CSE 421: Algorithms

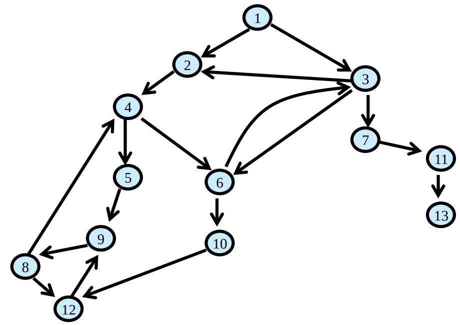
Winter 2014

Lecture 6: Graph traversal and topological sorting

Reading: Sections 3.3-3.6

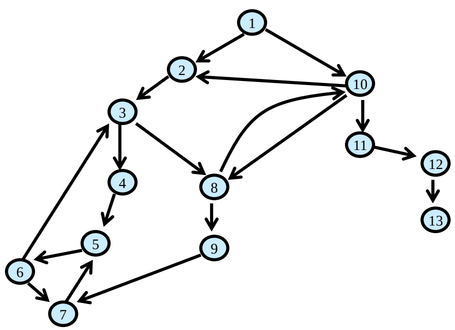
breadth-first search

- Completely explore the vertices in order of their distance from s
- Naturally implemented using a queue



depth-first search

- Completely explore the vertices in DFS order (duh)
- Naturally implemented using recursion



properties of BFS

- BFS(s) visits x if and only if there is a path in G from s to x.
- Edges followed to undiscovered vertices define a "breadth first spanning tree" of G
- Layer i in this tree, L_i
  - those vertices u such that the shortest path in G from the root s is of length i.
- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

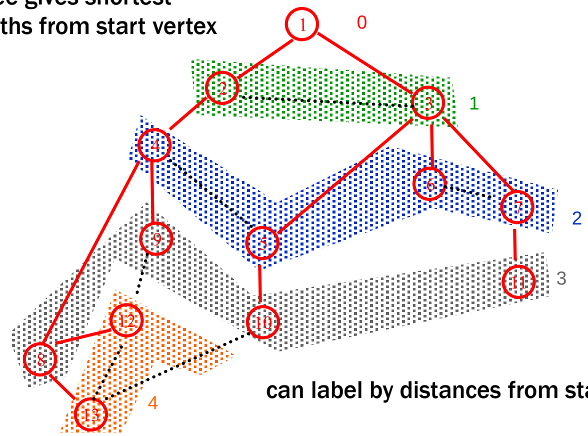
properties of BFS

On undirected graphs:

All non-tree edges join vertices on the same or adjacent layers.

BFS application: shortest paths

Tree gives shortest paths from start vertex



can label by distances from start

connected components

Want to answer questions of the form:

- **Given**: vertices u and v in G
- Is there a path from u to v?

connected components

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- **Given**: vertices u and v in G
- Is there a path from u to v?

Idea: create array A such that

$A[u]$ = smallest numbered vertex that is connected to u

- question reduces to whether $A[u] = A[v]$?

connected components

Want to answer questions of the form:

- Given: vertices u and v in G
- Is there a path from u to v?

Q: Why not create an array **Path**[u,v]?

Idea: create array A such that

$A[u]$ = smallest numbered vertex that is connected to u

- question reduces to whether $A[u] = A[v]$?

connected components

- initial state: all **v** unvisited

```
for s ← 1 to n do
    if state(s) ≠ "fully-explored" then
        BFS(s): setting A[u] ←s for each u found
            (and marking u visited/fully-explored)
    endif
endfor
```

- Total cost: $O(n + m)$
  - each vertex is touched once in this outer procedure and the edges examined in the different BFS runs are disjoint
  - works also with depth first search

DFS(u) – recursive version

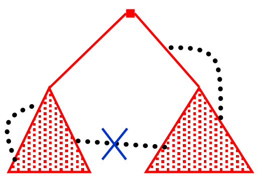
```
Global Initialization: mark all vertices "unvisited"
DFS(u)
  mark u "visited" and add u to R
  for each edge (u,v)
      if (v is "unvisited")
          DFS(v)
  mark u "fully-explored"
```

properties of DFS(s)

- Like BFS(s):
  - DFS(s) visits x if and only if there is a path in G from s to x
  - Edges into undiscovered vertices define a "depth first spanning tree" of G
- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- but…

non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- No cross edges.

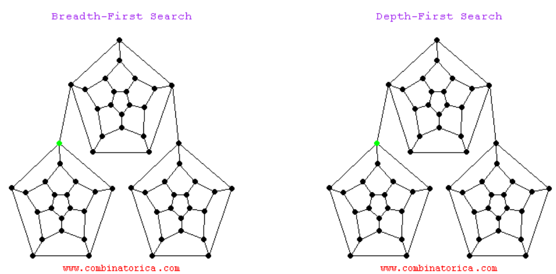


bipartite graph

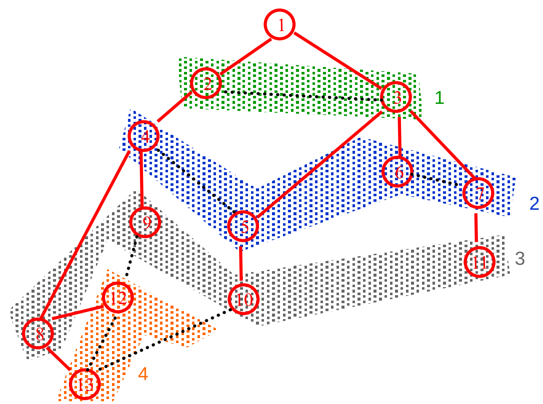
Definition:

Theorem:
Graph is bipartite iff does not contain an odd cycle.

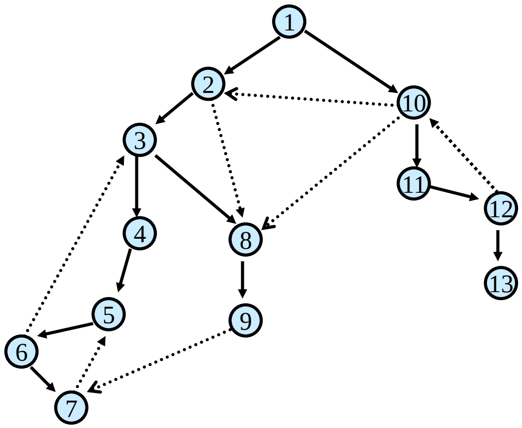
bfs vs dfs



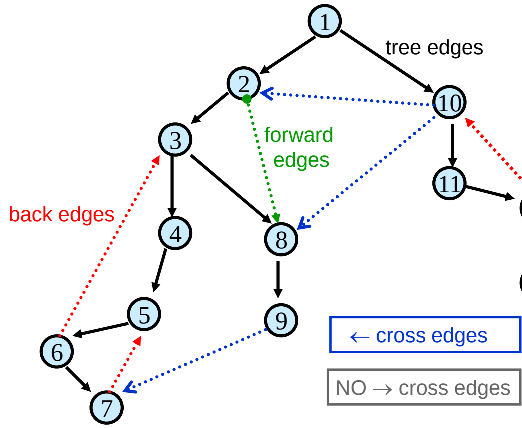
BFS for bipartite testing



DFS(v) for a directed graph



DFS(v)

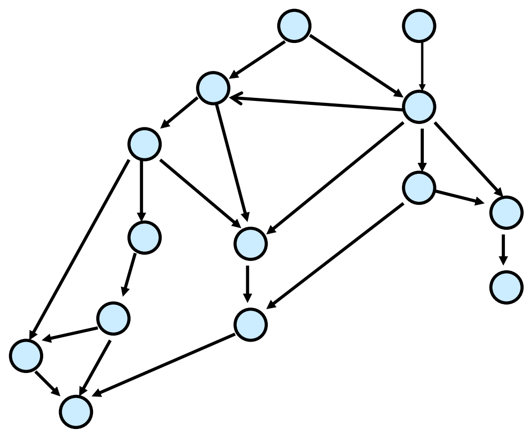


directed acyclic graphs

- A directed graph $G = (V, E)$ is **acyclic** if it has no **directed** cycles

- Terminology: A **d**irected **a**cyclic **g**raph is also called a **DAG**

directed acyclic graph



topological sort

- **Given:** a directed acyclic graph (DAG) **G=(V,E)**
- **Output:** **numbering** of the vertices of **G** with **distinct** numbers from **1** to **n** so edges only go from lower number to higher numbered vertices

- Applications
  - nodes represent tasks
  - edges represent precedence between tasks
  - topological sort gives a sequential schedule for solving them

topological sort

- **Given:** a directed acyclic graph (DAG) **G=(V,E)**
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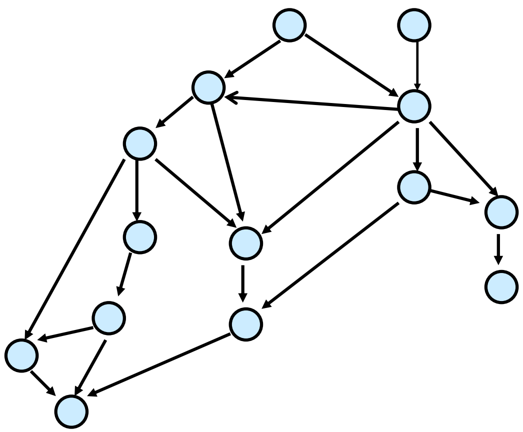
directed acyclic graph



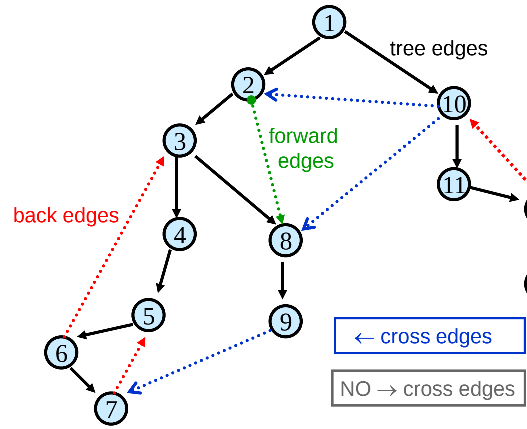
in-degree 0 vertices

Lemma: Every DAG has a vertex of in-degree 0

topological sort

- Can do using DFS

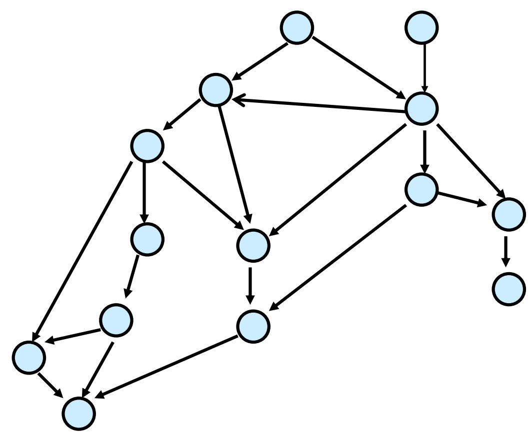
DFS(v)



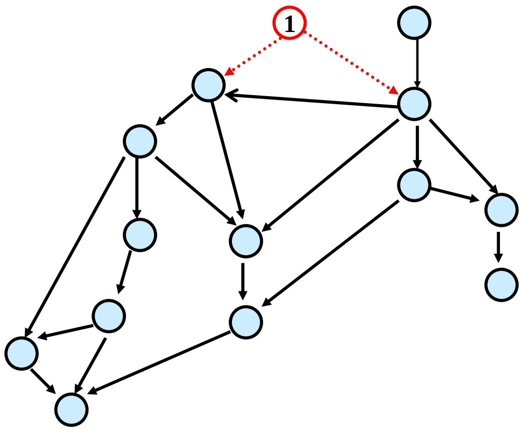
topological sort

- Can do using DFS
- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.

topological sort



topological sort



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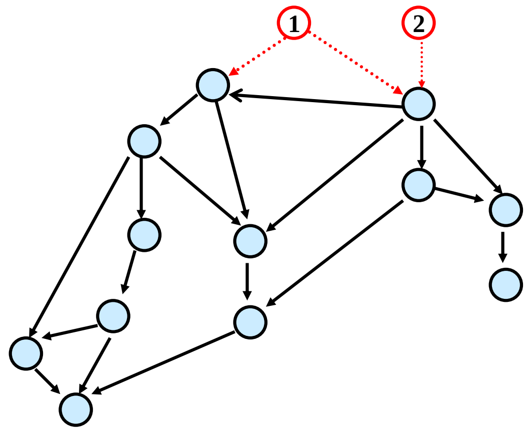
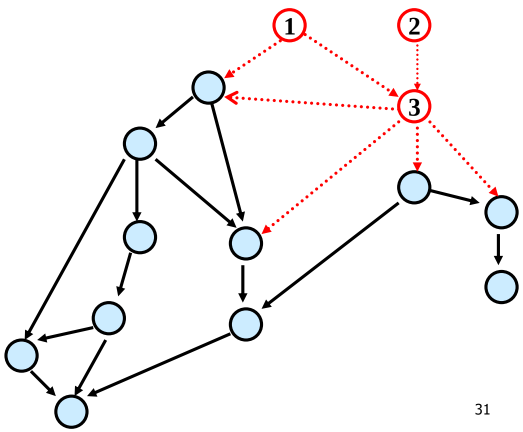


topological sort



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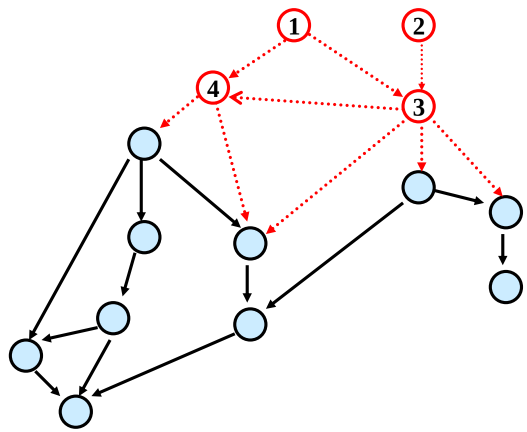

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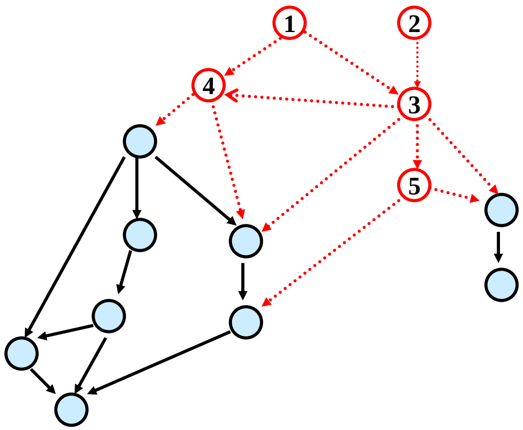

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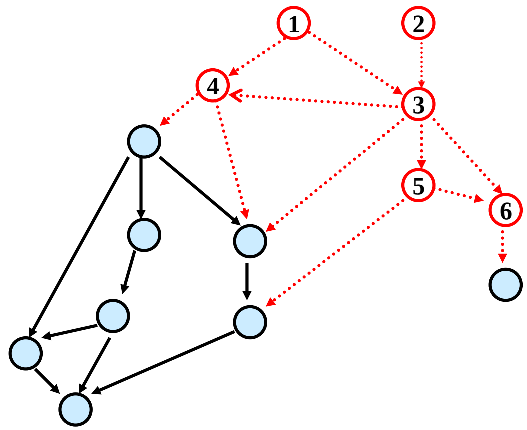

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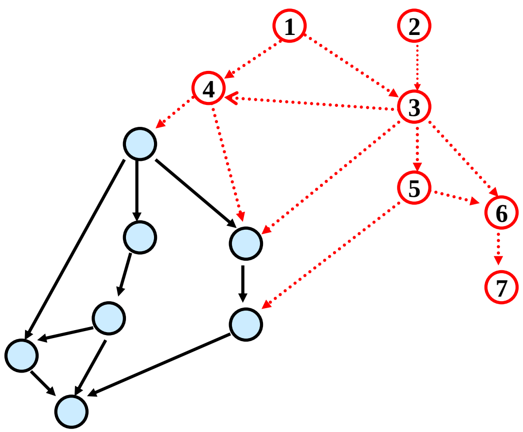

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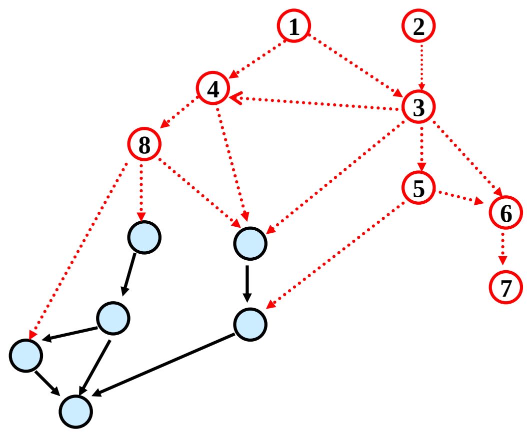

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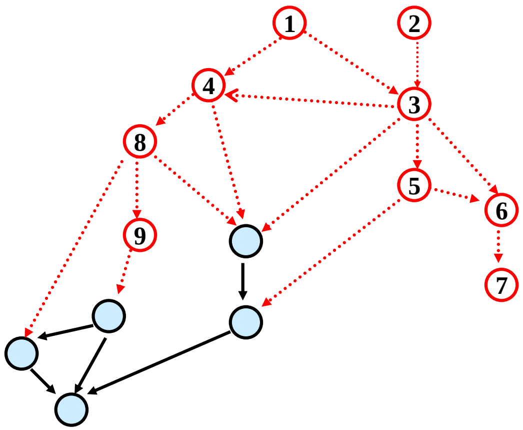

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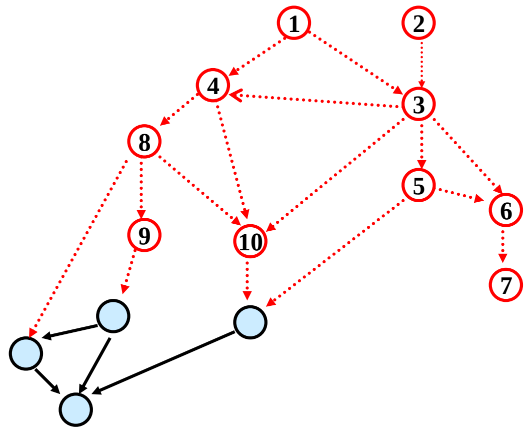

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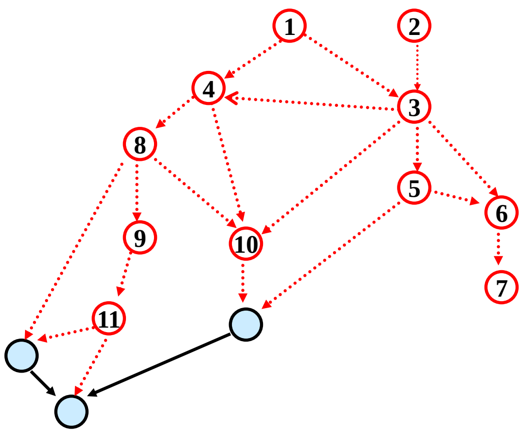

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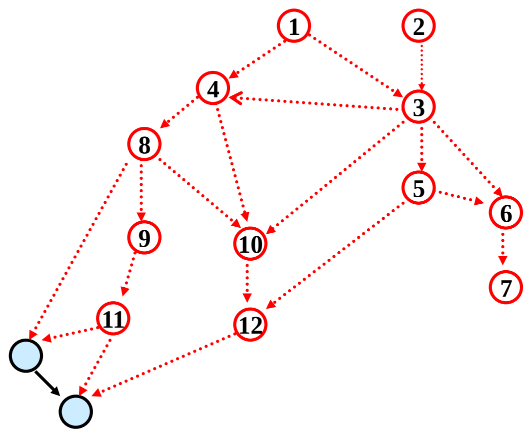

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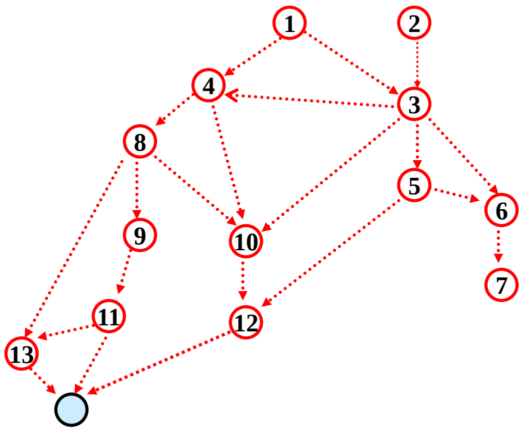

implementing topological sort

- Go through all edges, computing array with in-degree for each vertex $O(m+n)$
- Maintain a queue (or stack) of vertices of in-degree **0**
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by **1** and add them to the queue if their degree drops to **0**

Total cost:

topological sort