CSE 421
Algorithms

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Lecture 19

Dynamic Programming

Announcements

- Homework Deadlines
  - HW 7: Wednesday, November 18
  - HW 8: Wednesday, November 25
  - HW 9: Friday, December 4
  - HW 10: Friday, December 11
- Final Exam
  - Monday, December 14, 2:30-4:20 pm

One dimensional dynamic programming: Interval scheduling

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])

Two dimensional dynamic programming

K-segment linear approximation

$Opt_k[\ j\ ] = \min_i \{ Opt_{k-1}[\ i\ ] + E_{i,j} \}$ for $0 < i < j$

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Two dimensional dynamic programming

Subset sum and knapsack

Opt[j, K] = max(Opt[j – 1, K], Opt[j – 1, K – w_j] + w_j)

Opt[j, K] = max(Opt[j – 1, K], Opt[j – 1, K – w_j] + v_j)

4																	
3																	
2																	
1																	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Aside: Negative weights in subset sum

- Alternate formulation of Subset Sum dynamic programming algorithm
- Sum[i, K] = true if there is a subset of $\{w_1,\ldots w_k\}$ that sums to exactly K, false otherwise
- Sum [i, K] = Sum [i -1, K] **OR** Sum[i - 1, K - w_i]
- To allow for negative numbers, we need to fill in the array between K_{min} and K_{max}

Dynamic Programming Examples

- Examples
  - Optimal Billboard Placement
    - Text, Solved Exercise, Pg 307
  - Linebreaking with hyphenation
    - Compare with HW problem 6, Pg 317
  - String approximation
    - Text, Solved Exercise, Page 309

Billboard Placement

- Maximize income in placing billboards
  - $b_i = (p_i, v_i)$, v_i: value of placing billboard at position p_i
- Constraint:
  - At most one billboard every five miles
- Example
  - {(6,5), (8,6), (12, 5), (14, 1)}

Design a Dynamic Programming Algorithm for Billboard Placement

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Input $b_1, \ldots, b_n$, where $b_i = (p_i, v_i)$, position and value of billboard i

Solution

```
j = 0;          // j is five miles behind the current position
                // the last valid location for a billboard, if one placed at P[k]
for k := 1 to n
        while (P[ j ] < P[ k ] – 5)
                j := j + 1;
        j := j – 1;
        Opt[ k]  = Max(Opt[ k-1] , V[ k ] + Opt[ j ]);
```

String approximation

- Given a string S, and a library of strings B = $\{b_1, \ldots b_m\}$, construct an approximation of the string S by using copies of strings in B.

B = {abab, bbbaaa, ccbb, ccaacc}

S = abaccbbbaabbccbbccaabab

Formal Model

- Strings from B assigned to non-overlapping positions of S
- Strings from B may be used multiple times
- Cost of δ for unmatched character in S
- Cost of γ for mismatched character in S
  - MisMatch(i, j) – number of mismatched characters of b_j, when aligned starting with position i in s.

Design a Dynamic Programming Algorithm for String Approximation

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Target string $S = s_1s_2\ldots s_n$
Library of strings $B = \{b_1,\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Opt[k] = fun(Opt[0],…,Opt[k-1])

- How is the solution determined from sub problems?

Target string $S = s_1s_2\ldots s_n$
Library of strings $B = \{b_1,\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with b_j when aligned starting at position i of S.

Solution

```
for i := 1 to n
        Opt[k] = Opt[k-1] + δ;
        for j := 1 to |B|
                p = i – len(b_j);
                Opt[k] = min(Opt[k],  Opt[p-1] + γ MisMatch(p, j));
```

Longest Common Subsequence

- $C=c_1\ldots c_g$ is a subsequence of $A=a_1\ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

ocurranec

occurrence

attacggct

tacgacca

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

String Alignment Problem

- Align sequences with gaps

CAT TGA AT

CAGAT AGGA

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

LCS Optimization

- $A = a_1a_2\ldots a_m$
- $B = b_1b_2\ldots b_n$

- Opt[j, k] is the length of $LCS(a_1a_2\ldots a_j, b_1b_2\ldots b_k)$

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If a_j != b_k, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

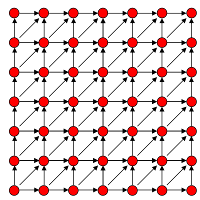
Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

Let a_j = x and b_k = y
Express as minimization

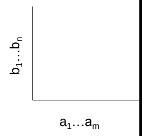
Dynamic Programming Computation



Code to compute Opt[j,k]

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else       {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Observations about the Algorithm

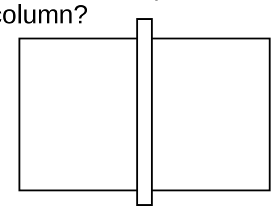
- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B),\ldots,LCS_{5,9}(A,B)$

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

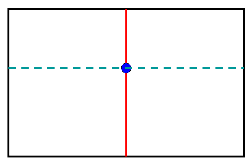
- From the left, compute $LCS(a_1 \ldots a_{m/2}, b_1 \ldots b_j)$
- From the right, compute $LCS(a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n)$
- Add values for corresponding j's
- Note – this is space efficient

Divide and Conquer

- $A = a_1,\ldots,a_m$ $\qquad B = b_1,\ldots,b_n$
- Find j such that
  - $LCS(a_1 \ldots a_{m/2}, b_1 \ldots b_j)$ and
  - $LCS(a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n)$ yield optimal solution
- Recurse

Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$



Prove by induction that $T(m,n) <= 2cmn$

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes