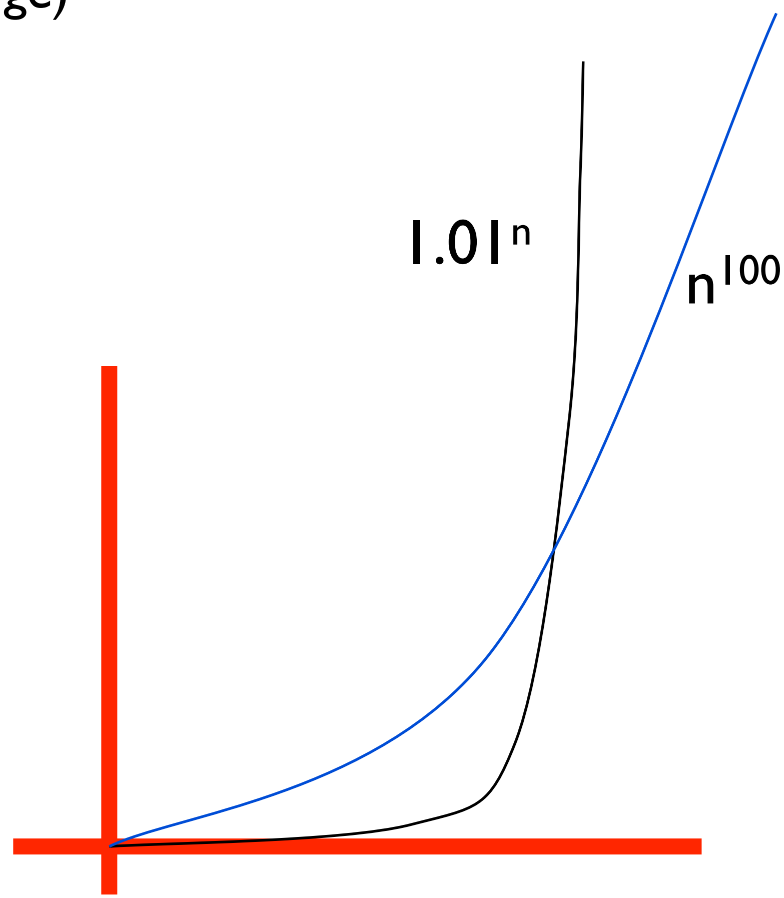
If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Useful, e.g., for analyzing programs with subroutines.

Proofs are left as exercises.

For all $r > 1$ (no matter how small)
and all $d > 0$, (no matter how large)
$n^d = O(r^n)$

In short, every exponential grows faster than every polynomial!



(proof below)

Example: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$

$$\boxed{\log_a b = x \text{ means } a^x = b}$$

definition

$$a^{\log_a b} = b$$

$$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$$

$$(\log_a b)(\log_b n) = \log_a n$$

$$c \log_b n = \log_a n \text{ for the constant } c = \log_a b$$

So:

