# CSE 421
# Algorithms

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Lecture 19

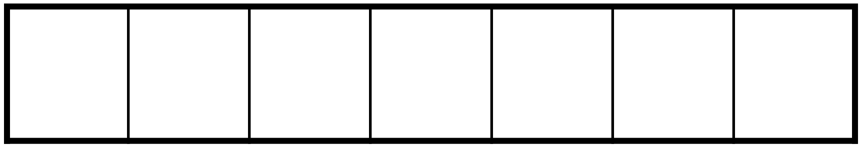
Dynamic Programming

# Announcements

- Homework Deadlines
  - HW 7: Wednesday, November 18
  - HW 8: Wednesday, November 25
  - HW 9: Friday, December 4
  - HW 10: Friday, December 11
- Final Exam
  - Monday, December 14, 2:30-4:20 pm

# One dimensional dynamic programming: Interval scheduling

$$\mathrm{Opt}[\,j\,] = \max\left(\mathrm{Opt}[\,j - 1], w_j + \mathrm{Opt}[\,p[\,j\,]\,]\right)$$



# Two dimensional dynamic programming

## K-segment linear approximation

$\mathrm{Opt}_k[\ j\ ] = \min_i \{ \mathrm{Opt}_{k-1}[\ i\ ] + E_{i,j} \}$ for $0 < i < j$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | | | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Two dimensional dynamic programming

## Subset sum and knapsack

$$Opt[\ j, K] = \max(Opt[\ j - 1, K], Opt[\ j - 1, K - w_j] + w_j)$$

$$Opt[\ j, K] = \max(Opt[\ j - 1, K], Opt[\ j - 1, K - w_j] + v_j)$$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | | | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Aside: Negative weights in subset sum

- Alternate formulation of Subset Sum dynamic programming algorithm
- Sum[i, K] = true if there is a subset of $\{w_1, \ldots w_k\}$ that sums to exactly K, false otherwise
- Sum [i, K] = Sum [i -1, K] **OR** Sum[i - 1, K - $w_i$]
- To allow for negative numbers, we need to fill in the array between $K_{min}$ and $K_{max}$

# Dynamic Programming Examples

- Examples
  - Optimal Billboard Placement
    - Text, Solved Exercise, Pg 307
  - Linebreaking with hyphenation
    - Compare with HW problem 6, Pg 317
  - String approximation
    - Text, Solved Exercise, Page 309

# Billboard Placement

- Maximize income in placing billboards
  - $b_i = (p_i, v_i)$, $v_i$: value of placing billboard at position $p_i$
- Constraint:
  - At most one billboard every five miles
- Example
  - {(6,5), (8,6), (12, 5), (14, 1)}

# Design a Dynamic Programming Algorithm for Billboard Placement

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Input $b_1$, …, $b_n$, where $b_i = (p_i, v_i)$, position and value of billboard i

# Solution

```
j = 0;               // j is five miles behind the current position
                      // the last valid location for a billboard, if one placed at P[k]
for k := 1 to n
          while (P[ j ] < P[ k ] – 5)
                    j := j + 1;
          j := j – 1;
          Opt[ k]  = Max(Opt[ k-1] , V[ k ] + Opt[ j ]);
```

# String approximation

- Given a string S, and a library of strings B = {$b_1$, …$b_m$}, construct an approximation of the string S by using copies of strings in B.

B = {abab, bbbaaa, ccbb, ccaacc}

S = abaccbbbaabbccbbccaabab

# Formal Model

- Strings from B assigned to non-overlapping positions of S
- Strings from B may be used multiple times
- Cost of $\delta$ for unmatched character in S
- Cost of $\gamma$ for mismatched character in S
  - MisMatch(i, j) – number of mismatched characters of $b_j$, when aligned starting with position i in s.

# Design a Dynamic Programming Algorithm for String Approximation

- Compute Opt[1], Opt[2], . . ., Opt[n]
- What is Opt[k]?

Target string $S = s_1s_2\ldots s_n$
Library of strings $B = \{b_{1,}\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with $b_j$ when aligned starting at position i of S.

# Opt[k] = fun(Opt[0],…,Opt[k-1])

- How is the solution determined from sub problems?

Target string S = $s_1s_2\ldots s_n$
Library of strings B = $\{b_{1,}\ldots,b_m\}$
MisMatch(i,j) = number of mismatched characters with $b_j$ when aligned starting at position i of S.

# Solution

```
for i := 1 to n
    Opt[k] = Opt[k-1] + $\delta$;
    for j := 1 to |B|
        p = i – len($b_j$);
        Opt[k] = min(Opt[k],  Opt[p-1] + $\gamma$ MisMatch(p, j));
```

# Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

**`ocurranec`**

**`occurrence`**

**`attacggct`**

**`tacgacca`**

# Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

# String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge $\delta_x$ if character x is unmatched
- Charge $\gamma_{xy}$ if character x is matched to character y

Note: the problem is often expressed as a minimization problem,
with $\gamma_{xx} = 0$ and $\delta_x > 0$

# LCS Optimization

- $A = a_1a_2\dots a_m$
- $B = b_1b_2\dots b_n$

- Opt[ j, k] is the length of LCS($a_1a_2\dots a_j$, $b_1b_2\dots b_k$)

# Optimization recurrence

If $a_j = b_k$,  Opt[ j,k ] = 1 + Opt[ j-1, k-1 ]

If $a_j$ != $b_k$,  Opt[ j,k] = max(Opt[ j-1,k], Opt[ j,k-1])

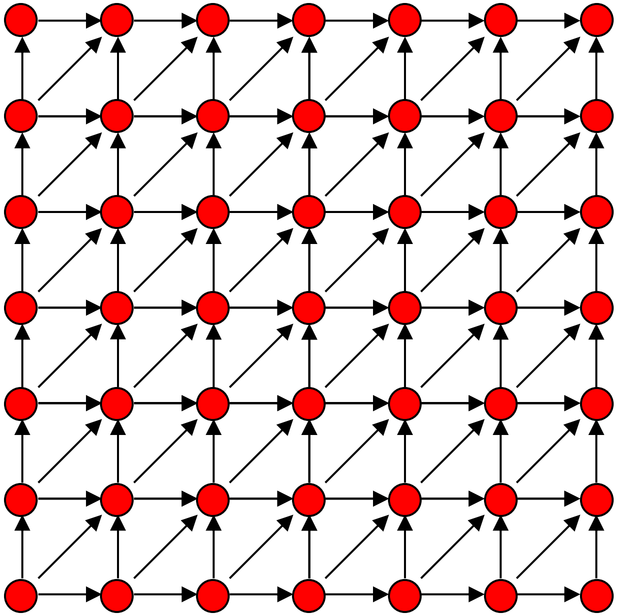
# Give the Optimization Recurrence for the String Alignment Problem

- Charge $\delta_x$ if character x is unmatched
- Charge $\gamma_{xy}$ if character x is matched to character y

Opt[ j, k] =

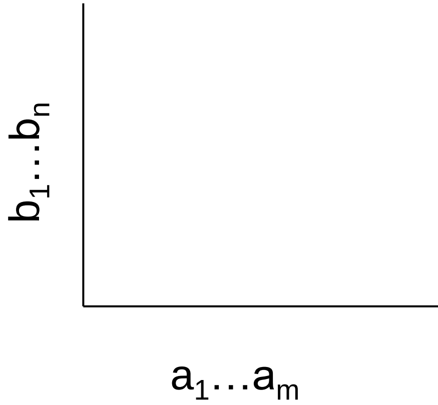
Let $a_j = x$ and $b_k = y$
Express as minimization

# Dynamic Programming Computation



# Code to compute Opt[j,k]

# Storing the path information



```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else       {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

# How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

# Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values

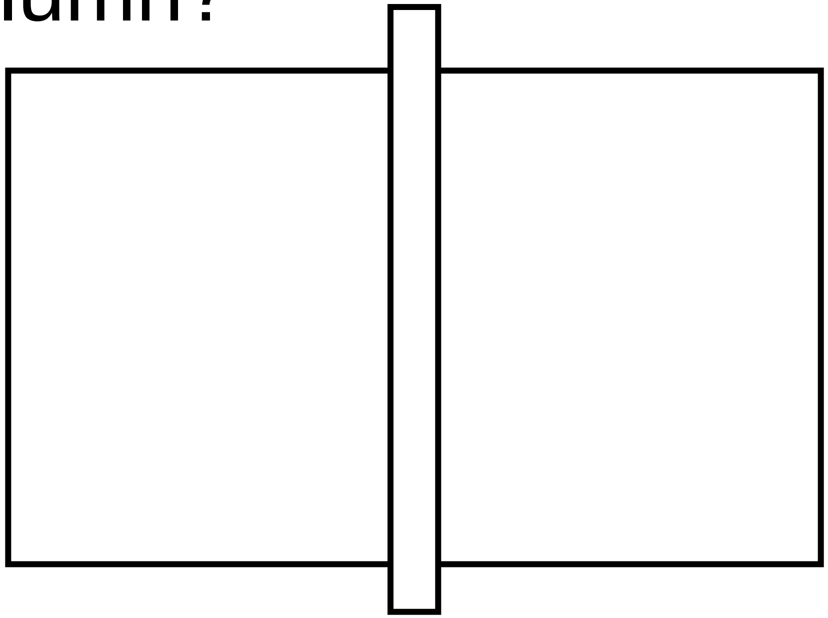
- The algorithm can be run from either end of the strings

# Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

# Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has $a_i$ matched with $b_j$

# Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$

- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS

B=RTSRRRSTST

Compute $\mathrm{LCS}_{5,0}(A,B)$, $\mathrm{LCS}_{5,1}(A,B),\ldots,\mathrm{LCS}_{5,9}(A,B)$

# A = RRSSRTTRTS
# B=RTSRRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

| j | left | right |
|---|---|---|
| 0 | 0 | 4 |
| 1 | 1 | 4 |
| 2 | 1 | 3 |
| 3 | 2 | 3 |
| 4 | 3 | 3 |
| 5 | 3 | 2 |
| 6 | 3 | 2 |
| 7 | 3 | 1 |
| 8 | 4 | 1 |
| 9 | 4 | 0 |

# Computing the middle column

- From the left, compute LCS($a_1 \ldots a_{m/2}, b_1 \ldots b_j$)
- From the right, compute LCS($a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n$)
- Add values for corresponding j's

- Note – this is space efficient

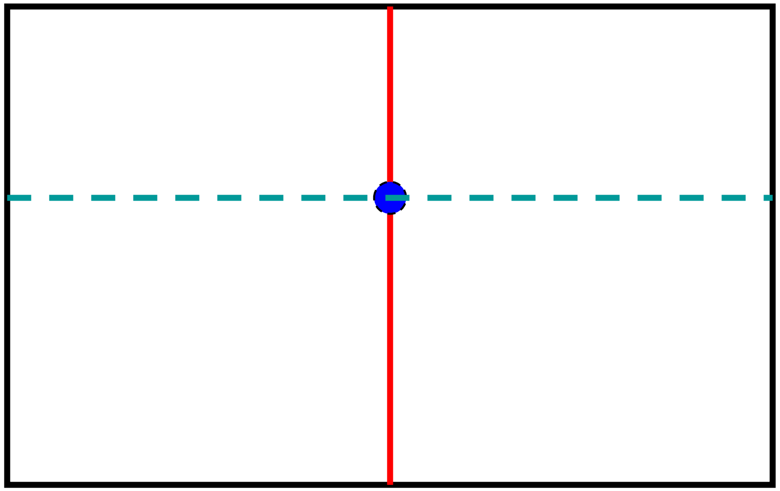
# Divide and Conquer

- $A = a_1,\ldots,a_m$ $B = b_1,\ldots,b_n$
- Find j such that
  - LCS($a_1\ldots a_{m/2}$, $b_1\ldots b_j$) and
  - LCS($a_{m/2+1}\ldots a_m$,$b_{j+1}\ldots b_n$) yield optimal solution

- Recurse

# Algorithm Analysis

- T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm



# Prove by induction that $T(m,n) <= 2cmn$

# Memory Efficient LCS Summary

- We can afford O(nm) time, but we can’t afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes