CSE 421: Algorithms

Winter 2014

Lecture 21: Edmonds-Karp and Project Selection

Reading:
Sections 7.3-7.5

Edmonds-Karp Algorithm

- Use a **shortest** augmenting path (via BFS in residual graph)

- Time: $O(n\,m^2)$

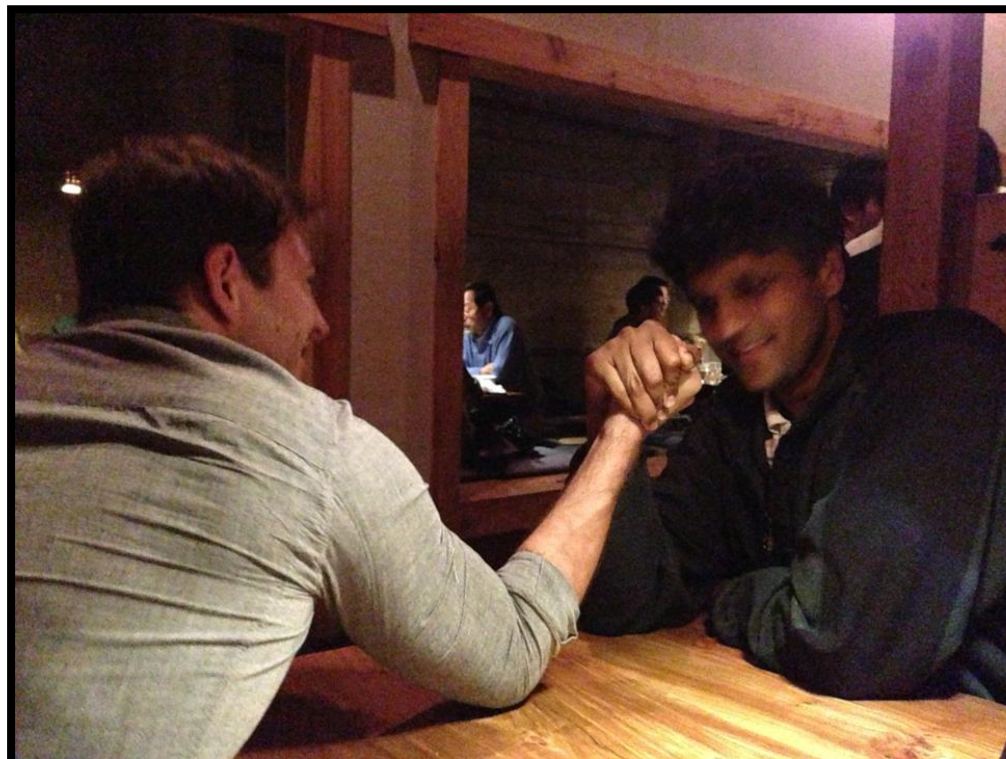
current best: Goldberg-Rao

15 yrs.

- **Time:** $O\left(\boldsymbol{m}\min\left\{m^{\frac{1}{2}}, n^{\frac{2}{3}}\right\}\log\left(\frac{n^2}{m}\right)\log U\right)$

Likely:
will be near-
linear time
alg

almost O(m+n)



bfs/shortest-path lemmas

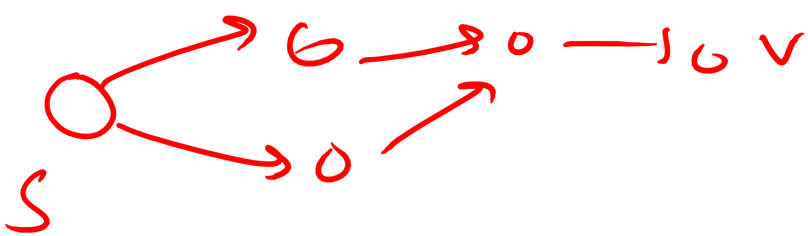
Distance from s in G_f is never reduced by:

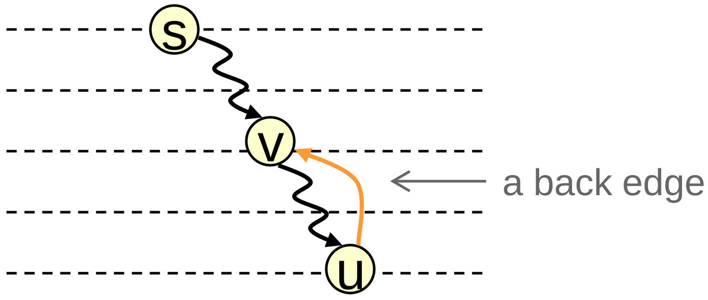
- Deleting an edge

  Proof: no new (hence no shorter) path created

- Adding an edge (u,v), provided v is nearer than u

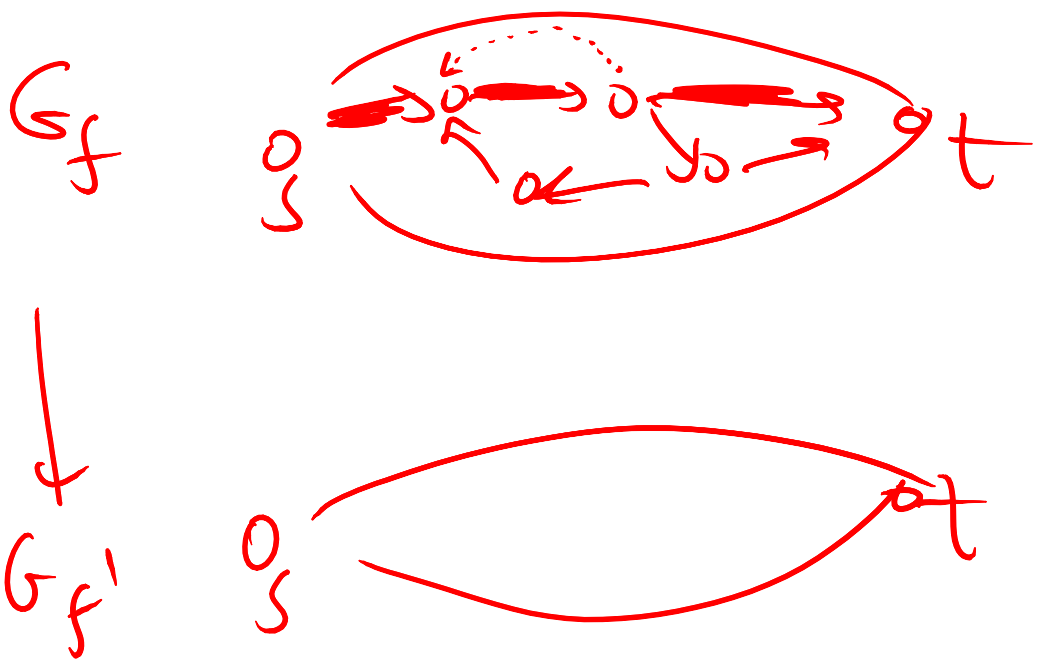
  Proof: BFS is unchanged, since v visited before (u,v) examined





key lemma

Let **f** be a flow, G_f the residual graph, and **P** a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

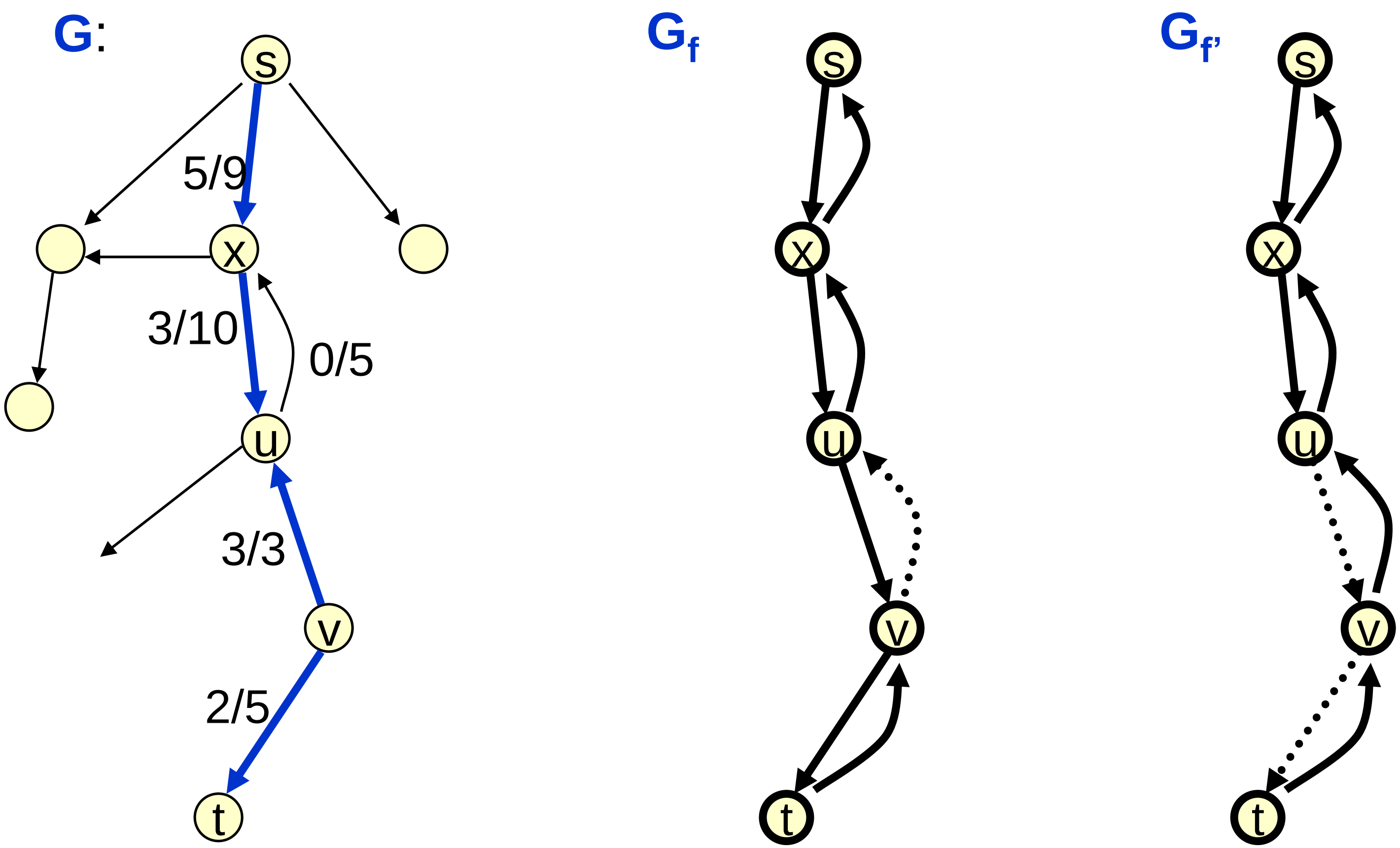


key lemma

Let f be a flow, G_f the residual graph, and P a shortest augmenting path. Then no vertex is closer to s after augmentation along P.

Proof: Augmentation along P only deletes forward edges, or adds back edges that go to previous vertices along P

augmentation vs BFS



theorem

The Edmonds-Karp Algorithm performs O(mn) flow augmentations.

Proof:

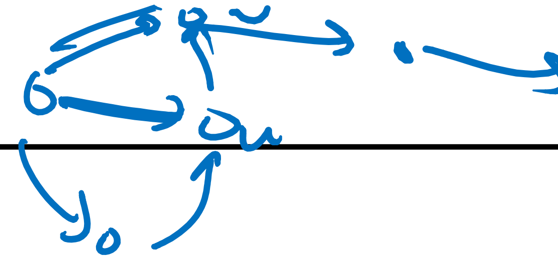
Call (u,v) critical for augmenting path P if it's closest to s having min residual capacity

It will disappear from G_f after augmenting along P

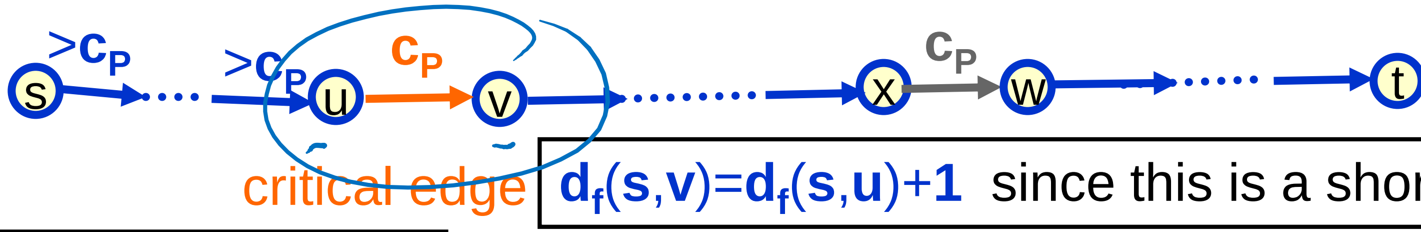
* In order for (u,v) to be critical again the (u,v) edge must re-appear in G_f but that will only happen when the distance to u has increased by 2 (next slide)

