CSE 421: Algorithms

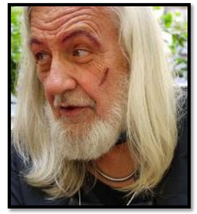
Winter 2014

Lecture 21: Edmonds-Karp and Project Selection

Reading:
Sections 7.3-7.5

Edmonds-Karp Algorithm

- Use a shortest augmenting path
  (via BFS in residual graph)
- Time: $O(n\,m^2)$



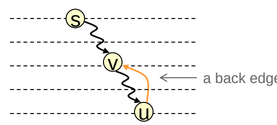
current best: Goldberg-Rao

- Time: $O\left(m \min\left\{m^{\frac{1}{2}}, n^{\frac{2}{3}}\right\} \log\left(\frac{n^2}{m}\right) \log U\right)$

bfs/shortest-path lemmas

Distance from s in G_f is never reduced by:

- Deleting an edge
  Proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  Proof: BFS is unchanged, since v visited before (u,v) examined



key lemma

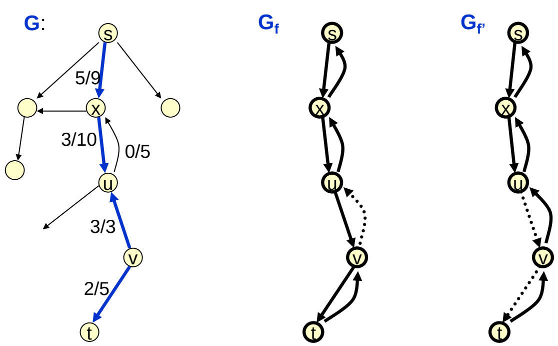
Let **f** be a flow, G_f the residual graph, and **P** a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

key lemma

Let **f** be a flow, G_f the residual graph, and **P** a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

Proof: Augmentation along **P** only deletes forward edges, or adds back edges that go to previous vertices along **P**

augmentation vs BFS



theorem

The Edmonds-Karp Algorithm performs $O(mn)$ flow augmentations.

Proof:

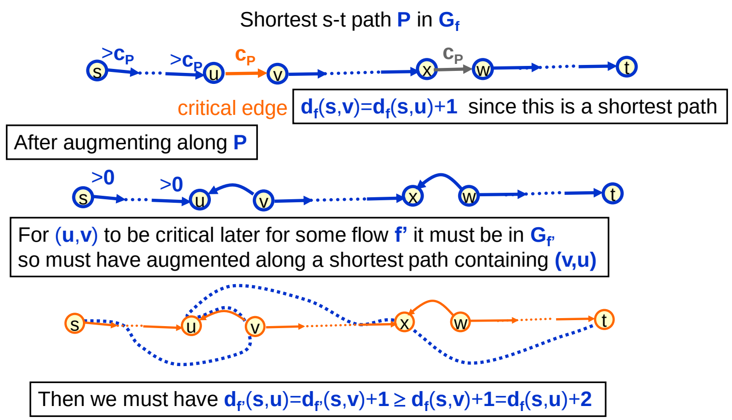
Call **(u,v)** critical for augmenting path **P** if it's closest to **s** having min residual capacity

It will disappear from G_f after augmenting along **P**

In order for **(u,v)** to be critical again the **(u,v)** edge must re-appear in G_f but that will only happen when the distance to **u** has increased by **2** (next slide)

It won't be critical again until farther from **s** so each edge critical at most **n/2** times

critical edges in G_f



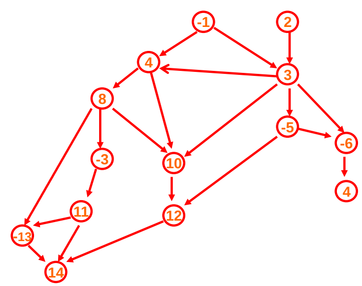
corollary

- **Edmonds-Karp runs in $O(nm^2)$ time**

project selection

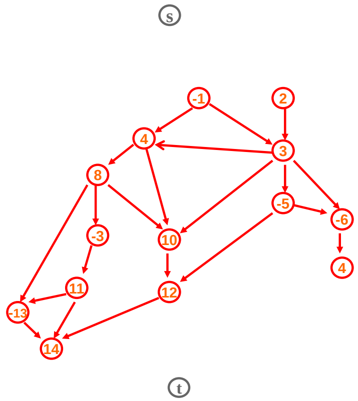
- **Given**
  - **a directed acyclic graph $G=(V,E)$ representing precedence constraints on tasks** (a task points to its **predecessors**)
  - **a profit value $p(v)$ associated with each task $v \in V$ (may be positive or negative)**
- **Find**
  - **a set $A \subseteq V$ of tasks that is closed under predecessors, i.e. if $(u,v) \in E$ and $u \in A$ then $v \in A$, that maximizes $\text{Profit}(A)=\sum_{v \in A} p(v)$**

project selection graph



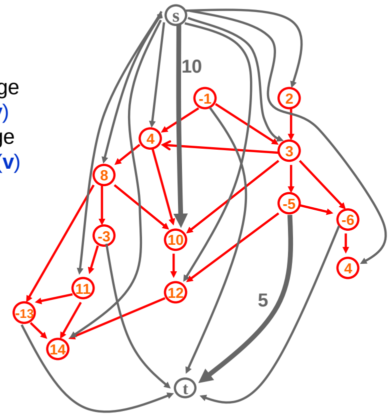
Each task points to its predecessor tasks

extended graph



extended graph G'

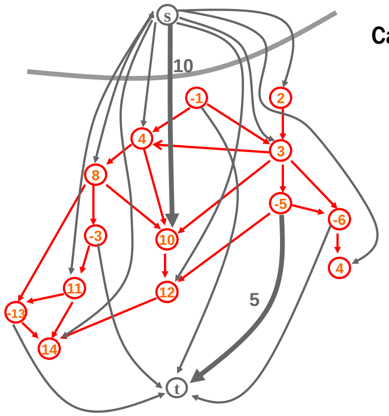
For each vertex **v**
If **p(v)≥0** add **(s,v)** edge
with capacity **p(v)**
If **p(v)<0** add **(v,t)** edge
with capacity **–p(v)**



extended graph G'

- **Want to arrange capacities on edges of G so that for minimum s-t-cut (S,T) in G', the set A=S-{s}**
  - **satisfies precedence constraints**
  - **has maximum possible profit in G**
- **Cut capacity with S={s} is just $C=\sum_{v:\, p(v)\geq 0} p(v)$**
  - **Profit(A) ≤ C for any set A**
- **To satisfy precedence constraints don't want any original edges of G going forward across the minimum cut**
  - **That would correspond to a task in A=S-{s} that had a predecessor not in A=S-{s}**
- **Set capacity of each of the edges of G to C+1**
  - **The minimum cut has size at most C**

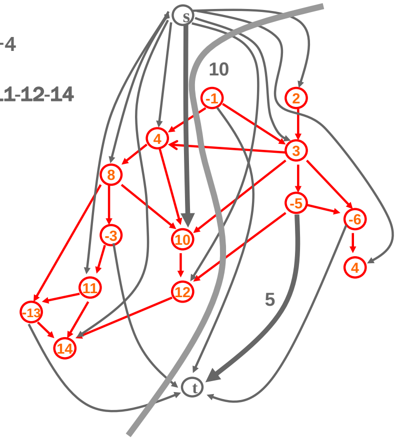
extended graph G'



Capacity C

extended graph G'

Cut value
=13+3+2+3+4
=13+3
+C-4-8-10-11-12-14



proof of claim

- A=S-{s} satisfies precedence constraints
  - No edge of G crosses forward out of A since those edges have capacity C+1
  - Only forward edges cut are of the form (v,t) for $v \in A$ or (s,v) for $v \notin A$
  - The (v,t) edges for $v \in A$ contribute
    $\sum_{v \in A: p(v)<0} -p(v) = -\sum_{v \in A: p(v)<0} p(v)$
  - The (s,v) edges for $v \notin A$ contribute
    $\sum_{v \notin A: p(v) \geq 0} p(v) = C - \sum_{v \in A: p(v) \geq 0} p(v)$
  - Therefore the total capacity of the cut is

    $c(S,T) = C - \sum_{v \in A} p(v) = C - \text{Profit}(A)$

project selection

- **Claim**: Any s-t-cut (S,T) in G' such that A=S-{s} satisfies precedence constraints has capacity
  $c(S,T) = C - \sum_{v \in A} p(v) = C - \text{Profit}(A)$
- **Corollary**: A minimum cut (S,T) in G' yields an optimal solution A=S-{s} to the profit selection problem
- **Algorithm**: Compute maximum flow f in G', find the set S of nodes reachable from s in G'_f and return S-{s}