CSE 421
Algorithms

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Lecture 20

Memory Efficient Dynamic Programming / Shortest Paths

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

LCS(BARTHOLEMEWSIMPSON, KRUSTYTHECLOWN)
= RTHOWN

LCS Optimization

- $A = a_1a_2 \ldots a_m$
- $B = b_1b_2 \ldots b_n$

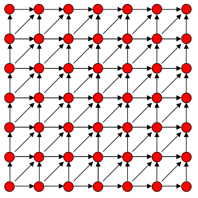
- Opt[j, k] is the length of LCS($a_1a_2 \ldots a_j$, $b_1b_2 \ldots b_k$)

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Dynamic Programming Computation

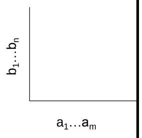


Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
      Opt[ i, j ] := Opt[ i-1, j ];
    else
      Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
    for j := 1 to n
        if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
        else if Opt[i-1, j] >= Opt[i, j-1]
            {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
        else        {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Observations about the Algorithm

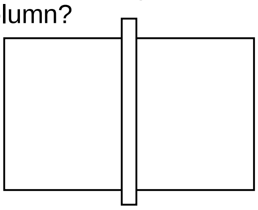
- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

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Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

- From the left, compute $LCS(a_1…a_{m/2},b_1…b_j)$
- From the right, compute $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$
- Add values for corresponding j's

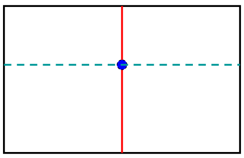
- Note – this is space efficient

Divide and Conquer

- $A = a_1,…,a_m$ $B = b_1,…,b_n$
- Find j such that
  - $LCS(a_1…a_{m/2}, b_1…b_j)$ and
  - $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$ yield optimal solution
- Recurse

Algorithm Analysis

- T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm



Prove by induction that
T(m,n) <= 2cmn

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes

Shortest Paths with Dynamic Programming

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- General case – handling negative edges
- If there exists a negative cost cycle, the shortest path is not defined
- Bellman-Ford Algorithm
  - O(mn) time for graphs with negative cost edges

Lemma

- If a graph has no negative cost cycles, then the shortest paths are simple paths
- Shortest paths have at most n-1 edges

Shortest paths with a fixed number of edges

- Find the shortest path from v to w with exactly k edges

Express as a recurrence

- $Opt_k(w) = \min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w) = 0$ if v=w and infinity otherwise

Algorithm, Version 1

```
foreach w
    M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
    foreach w
        M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

Algorithm, Version 2

```
foreach w
    M[0, w] = infinity;
M[0, v] = 0;
for i = 1 to n-1
    foreach w
        M[i, w] = min(M[i-1, w], min_x(M[i-1,x] + cost[x,w]))
```

Algorithm, Version 3

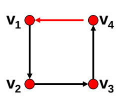
```
foreach w
    M[w] = infinity;
M[v] = 0;
for i = 1 to n-1
    foreach w
        M[w] = min(M[w], min_x(M[x] + cost[x,w]))
```

Correctness Proof for Algorithm 3

- Key lemma – at the end of iteration i, for all w, M[w] <= M[i, w];
- Reconstructing the path:
  - Set P[w] = x, whenever M[w] is updated from vertex x

If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] >= M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let $v_1, v_2, \ldots v_k$ be a cycle in the pointer graph with (v_k, v_1) the last edge added
  - Just before the update
    - $M[v_j] >= M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1,v_2) + cost(v_2,v_3) + \ldots + cost(v_k, v_1)$

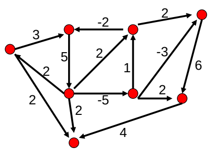


Negative Cycles

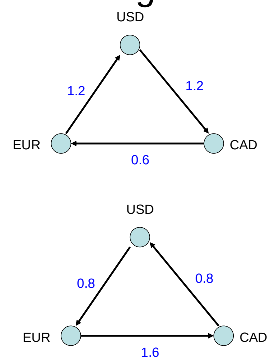
- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

Finding negative cost cycles

- What if you want to find negative cost cycles?



Foreign Exchange Arbitrage



	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----