It won't be critical again until farther from s so each edge critical at most n/2 times $\Rightarrow \leq \frac{mn}{2}$ augmentations

critical edges in G_f

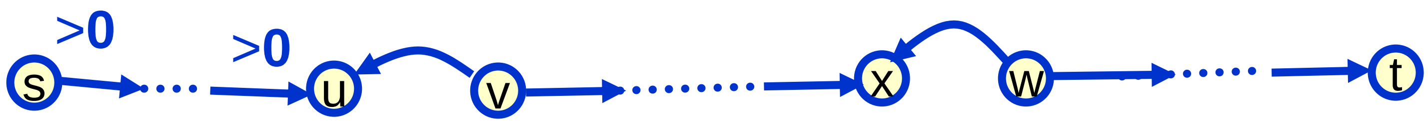


Shortest s-t path **P** in $\mathbf{G_f}$

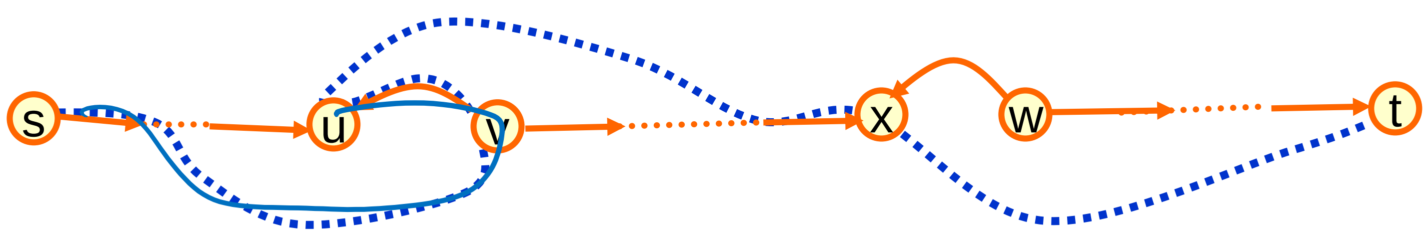


$d_f(s,v)=d_f(s,u)+1$ since this is a shortest path

After augmenting along **P**



For **(u,v)** to be critical later for some flow **f'** it must be in $\mathbf{G_{f'}}$
so must have augmented along a shortest path containing **(v,u)**



