CSE 421
Algorithms

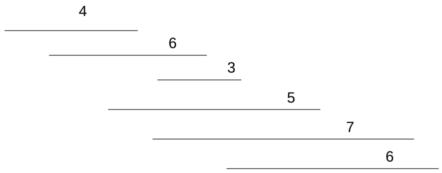
Richard Anderson

Lecture 17

Dynamic Programming

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Optimality Condition

- Opt[j] is the maximum weight independent set of intervals $I_1, I_2, \ldots, I_j$
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

```
MaxValue(j) =
  if j = 0 return 0
  else
      return max( MaxValue(j-1),
                  wj + MaxValue(p[ j ]))
```

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

```
MaxValue(j) =
  if j = 0 return 0;
  else if M[ j ] != -1 return M[ j ];
  else
      M[ j ] = max(MaxValue(j-1), wj + MaxValue(p[ j ]));
      return M[ j ];
```

Iterative Algorithm

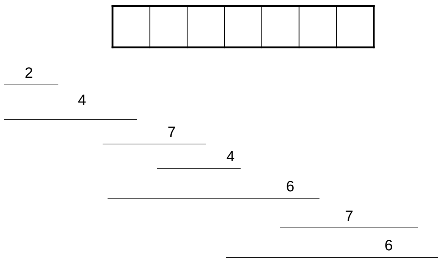
Express the MaxValue algorithm as an iterative algorithm

```
MaxValue {


}
```

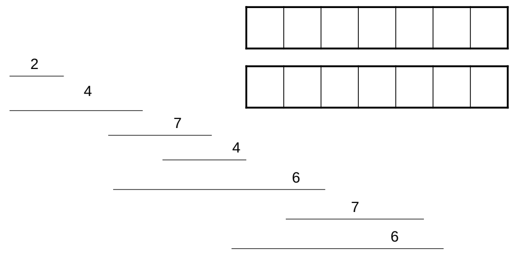
Fill in the array with the Opt values

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])



Computing the solution

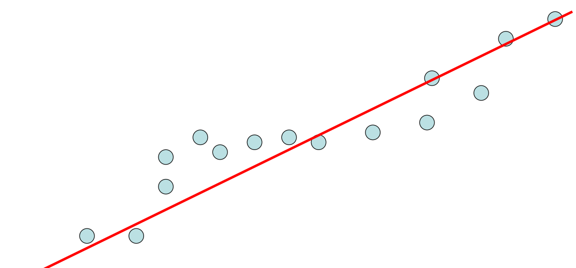
Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])
Record which case is used in Opt computation



Dynamic Programming

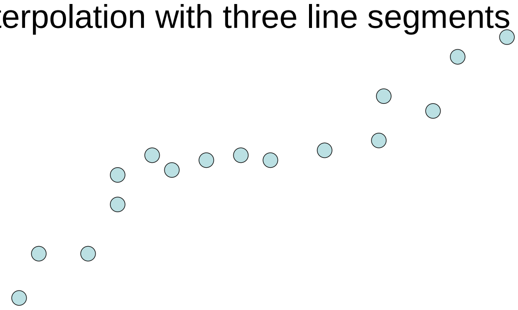
- The most important algorithmic technique covered in CSE 421
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Optimal linear interpolation

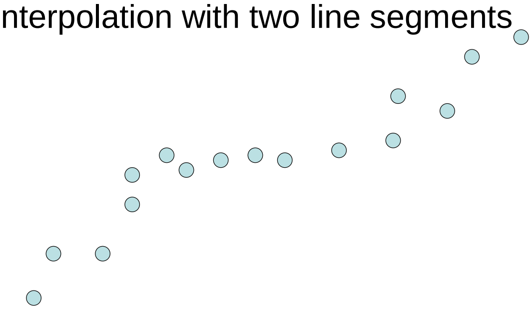


Error = $\sum (y_i - ax_i - b)^2$

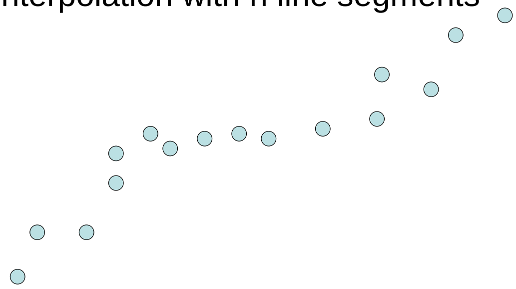
What is the optimal linear interpolation with three line segments



What is the optimal linear interpolation with two line segments

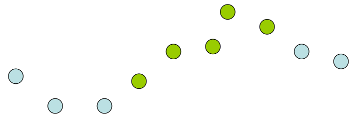


What is the optimal linear interpolation with n line segments



Notation

- Points $p_1, p_2, \ldots, p_n$ ordered by x-coordinate ($p_i = (x_i, y_i)$)
- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$



Optimal interpolation with two segments

- Give an equation for the optimal interpolation of $p_1,\ldots,p_n$ with two line segments
- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$

Optimal interpolation with k segments

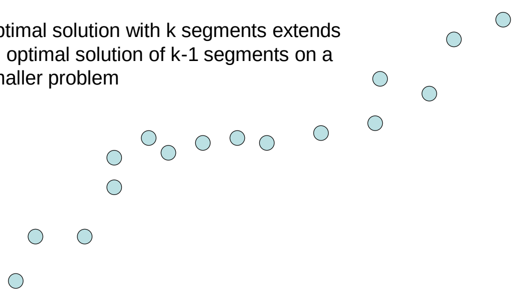
- Optimal segmentation with three segments
  - $\text{Min}_{i,j}\{E_{1,i} + E_{i,j} + E_{j,n}\}$
  - $O(n^2)$ combinations considered
- Generalization to k segments leads to considering $O(n^{k-1})$ combinations

Opt_k[j] : Minimum error approximating $p_1\ldots p_j$ with k segments

How do you express Opt_k[j] in terms of $\text{Opt}_{k-1}[1],\ldots,\text{Opt}_{k-1}$[j]?

Optimal sub-solution property

Optimal solution with k segments extends an optimal solution of k-1 segments on a smaller problem



Optimal multi-segment interpolation

Compute Opt[k, j] for 0 < k < j < n

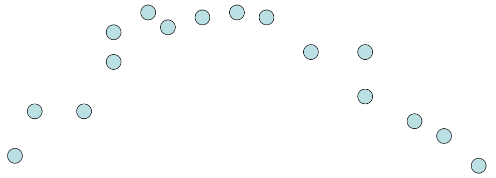
```
for j := 1 to n
   Opt[ 1, j] = E_{1,j};
for k := 2 to n-1
   for j := 2 to n
       t := E_{1,j}
       for i := 1 to j -1
          t = min (t, Opt[k-1, i ] + E_{i,j})
       Opt[k, j] = t
```

Determining the solution

- When Opt[k,j] is computed, record the value of i that minimized the sum
- Store this value in a auxiliary array
- Use to reconstruct solution

Variable number of segments

- Segments not specified in advance
- Penalty function associated with segments
- Cost = Interpolation error + C x #Segments



Penalty cost measure

- Opt[j] = min($E_{1,j}$, $\min_i$(Opt[i] + $E_{i,j}$ + P))