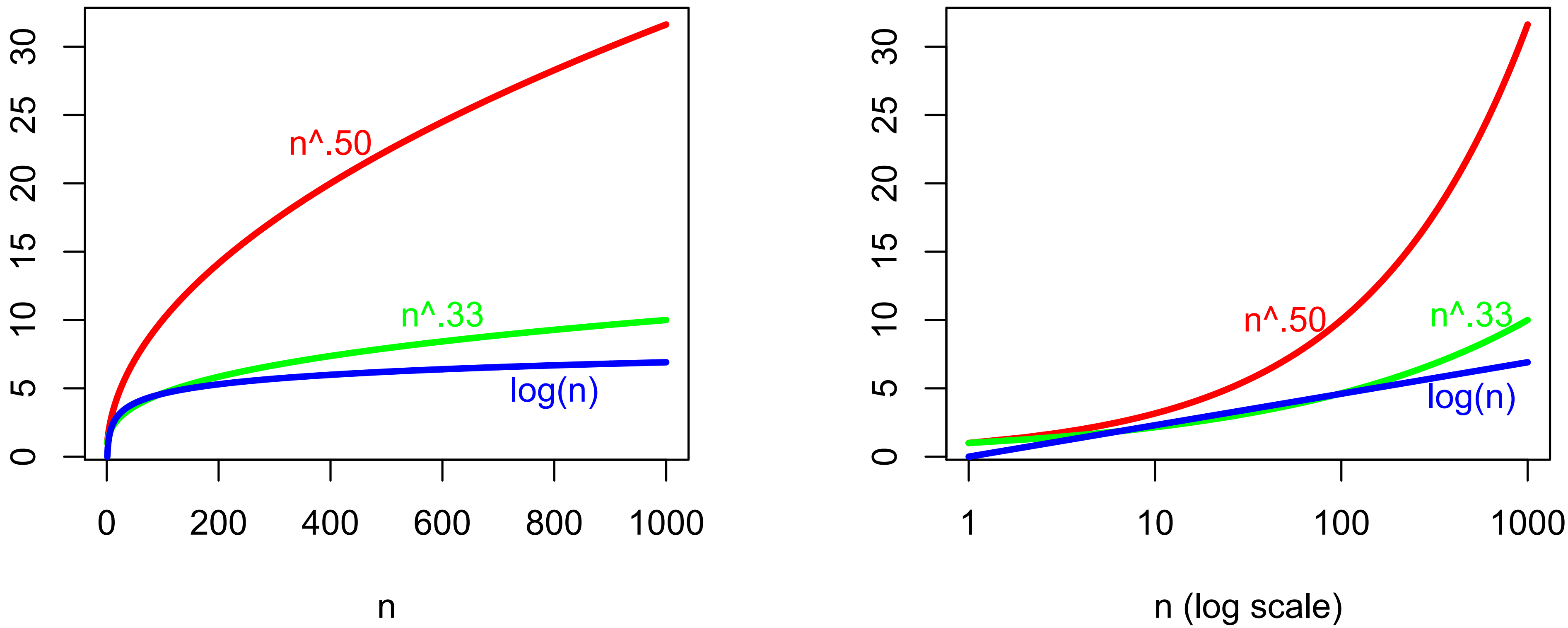
$$\log_b n = \Theta(\log_a n) = \Theta(\log n)$$

Corollary: base of a log factor is usually irrelevant, asymptotically. E.g. "O(n log n)" [but $n^{\log_2 8} \neq O(n^{\log_8 8})$]

Logarithms:

For all $x > 0$, *(no matter how small)* $\log n = O(n^x)$

log grows slower than every polynomial



$f(n)$ is $o(g(n))$ iff $\lim_{n\to\infty} f(n)/g(n)=0$

that is, $g(n)$ *dominates* $f(n)$

If $a \le b$ then n^a is $O(n^b)$

If $a < b$ then n^a is $o(n^b)$

$f(n) = O(g(n))$ vs $f(n) = o(g(n))$ are analogs to $\le$ vs $<$

Note:
if $f(n)$ is $\Theta(g(n))$ then it cannot be $o(g(n))$

$n^2 = o(n^3)$ [Use algebra]:

$$\lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$$

$n^3 = o(e^n)$ [Use L'Hospital's rule 3 times]:

$$\lim_{n \to \infty} \frac{n^3}{e^n} = \lim_{n \to \infty} \frac{3n^2}{e^n} = \lim_{n \to \infty} \frac{6n}{e^n} = \lim_{n \to \infty} \frac{6}{e^n} = 0$$

For all $r > 1$ (no matter how small)

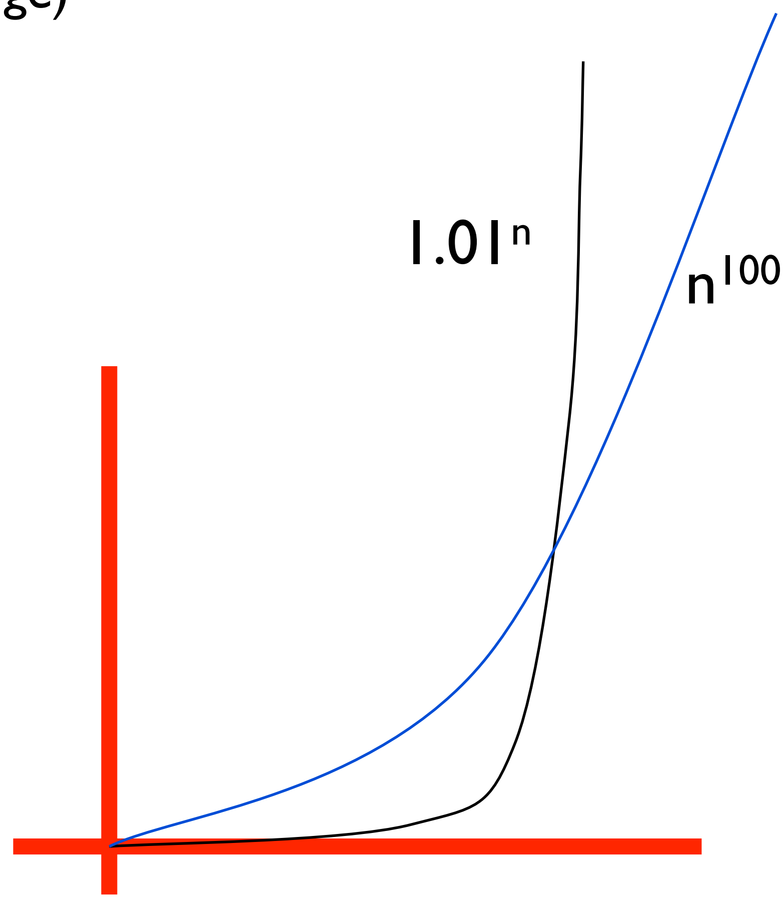
and all $d > 0$, (no matter how large)

$n^d = O(r^n)$

$n^d = o(r^n)$, even

Exercise: prove this, using tricks from previous slide

In short, every exponential grows faster than every polynomial!



Given two functions f(n) and g(n), if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} c & \text{for some constant } c>0 \\ 0 & \end{cases},$$

$$\text{then } \begin{cases} f(n) = \Theta(g(n)) \\ f(n) = o(g(n)) \; [\Rightarrow O(g(n))] \end{cases}, \text{ respectively.}$$

Inconclusive if the limit doesn't exist. E.g., no limit for f/g at right, but $g(n) \le f(n) = O(f(n))$

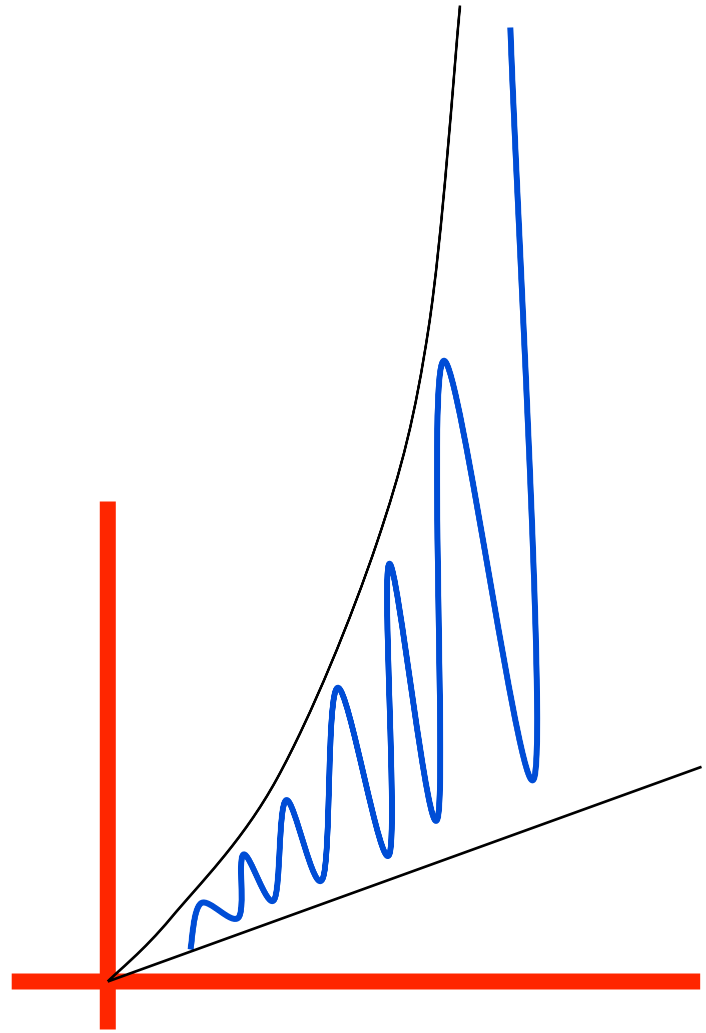
$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n^2 & \text{otherwise} \end{cases}$$

$$g(n) = n$$

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

$f(n) \neq \Theta(n^a)$ for any a.