Then we must have $d_{f'}(s,u)=d_{f'}(s,v)+1 \geq d_f(s,v)+1=d_f(s,u)+2$

corollary

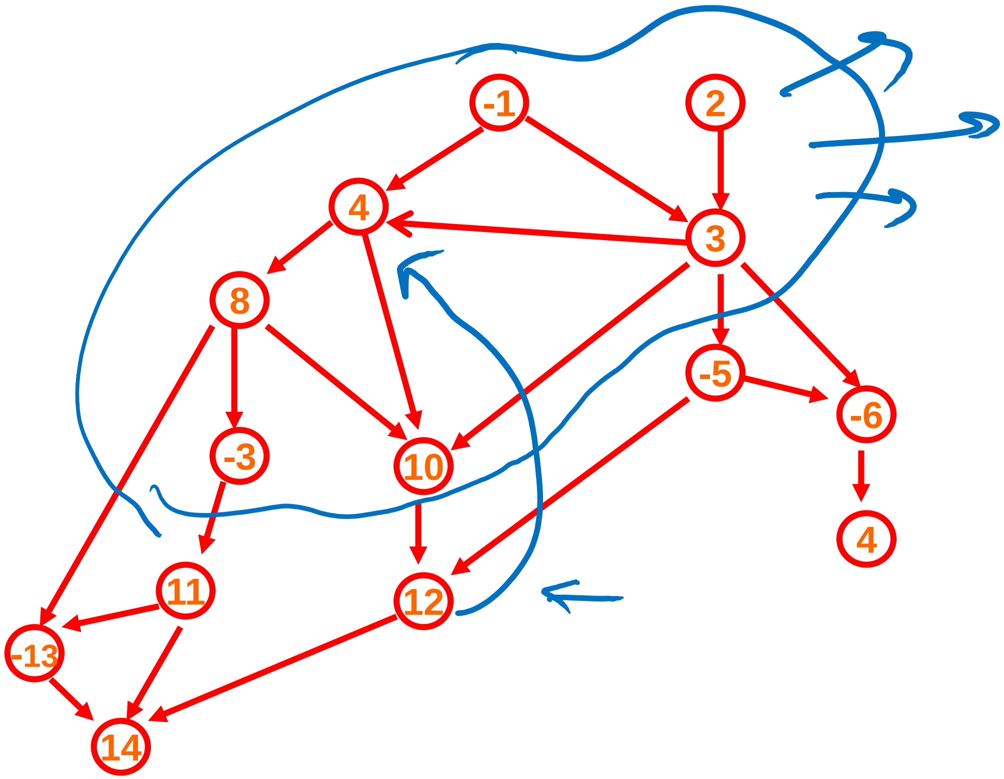
- Edmonds-Karp runs in $O(nm^2)$ time

project selection

- **Given**
  - a directed acyclic graph $G=(V,E)$ representing precedence constraints on tasks (a task points to its **predecessors**)
  - a profit value $p(v)$ associated with each task $v \in V$ (may be positive or negative)

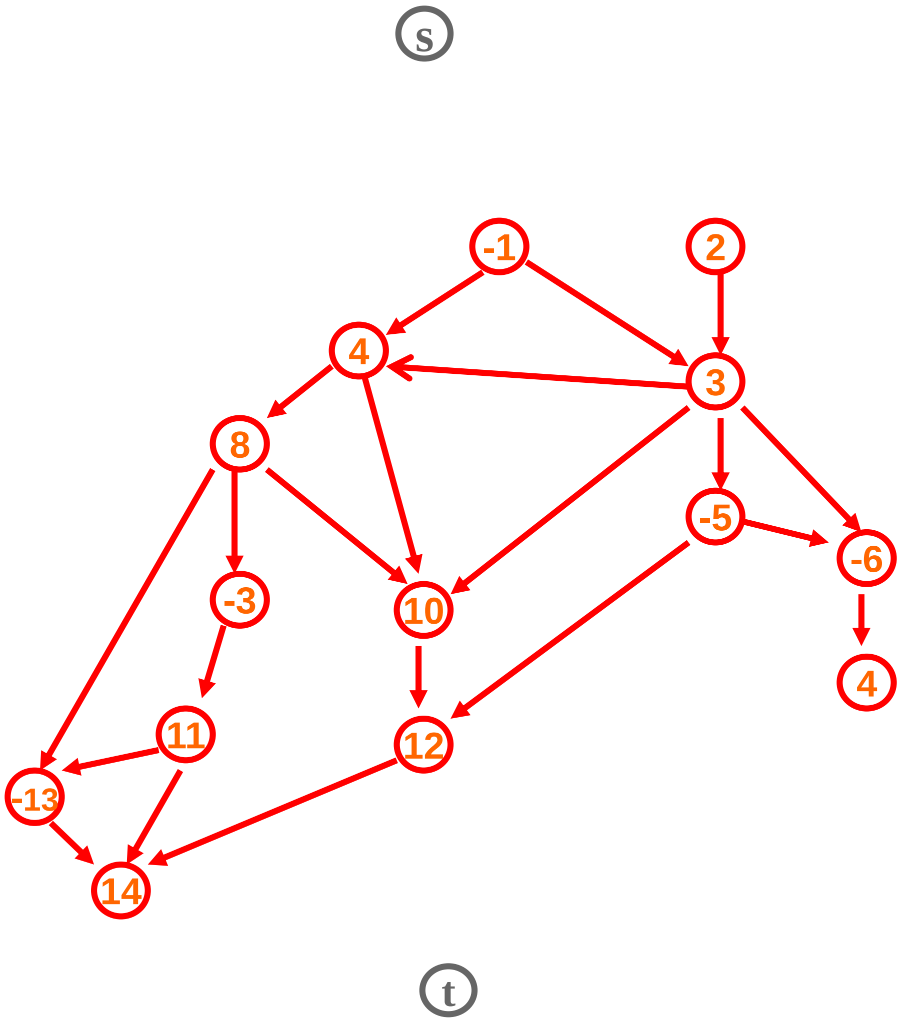
- **Find**
  - a set $A \subseteq V$ of tasks that is closed under predecessors, i.e. if $(u,v) \in E$ and $u \in A$ then $v \in A$, that maximizes $\text{Profit}(A) = \sum_{v \in A} p(v)$

project selection graph



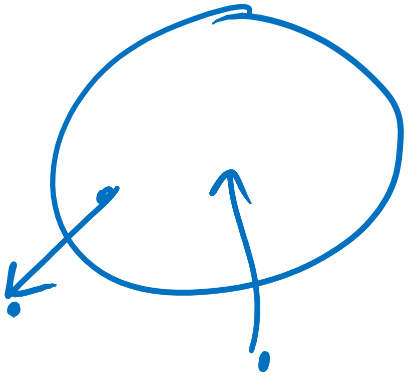
Each task points to its predecessor tasks

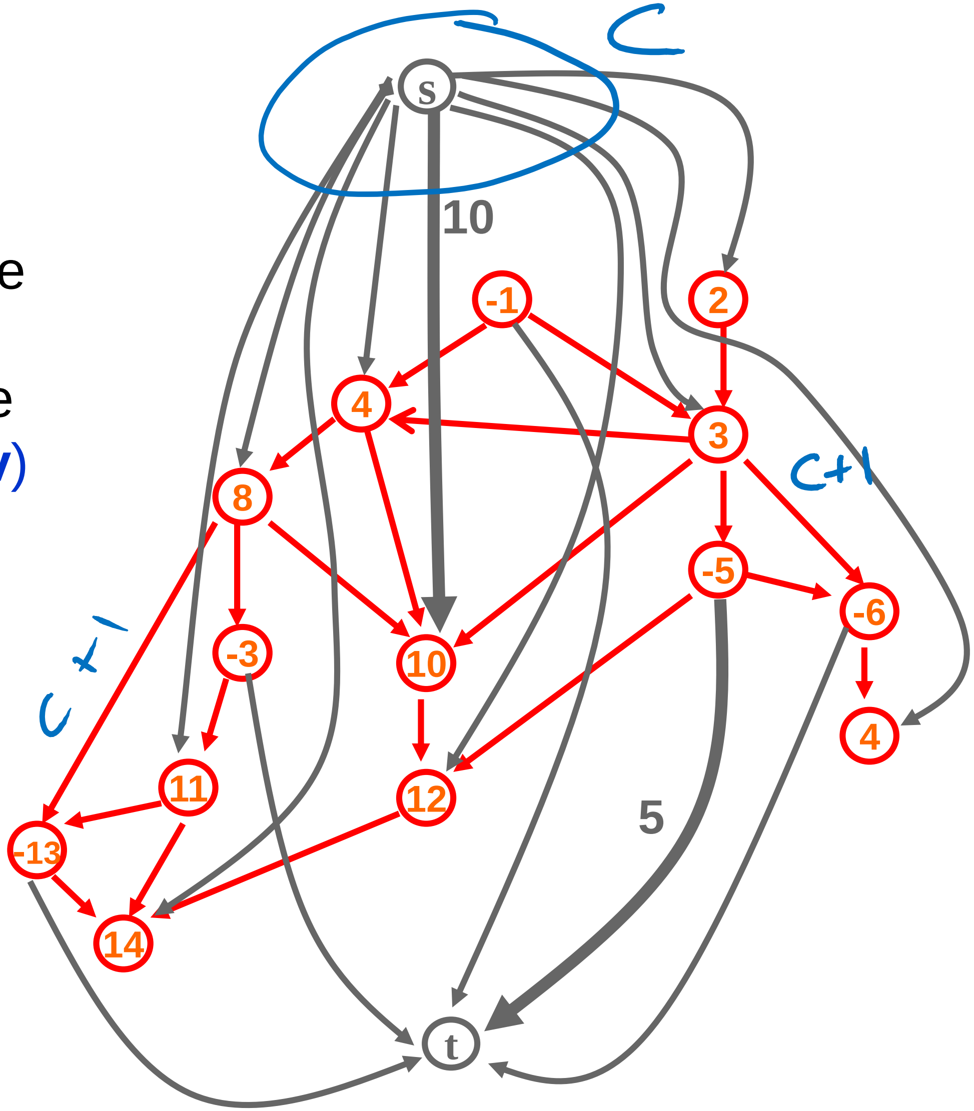
extended graph



extended graph G'

For each vertex **v**
If **p(v)≥0** add **(s,v)** edge
with capacity **p(v)**
If **p(v)<0** add **(v,t)** edge
with capacity **–p(v)**

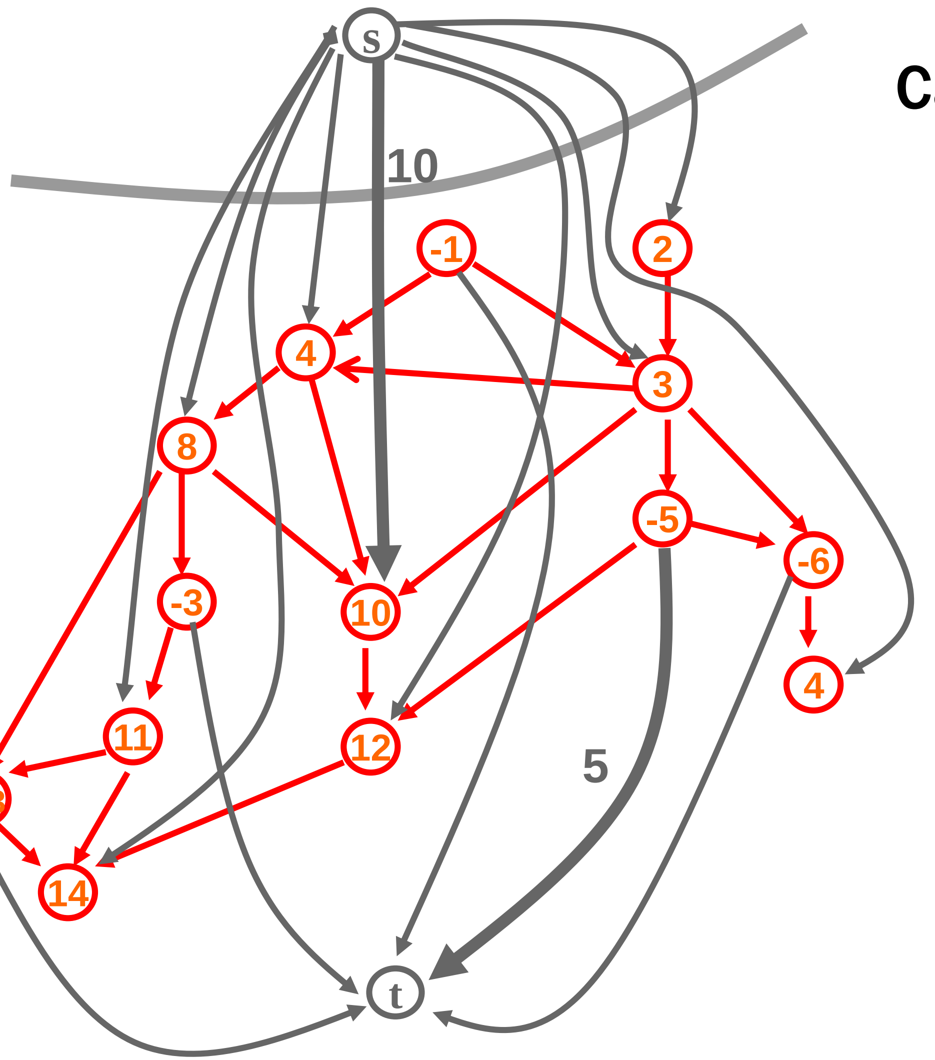




extended graph G'

- Want to arrange capacities on edges of G so that for minimum s-t-cut (S,T) in G', the set A=S-{s}
  - satisfies precedence constraints
  - has maximum possible profit in G
- Cut capacity with S={s} is just $C=\sum_{v:\, p(v)\geq 0} p(v)$
  - $\text{Profit}(A) \leq C$ for any set A
- To satisfy precedence constraints don't want any original edges of G going forward across the minimum cut
  - That would correspond to a task in A=S-{s} that had a predecessor not in A=S-{s}
- Set capacity of each of the edges of G to C+1
  - The minimum cut has size at most C

extended graph G'



Capacity C

extended graph G'

cut value = − (stuff we spend) + C − (money we make)

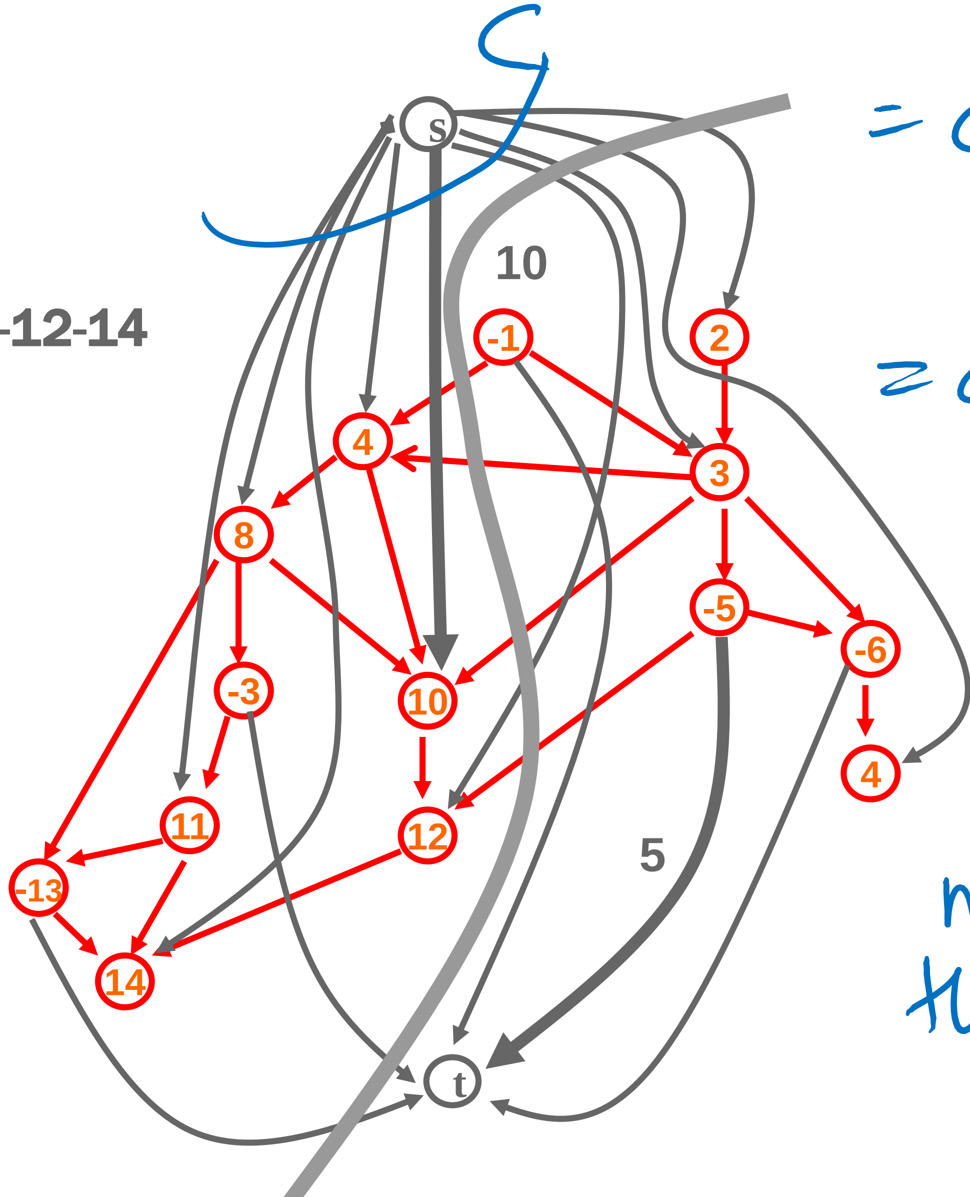
Cut value

=13+3+2+3+4

=13+3

+C-4-8-10-11-12-14

$C = \sum_{v: p(v) > 0} p(v)$



= C − make + spend

= C − Profit(A)

⇓

min cut maximizes the profit.

project selection

- **Claim**: Any s-t-cut (S,T) in G' such that $A=S-\{s\}$ satisfies precedence constraints has capacity
  $c(S,T)=C - \sum_{v\in A} p(v) = C - \text{Profit}(A)$

- **Corollary**: A minimum cut (S,T) in G' yields an optimal solution $A=S-\{s\}$ to the profit selection problem

- **Algorithm**: Compute maximum flow f in G', find the set S of nodes reachable from s in G'_f and return $S-\{s\}$

proof of claim

- $A=S-\{s\}$ satisfies precedence constraints
  - No edge of G crosses forward out of A since those edges have capacity $C+1$
  - Only forward edges cut are of the form (v,t) for $v\in A$ or (s,v) for $v\notin A$
  - The (v,t) edges for $v\in A$ contribute

    $$\sum_{v\in A:p(v)<0} -p(v) = -\sum_{v\in A:p(v)<0} p(v)$$
  - The (s,v) edges for $v\notin A$ contribute

    $$\sum_{v\notin A:\ p(v)\geq 0} p(v)=C-\sum_{v\in A:\ p(v)\geq 0} p(v)$$
  - Therefore the total capacity of the cut is

    $$c(S,T) = C - \sum_{v\in A} p(v) = C\text{-Profit}(A)$$