CSE 421
Algorithms

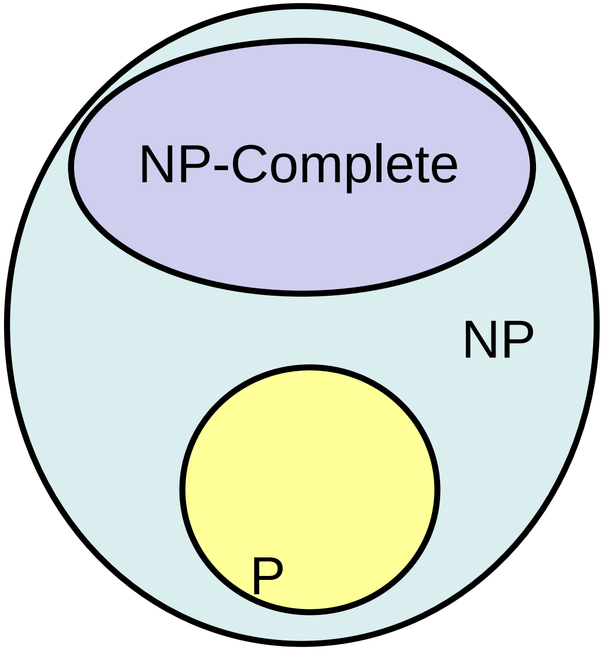
Richard Anderson

Lecture 27

NP-Completeness Proofs

NP-Completeness

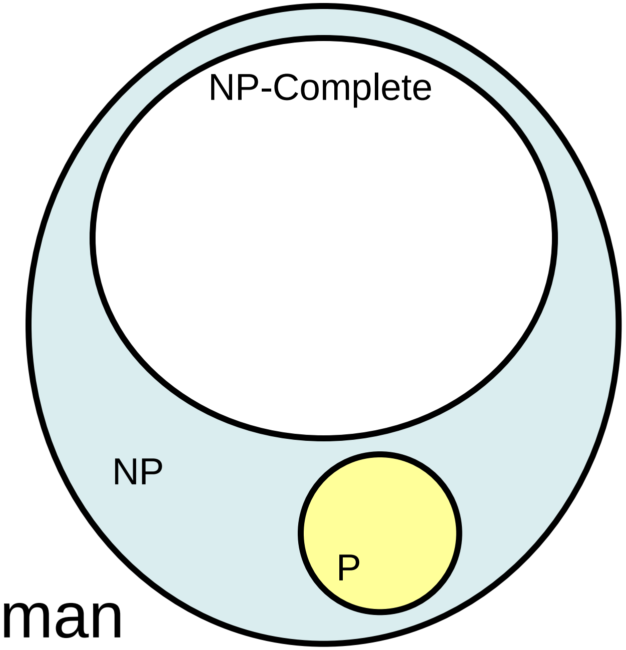
- A problem X is NP-complete if
  - X is in NP
  - For every Y in NP, $Y <_P X$



Cook’s Theorem

- The Circuit Satisfiability Problem is NP-Complete
- Proof ideas
  - Let A be an arbitrary problem in NP
  - Show that an instance x of A can be transformed in polynomial time into an instance y of Circuit SAT, such that x is a true instance of A iff y is a satisfiable circuit
  - A $<_P$ Circuit SAT

Populating the NP-Completeness Universe



- Circuit SAT $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

Satisfiability

Literal: A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

Clause: A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

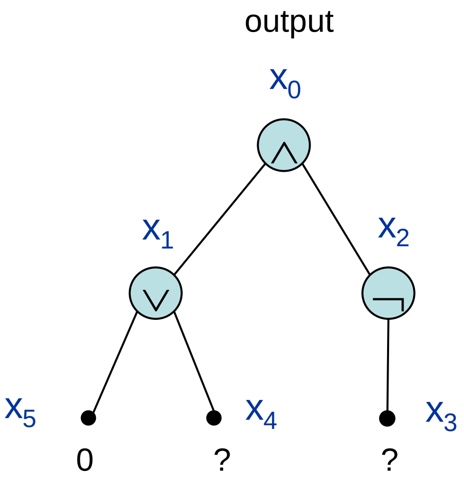
Yes: x_1 = true, x_2 = true x_3 = false.

3-SAT is NP-Complete

Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT $\leq_P$ 3-SAT since 3-SAT is in NP.

- Let K be any circuit.
- Create a 3-SAT variable x_i for each circuit element i.
- Make circuit compute correct values at each node:
  - $x_2 = \neg x_3$ $\Rightarrow$ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
  - $x_1 = x_4 \vee x_5$ $\Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
  - $x_0 = x_1 \wedge x_2$ $\Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$
- Hard-coded input values and output value.
  - $x_5 = 0$ $\Rightarrow$ add 1 clause: $\overline{x_5}$
  - $x_0 = 1$ $\Rightarrow$ add 1 clause: x_0
- Final step: turn clauses of length < 3 into clauses of length exactly 3. ▪



Proving a problem A is NP Complete

- Show A is in NP (usually easy)
- Choose an NP complete problem B
  - Convert an instance of B into an *equivalent* instance of A

3 Satisfiability Reduces to Independent Set

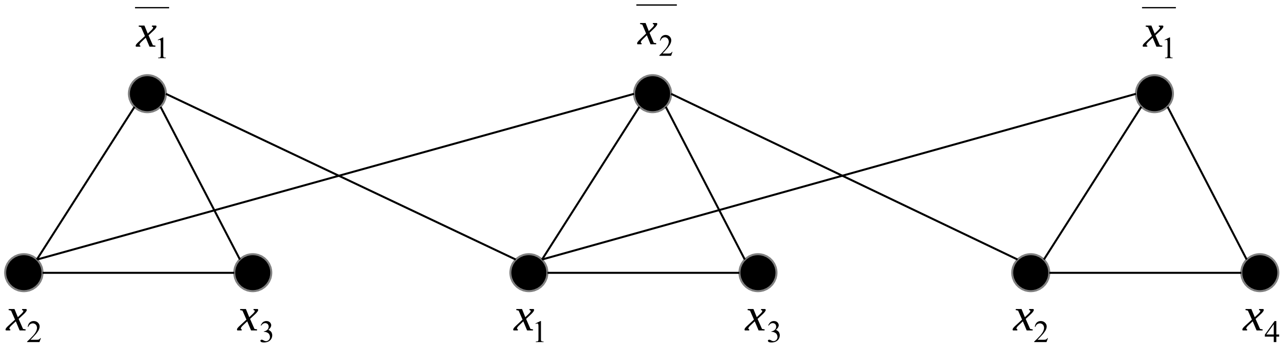
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

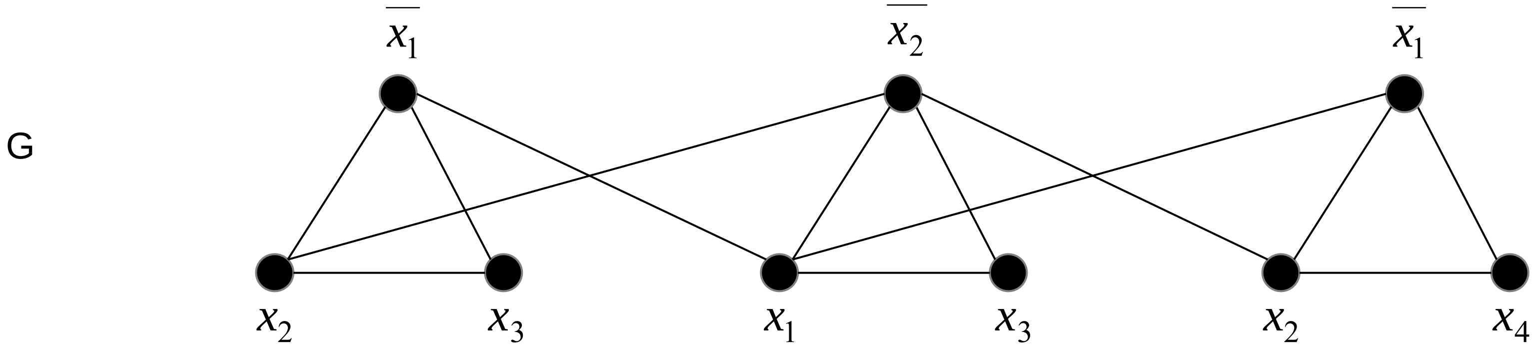
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size k = |Φ| iff Φ is satisfiable.

Pf. ⇒ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. ← and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf ⇐ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

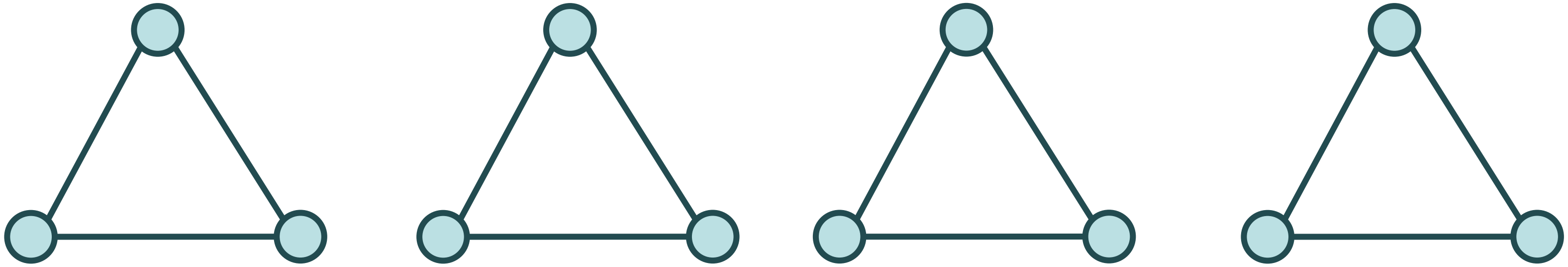


k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

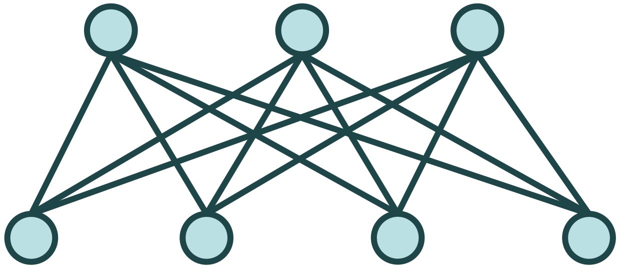
Analysis of 3-SAT to IS reduction

- Clause satisfaction
  - Only one literal per clause can be selected, so to get k literals, a literal from every clause must be selected



Analysis of 3-SAT to IS reduction

- Truth setting
  - X is true if at least one X is in the independent set
  - X is false if at least one $\overline{X}$ is in the independent set
- Truth consistency
  - Edges between all copies of X and $\overline{X}$ ensure variables are true or false



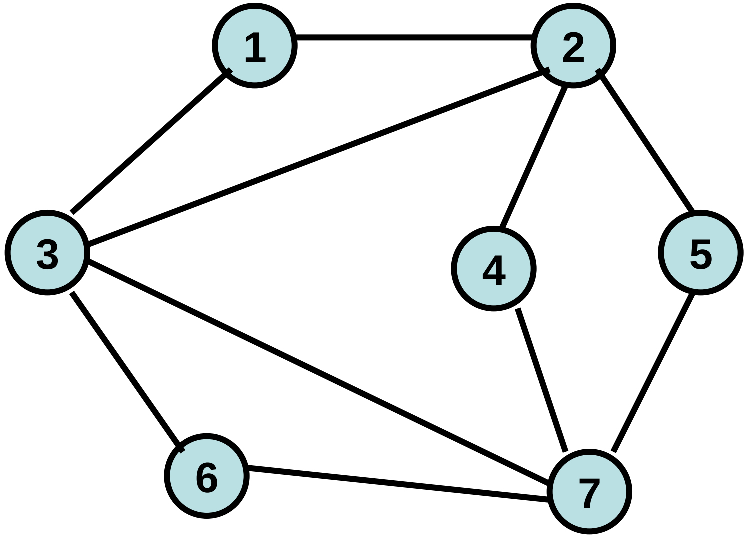
IS $<_P$ VC

- Lemma: A set S is independent iff V-S is a vertex cover

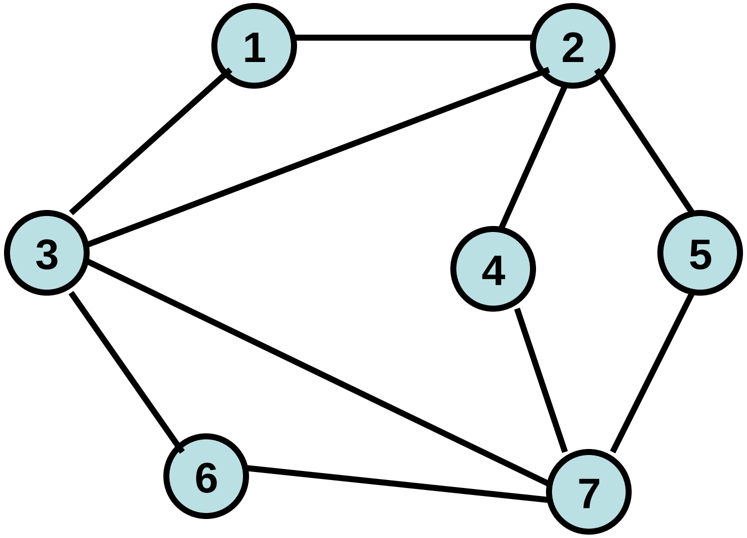
- To reduce IS to VC, we show that we can determine if a graph has an independent set of size K by testing for a Vertex cover of size n - K

IS $<_P$ VC

Find a maximum independent set S

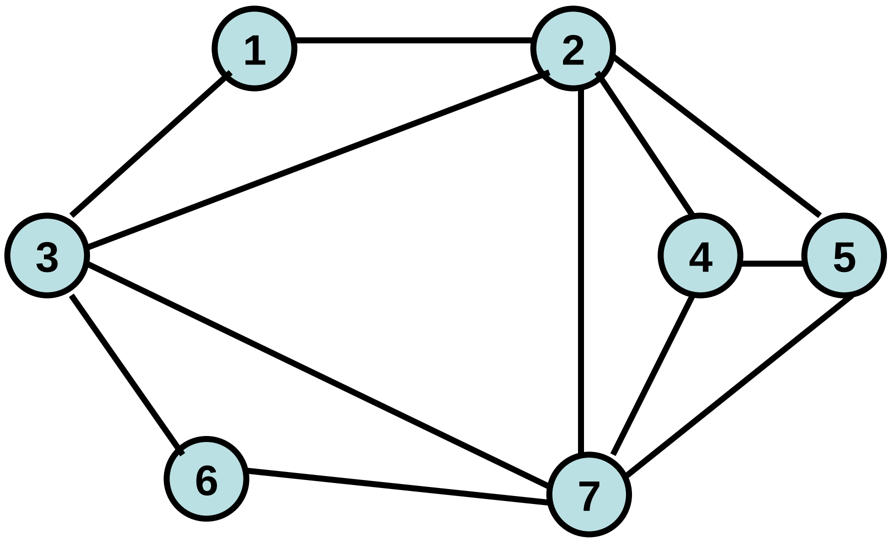


Show that V-S is a vertex cover



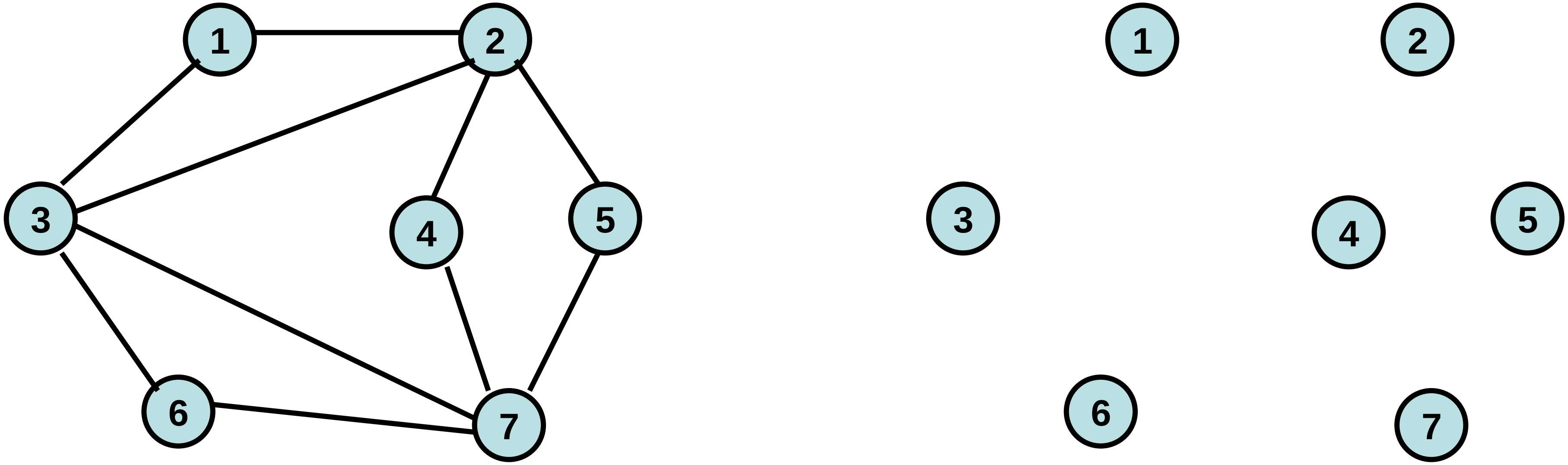
Clique

- Clique
  - Graph G = (V, E), a subset S of the vertices is a clique if there is an edge between every pair of vertices in S



Complement of a Graph

- Defn: G’=(V,E’) is the complement of G=(V,E) if (u,v) is in E’ iff (u,v) is not in E



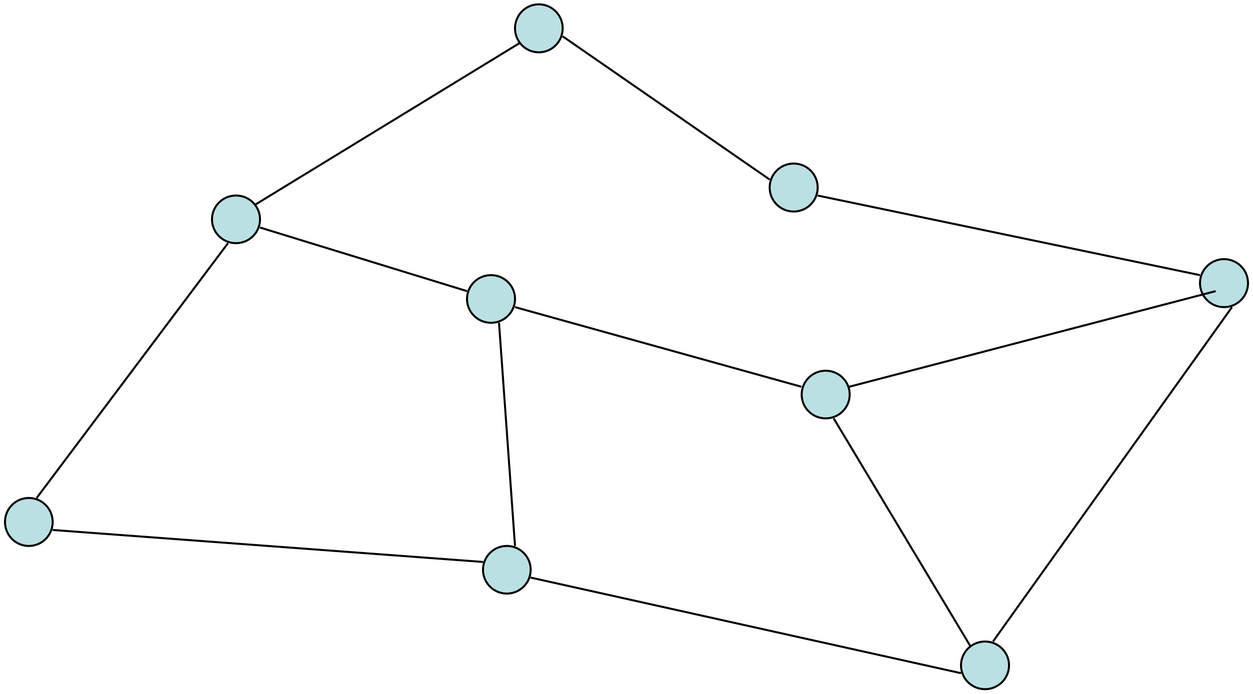
IS $<_P$ Clique

- Lemma: S is Independent in G iff S is a Clique in the complement of G

- To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size K iff the original graph has an independent set of size K

Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph



Thm: Hamiltonian Circuit is NP Complete

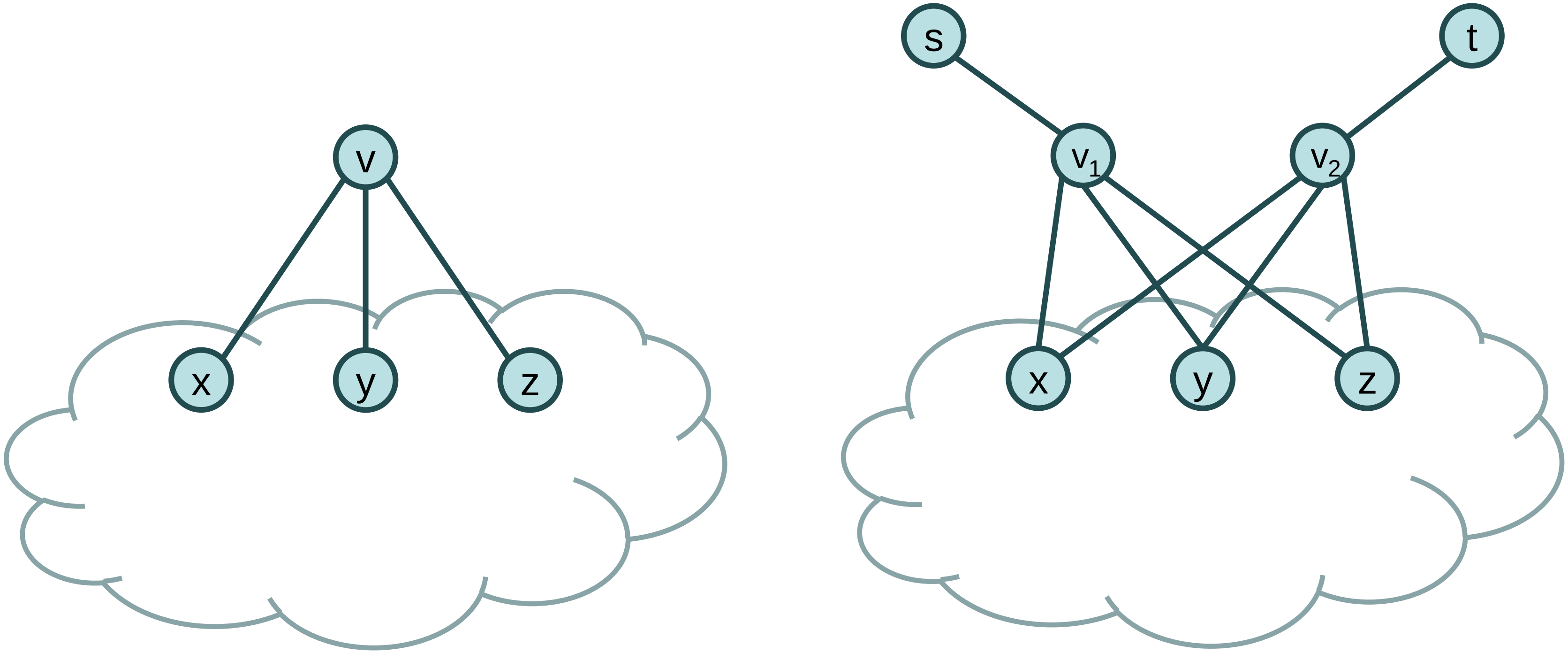
- Reduction from 3-SAT

Hamiltonian Path

- Is there a simple path that visits all the vertices?
- Is there a simple path from s to t that visits all the vertices?

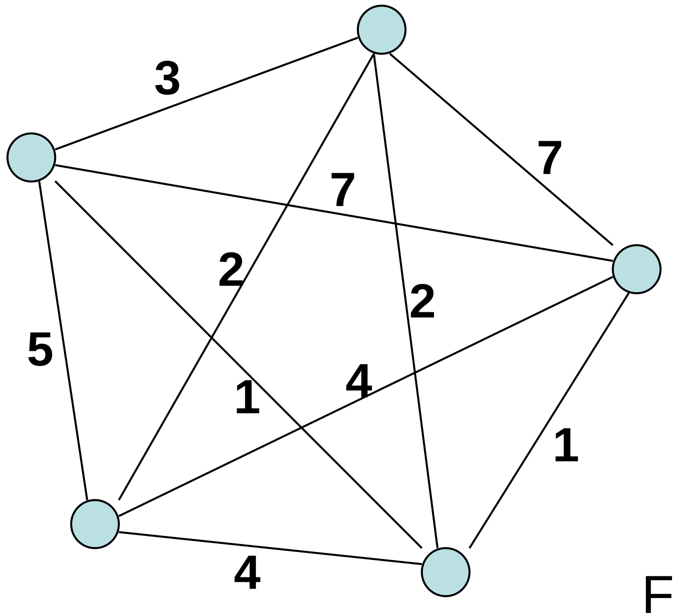
Reduce HC to HP

G_2 has a Hamiltonian Path iff G_1 has a Hamiltonian Circuit



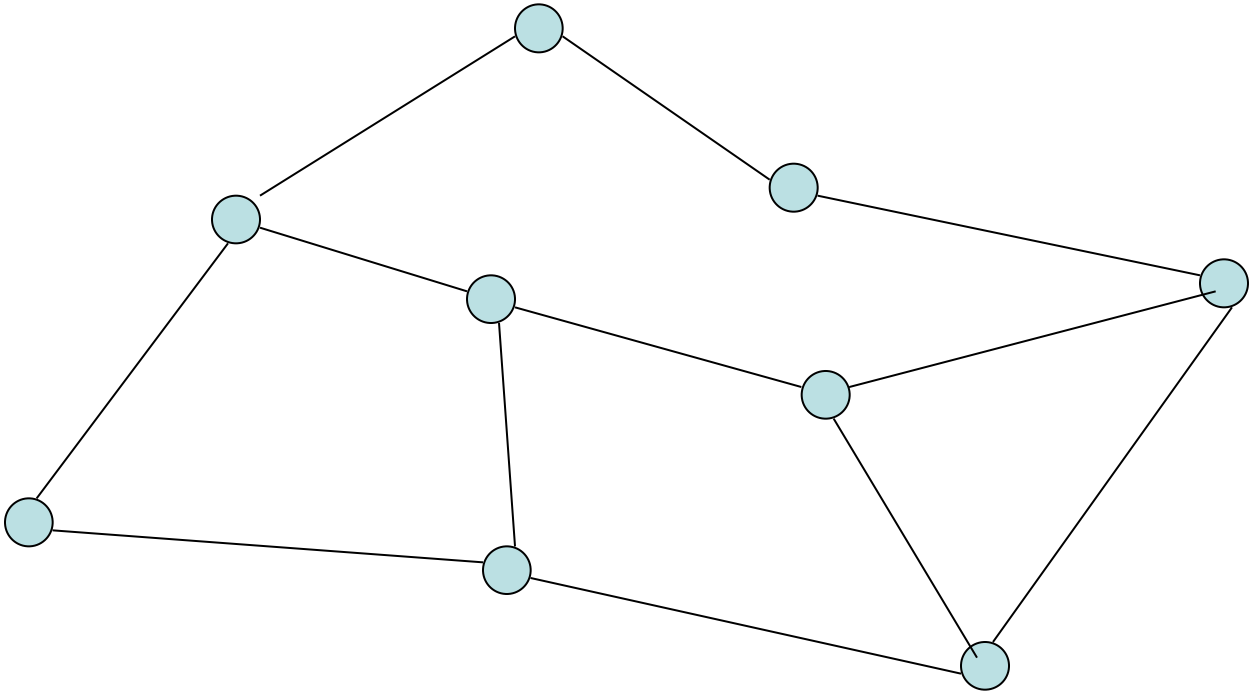
Traveling Salesman Problem

- Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)



Find the minimum cost tour

Thm: HC $<_P$ TSP



Graph Coloring

- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring