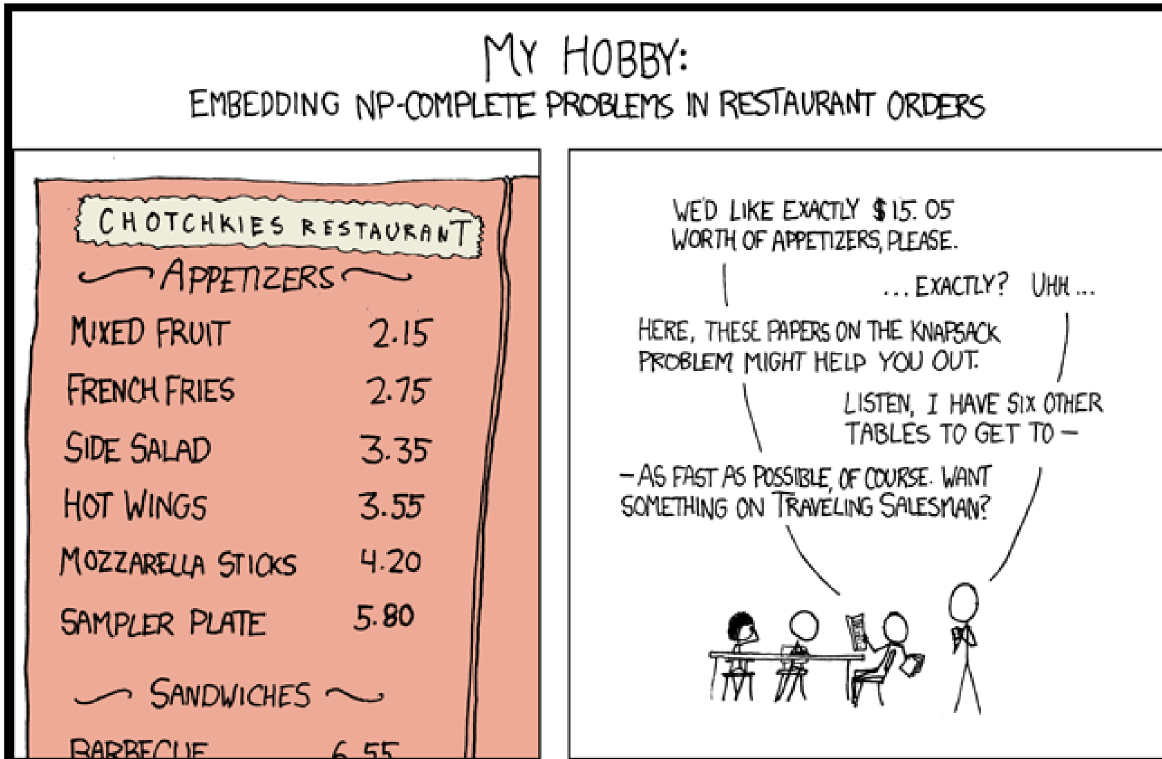
CSE 421: Algorithms

Winter 2014

Lecture 23: P, NP, and reductions

Reading:
Sections 8.3-8.7



polynomial time

Define P (polynomial-time) to be

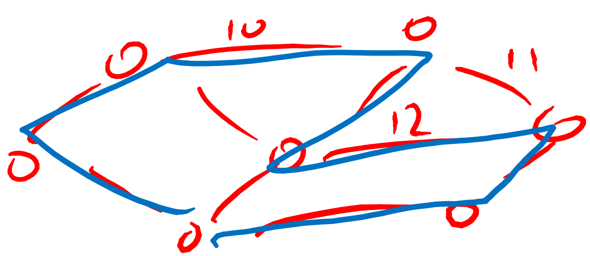
- the set of all **decision problems** solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.

decision problems: output YES/NO

beyond P?

- There are many other natural, practical problems for which we don't know any polynomial-time algorithms

- For example: decisionTSP
  - Given a weighted graph **G** and an integer **k**, does there exist a tour that visits all vertices in **G** having total weight at most **k**?