Fortunately, such nasty cases are rare



$n \log n \neq \Theta(n^a)$ for any a, either, but at least it's simpler.

"Theorem": $\sum_{1 \le i \le n} i = O(n)$

"Proof:" (by induction on n)

basis: $\sum_{1 \le i \le 1} i = 1 = O(1)$

induction step:

$$\begin{aligned} \sum_{1 \le i \le n} i &= \left(\sum_{1 \le i \le n-1} i\right) + n \\ &= O(n-1) + n \qquad \text{(by ind. hyp.)} \\ &= O(n) \end{aligned}$$

FALSE!

Q. Where's the flaw??

A. Never use "big-O" like this in an induction; instead, explicitly show the implicit constant "c"; in the above "proof," you'll see "c" become "c+1"…

2 + 2 is 4	$2n^2 + 5\,n$ is $O(n^3)$
2 + 2 = 4	$2n^2 + 5\,n = O(n^3)$
4 = 2 + 2	$\cancel{O(n^3) = 2n^2 + 5\,n}$
All dogs are mammals	~~All mammals are dogs~~

Bottom line:

OK to put big-O in R.H.S. of equality, but not left.

Better, but less common, notation: $T(n) \in O(f(n))$.

I.e., $O(f(n))$ is the *set of all functions* that grow no more rapidly than some constant times f.

Replace “=” by “$\in$” or “$\subseteq$” as appropriate: e.g.:

$2n^2 + 5\,n \in O(n^2) \subseteq O(n^3)$

Polynomial Time

P: The set of problems solvable by algorithms with running time $O(n^d)$ for some constant d

(d is a constant independent of the input size n)

Nice scaling property: there is a constant c s.t. *doubling* n, time increases only by a factor of c.

(E.g., $c \sim 2^d$)

Contrast with exponential: For any constant c, there is a d such that $n \rightarrow n+d$ increases time by a factor of more than c.

(E.g., c = 100 and d = 7 for 2^n vs 2^{n+7})

polynomial vs exponential growth

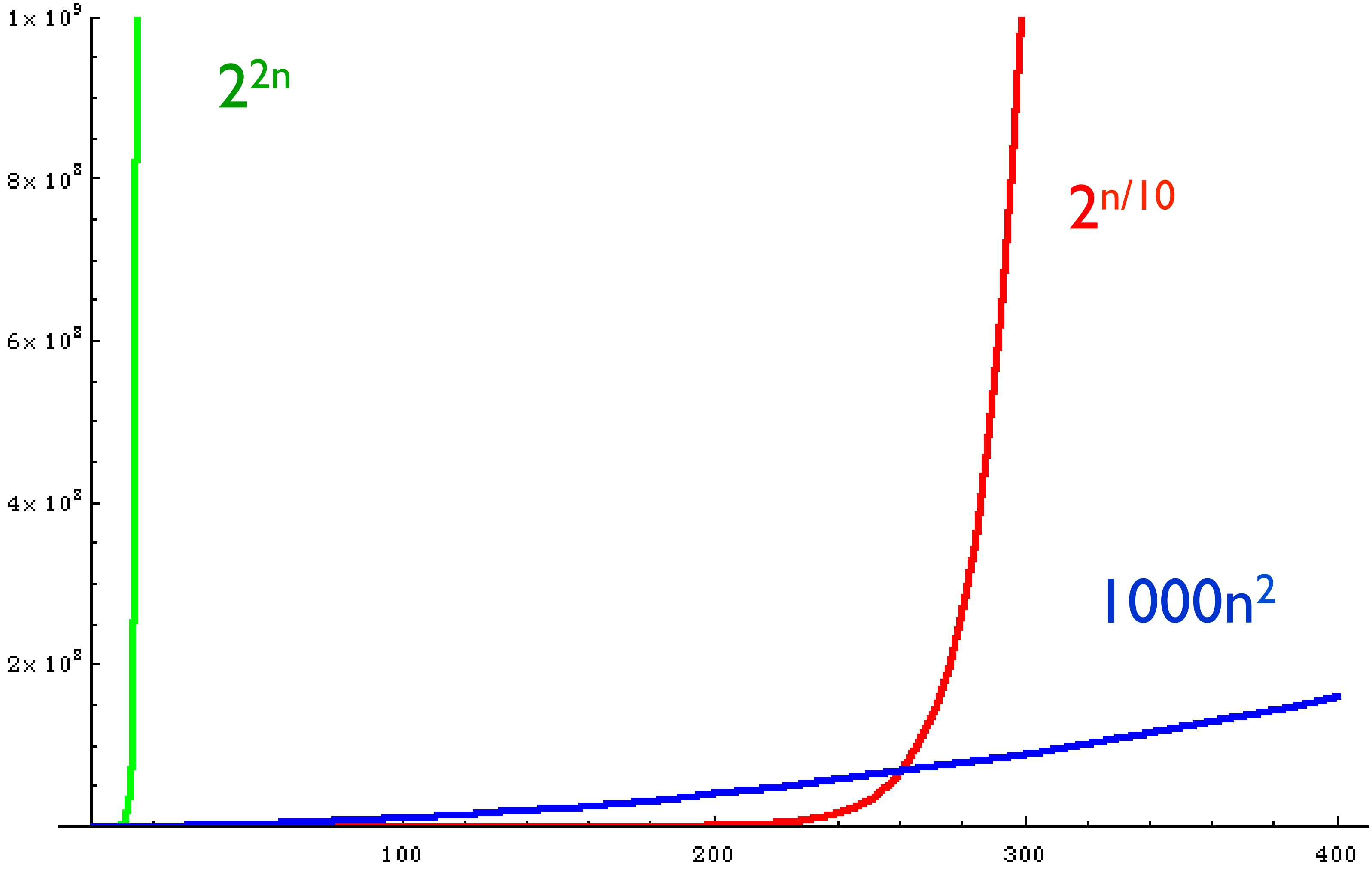


Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

not only get very big, but do so abruptly, which likely yields erratic performance on small instances

Next year's computer will be 2x faster. If I can solve problem of size n_0 today, how large a problem can I solve in the same time next year?

Complexity	Size Increase	E.g. $T=10^{12}$		
$O(n)$	$n_0 \rightarrow 2n_0$	10^{12}	$\rightarrow$	2×10^{12}
$O(n^2)$	$n_0 \rightarrow \sqrt{2}\, n_0$	10^6	$\rightarrow$	1.4×10^6
$O(n^3)$	$n_0 \rightarrow \sqrt[3]{2}\, n_0$	10^4	$\rightarrow$	1.25×10^4
$2^{n/10}$	$n_0 \rightarrow n_0+10$	400	$\rightarrow$	410
2^n	$n_0 \rightarrow n_0+1$	40	$\rightarrow$	41

Point is not that n^{2000} is a nice time bound, or that the differences among n and 2n and n^2 are negligible.

Rather, simple theoretical tools may not easily capture such differences, whereas exponentials are qualitatively different from polynomials, so more amenable to theoretical analysis.

- “My problem is in P” is a starting point for a more detailed analysis
- “My problem is *not* in P” may suggest that you need to shift to a more tractable variant, or otherwise readjust expectations

Summary

Big O is a *math* notation defining an *upper bound* on *growth rate* of a function (typically a function lacking a simple analytic formula.)

In CS, that function is often the *worst case run time* of some algorithm (as a function of input size, *n*, where "worst case" means max time over all inputs of size *n*.)

BUT, it can also be used for other functions, like best- or average-case time/space/…, so be clear/careful re defn.

Big Ω is analogous math notation for lower bounds

Big Θ: upper and lower bounds simultaneously

These notations deliberately define growth rate only *up to a (hidden) constant factor,* essentially because (a) scaling matters more than the constant, and (b) the constant is strongly technology-dependent (language, code, compiler, processor, …) making it much more work to pin down.

So, a typical initial goal for algorithm analysis is to find a

reasonably tight, ← i.e., Θ if possible

asymptotic, ← i.e., O or Θ

bound on ← usually upper bound

worst case running time

as a function of problem size

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones – so you can concentrate on the good ones!

As one important example, poly time algorithms are almost always preferable to exponential time ones.