$\overset{?}{\in} P$

satisfiability

- Boolean variables $x_1,\ldots,x_n$
  - taking values in $\{0,1\}$. 0=false, 1=true
- Literals
  - x_i or $\neg x_i$ for $i=1,\ldots,n$
- Clause
  - a logical OR of one or more literals
  - e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$
- CNF formula $C_1 \wedge C_2 \wedge \cdots \wedge C_m$
  - a logical AND of a bunch of clauses

$C_i = (x_{17} \vee x_3 \vee \neg x_2)$

- k-CNF formula $k=3$
  - All clauses have exactly k variables

$(x_3 \vee \neg x_4 \vee x_7) \wedge (x_2 \vee x_1) \wedge (x_{12} \vee \neg x_1)$

satisfiability

$(a \wedge a \to b) \to b$ $x_1 = x_3 = 1$ $x_2 = 0$ $x_4 = 0$

$p \to q$ $\neg p \vee q$

$x_1 = x_2 = x_3 = 1, x_4 = 0$

- CNF formula example

  $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$

- If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - the one above is, the following isn't $x_3 = 0$

$\times$ – $x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$ $x_2 = 0$ $x_1 = 0$

- 3-SAT: Given a CNF formula F with 3 variables per clause, is it satisfiable?

$\overset{?}{\in} P$

common property of these problems

- There is a special piece of information, a **short certificate** or proof, that allows you to **efficiently verify** (in polynomial-time) that the YES answer is correct. This certificate might be very hard to find

- e.g.
  - **DecisionTSP:** (G, k) is there a tour of weight ≤ k?
  - **Independent-Set, Clique:** Give me the clique or ind. set
  - **3-SAT:** satisfy assignment

TAUTOLOGY: Does every assgn satisfy F?

The complexity class NP (non-deterministic polynomial)

$\boldsymbol{NP}$ consists of all decision problems where

- You can **verify** the YES answers efficiently (in polynomial time) given a short (polynomial-size) **certificate**

and

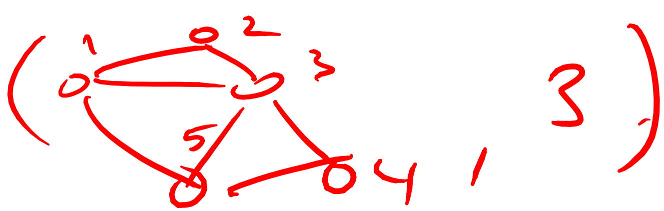
- **No certificate** can fool your polynomial time verifier into saying YES for a NO instance

more precise definition of $\mathcal{NP}$

- A decision problem is in $\mathcal{NP}$ iff there is a polynomial time procedure **verify(.,.)**, and an integer **k** such that
  - for every input **x** to the problem that is a **YES** instance there is a certificate **t** with $|t| \leq |x|^k$ such that **verify(x,t) = YES**

  and

  - for every input **x** to the problem that is a **NO** instance there does **not** exist a certificate **t** with $|t| \leq |x|^k$ such that **verify(x,t) = YES**



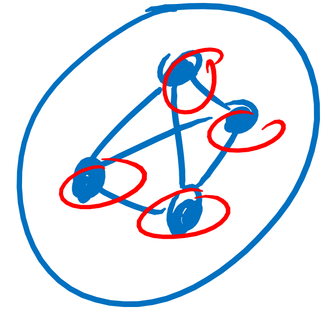
$t = \{3, 4, 5\}$

$t = \{1, 3, 5\}$

CLIQUE is in NP

$X = (G, k)$ instance

$a \to b \iff \neg b \to \neg a$



procedure **verify**(x,t)

if **x** is a well-formed representation of a graph **G** = (**V**, **E**) and an integer **k**,

and

t is a well-formed representation of a vertex subset **U** of **V** of size **k**,

and

U is a clique in **G**,

then output "**YES**"

else output "**I'm unconvinced**"

(G,k) is a YES instance $\iff \exists t$ s.t. verify$((G,k), t)$ says YES.

(G,k) NO instance $\Rightarrow \forall t$ verify$((G,k), t)$ says NO.

is it correct?

✓

keys to showing a problem is in NP

- What's the output? (must be YES/NO)
- What must the input look like?
- Which inputs need a YES answer?
  - Call such inputs YES inputs/YES instances

- For every given YES input, is there a certificate that would help?
  - OK if some inputs need no certificate
- For any given NO input, is there a fake certificate that would trick you?

solving NP problems without hints

The only **obvious algorithm** for most of these problems is **brute force**:

- try all possible certificates and check each one to see if it works.
- *Exponential* time:

  2^n truth assignments for n variables

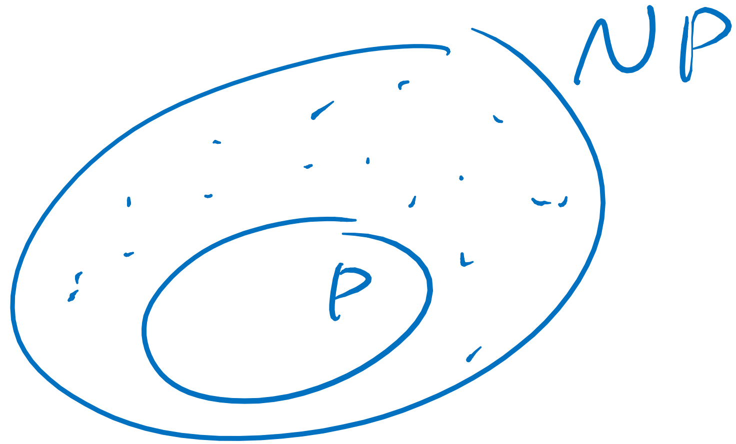
  $n!$ possible TSP tours of n vertices

  $\binom{n}{k}$ possible k element subsets of n vertices

  etc.

what we know

- Nobody knows if all problems in **NP** can be done in polynomial time, i.e. does **P = NP ?**
  - one of the most important open questions in all of science.
  - huge practical implications



- Every problem in **P** is in **NP**

- Every problem in **NP** can be solved in exponential time

P vs. NP

solving NP problems in exponential time

Every problem in NP has alg. running in time $2^{O(n^c)}$ for some $c > 0$.

$2^{n^k} \cdot n^a$

Try verify(x,t) $\forall\ |t| \le |x|^k$

$A \in NP$ Then A has a verifier verify(x,t)

x a YES inst. $\Rightarrow \exists\ |t| \le |x|^k$ s.t. verify(x,t) accepts.

NP-hardness & NP-completeness

- Alternative approach to proving problems not in **P**
  - show that they are at least as hard as any problem in **NP**

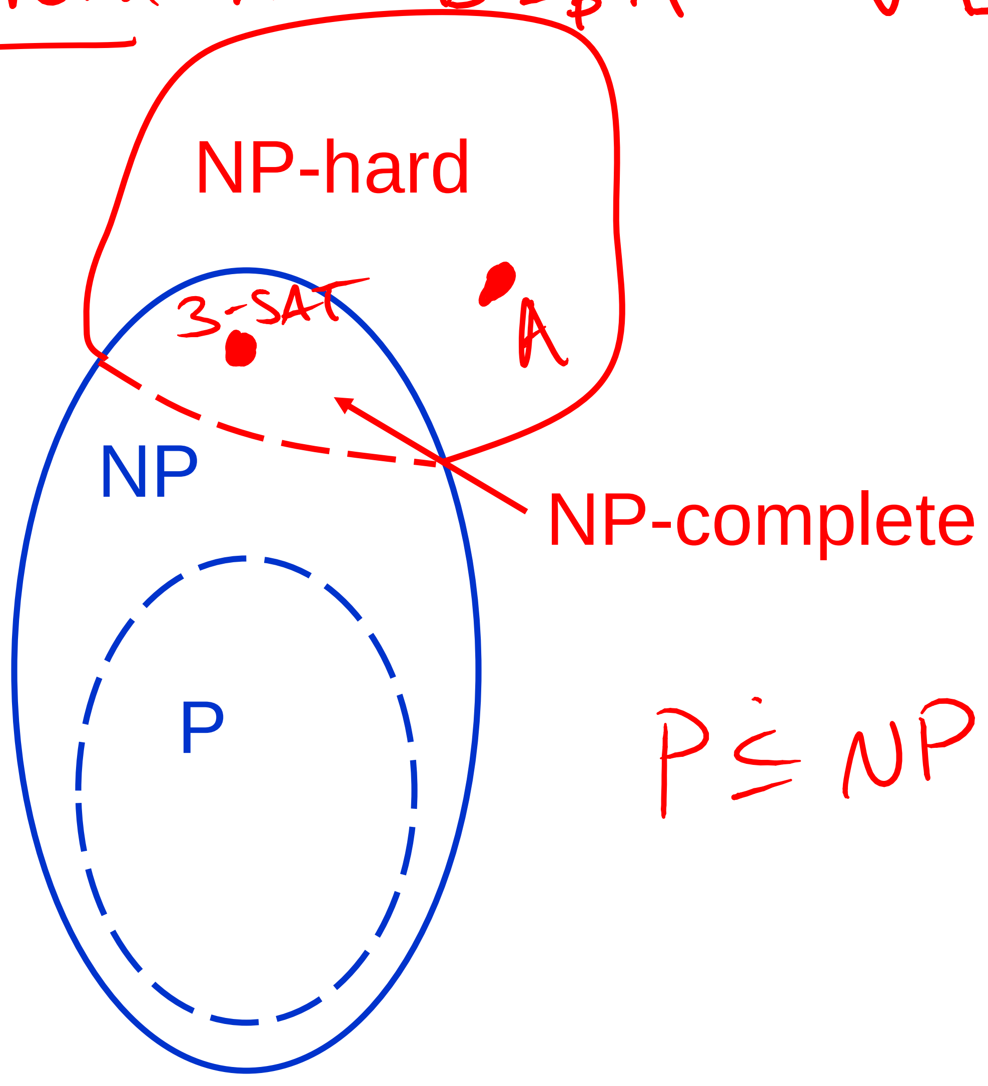
- Rough definition:
  - A problem is NP-hard iff it is at least as hard as any problem in **NP**
  - A problem is NP-complete iff it is both

    NP-hard

    in **NP**

P and NP

A is NP-hard if $B \leq_p A \quad \forall B \in NP$.



$P \dot{\leq} NP$

NP-hardness & NP-completeness

- **Definition:** A problem **B** is NP-hard iff every problem $A \in \mathit{NP}$ satisfies $A \leq_P B$

- **Definition:** A problem **B** is NP-complete iff **A** is NP-hard and $A \in \mathit{NP}$

- Even though we seem to have lots of hard problems in NP it is not obvious that such super-hard problems even exist!

Cook-Levin Theorem

- Theorem (Cook 1971, ~~Levin 1973~~):

  3-SAT is NP-complete.

- Recall
  - CNF formula

    $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
  - If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - 3-SAT: Given a 3-CNF formula F, is it satisfiable?

implications of the Cook-Levin theorem?

- There is at least one interesting super-hard problem in NP
- Is that such a big deal?
- Yes, a jumping off point.
  - There are lots of other problems that can be solved if we had a polynomial-time algorithm for 3-SAT
  - Many of these problems are exactly as hard as 3-SAT

A useful property of polynomial-time reductions

- **Theorem:** If $A \leq_P B$ and $B \leq_P C$ then $A \leq_P C$
- **Proof idea:** (Using $\leq_P^1$)
  - Compose the reduction f from A to B with the reduction g from B to C to get a new reduction $h(x)=g(f(x))$ from A to C.
  - The general case is similar and uses the fact that the composition of two polynomials is also a polynomial

A useful property of polynomial-time reductions

- **Theorem:** If $\mathbf{A} \leq_\mathbf{P} \mathbf{B}$ and $\mathbf{B} \leq_\mathbf{P} \mathbf{C}$ then $\mathbf{A} \leq_\mathbf{P} \mathbf{C}$
- **Proof idea:**

$p(x)$ $q(x)$ $q(p(x))$.

Cook-Levin theorem & implications

- Theorem (Cook 1971, Levin 1973):
  3-SAT is NP-complete (for proof see CSE 431)

- Corollary: B is NP-hard $\Leftrightarrow$ 3-SAT $\le_P$ B
  (or A $\le_P$ B for any NP-complete problem A)

$$\underbrace{A \le_p 3\text{-SAT}}_{\text{thm}} \le_p B$$

- Proof:
  - If B is NP-hard then every problem in NP polynomial-time reduces to B, in particular 3-SAT does since it is in NP
  - For any problem A in NP, A $\le_P$ 3-SAT and so if
    3-SAT $\le_P$ B we have A $\le_P$ B.
    therefore B is NP-hard if 3-SAT $\le_P$B

3-SAT $\leq_P$ Independent-Set

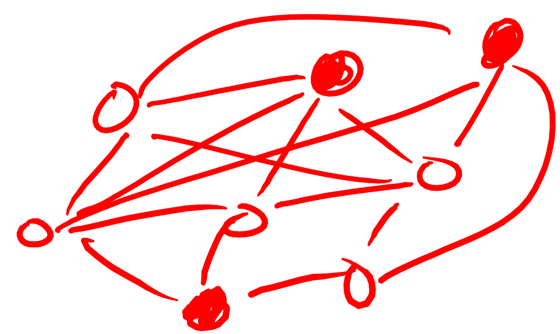
$A \in NP$

$A \leq_p 3\text{-SAT} \leq_p \text{Ind-Set}$

- **A Tricky Reduction:**
  - **mapping CNF formula F to a pair <G,k>**
  - **Let m be the number of clauses of F**
  - **Create a vertex in G for each literal in F**
  - **Join two vertices u, v in G by an edge iff**

    u and v correspond to literals in the same clause of F, (green edges) or

    u and v correspond to literals x and $\neg$x (or vice versa) for some variable x. (red edges)
  - **Set k=m**
  - **Clearly polynomial-time**



3-SAT $\leq_P$ Independent-Set

$x_1 = x_2 = x_3 = 1$

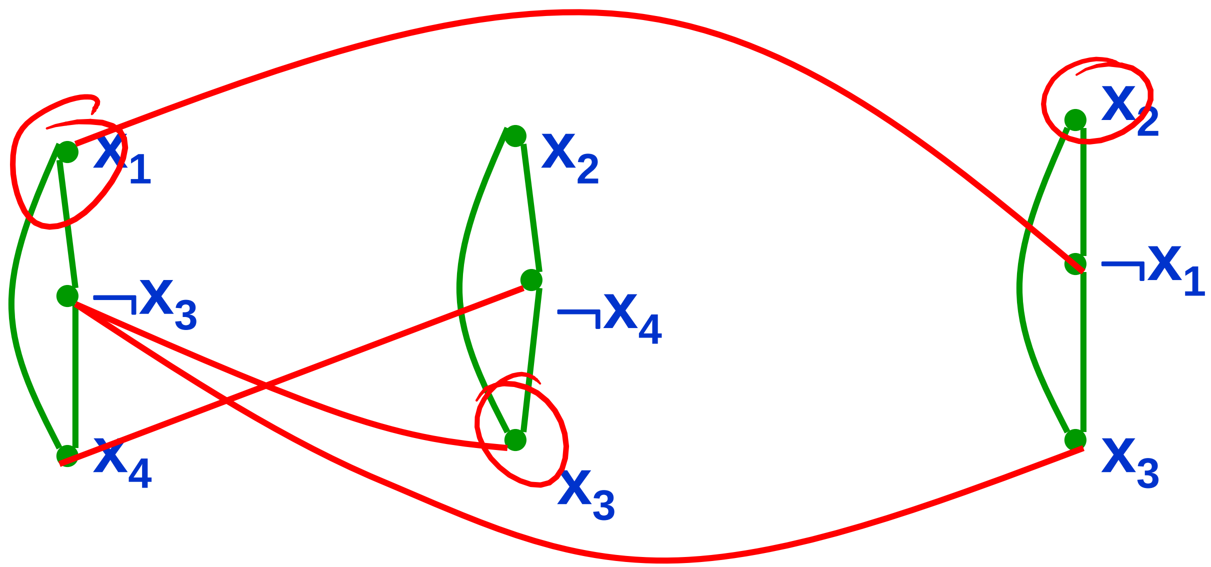
$x_4 = 0$

$F \xrightarrow{P} (G, k)$

$m = \#$ clauses in F

$k = m$

F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$



3-SAT $\leq_P$ Independent-Set

- **Correctness:**
  - **If F is satisfiable then there is some assignment that satisfies at least one literal in each clause.**
  - Consider the set **U** in **G** corresponding to the **first satisfied literal in each clause.**

    |U|=m

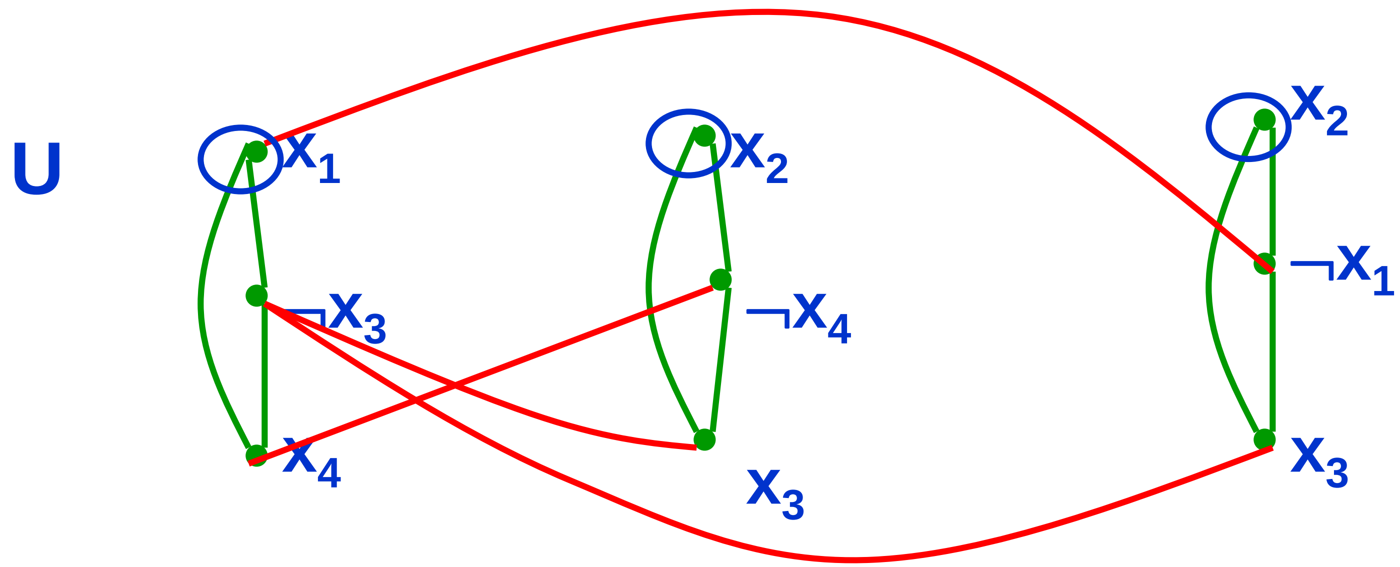
    Since **U** has only one vertex per clause, no two vertices in **U** are joined by green edges

    Since a truth assignment never satisfies both **x** and **¬x**, **U** doesn't contain vertices labeled both **x** and **¬x** and so no vertices in **U** are joined by red edges

    Therefore **G** has an independent set, **U**, of size at least **m**
  - **Therefore (G,m) is a YES for independent set.**

3-SAT $\leq_P$ Independent-Set

$$\begin{array}{l} \quad\;\; 1 \quad\;\; 0 \quad\;\; 1 \qquad\;\; 1 \quad\;\; 0 \quad\;\; 1 \qquad\;\; 1 \quad\;\; 0 \quad\;\; 1 \\ F: \; (x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3) \end{array}$$



Given assignment $x_1=x_2=x_3=x_4=1$,

U is as circled

3-SAT $\leq_P$ Independent-Set

- **Correctness continued:**
  - **If (G,m) is a YES for Independent-Set then there is a set U of m vertices in G containing no edge.**

    Therefore **U** has precisely one vertex per clause because of the green edges in **G.**

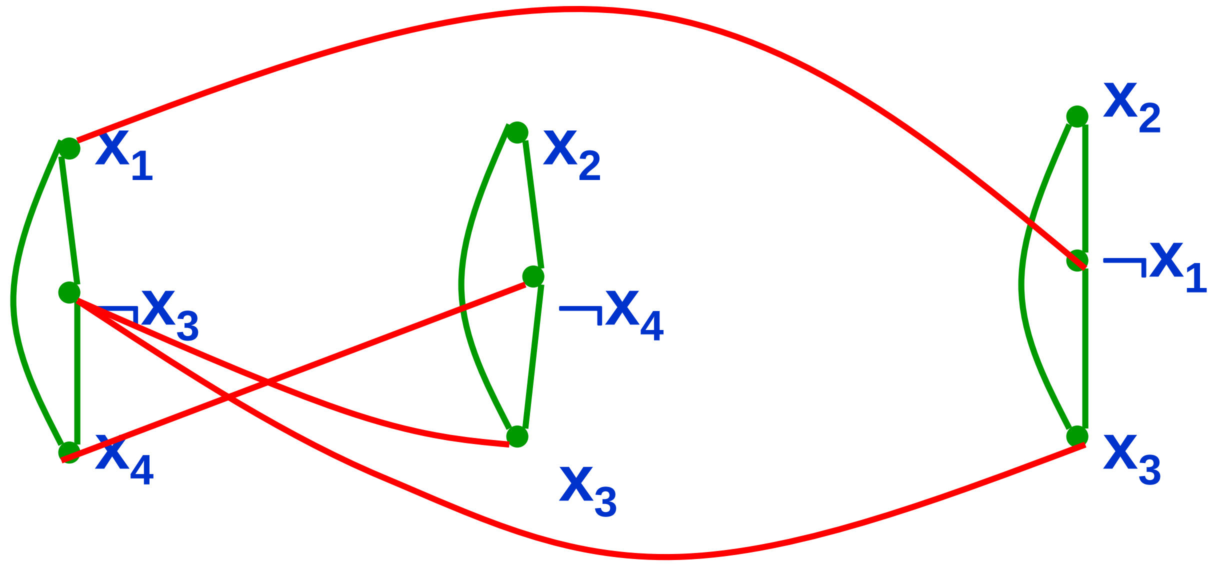
    Because of the red edges in **G**, **U** does not contain vertices labeled both **x** and **¬x**

    Build a truth assignment **A** that makes all literals labeling vertices in **U** true and for any variable not labeling a vertex in **U**, assigns its truth value arbitrarily.

    By construction, **A** satisfies **F**
  - **Therefore F is a YES for 3-SAT.**

3-SAT $\leq_P$ Independent-Set

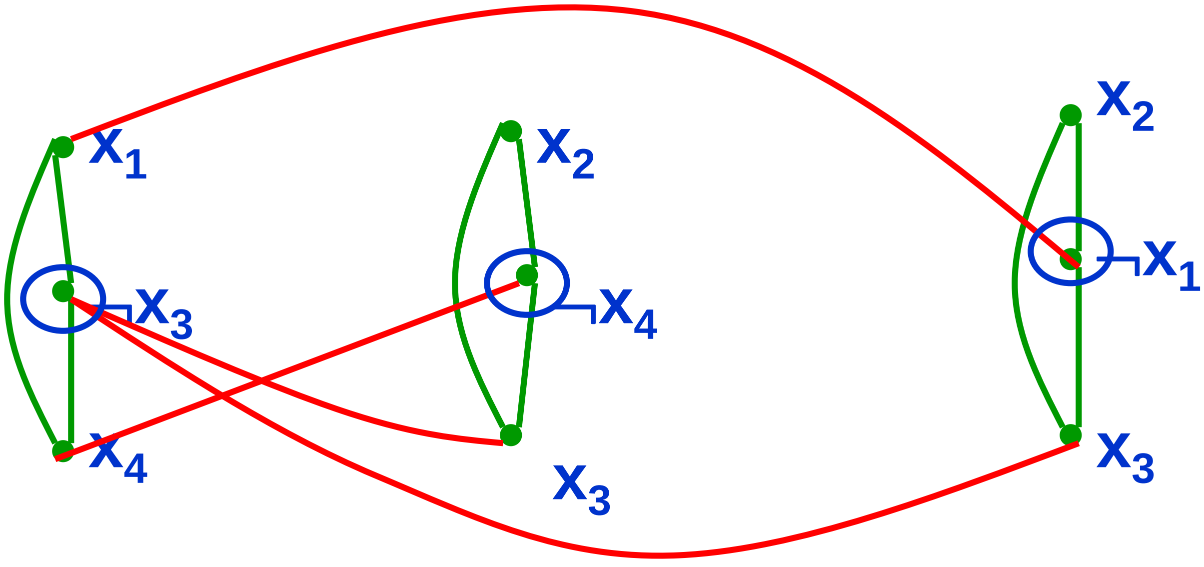
F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$



3-SAT $\leq_P$ Independent-Set

0 1 0 ? 1 0 ? 1 0

F: $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$



Given **U**, satisfying assignment is $x_1 = x_3 = x_4 = 0$, $x_2 = 0$ or 1

Independent-Set is NP-complete

- We just showed that Independent-Set is NP-hard and we already knew Independent-Set is in NP.

- Corollary: Clique is NP-complete
  - We showed already that Independent-Set $\leq_P$ Clique and Clique is in NP.