1 HW #2, Extra credit: Spectral connectivity

[To get significant points for this problem, you should at least complete it up to part (e).]

There are many fascinating ways to think about graphs. For this problem, you will need to remember some Linear Algebra. We will show how to associate a matrix to a graph and then argue that the eigenvalues and eigenvectors of the matrix (which can be computed efficiently!) tell us interesting things about the graph. For instance, if the graph is a social network, these techniques can tell us how it clusters into communities.

Consider an undirected graph $G = (V, E)$ with $n = |V|$. We can associate to G its *adjacency matrix* which is an $n \times n$ matrix A such that $A_{ij} = 1$ if there is an edge between vertices i and j and $A_{ij} = 0$ otherwise. Note that since G is undirected, the matrix A is symmetric.

Now let D be the $n \times n$ diagonal matrix with $D_{ii} = \deg(i)$ and $D_{ij} = 0$ if $i \neq j$. This is called the *degree matrix* of G. Finally, we define a very important object: The *Laplacian of* G is the matrix $L = D - A$. Note that if $v \in \mathbb{R}^n$ is a vector, then

$$(Lv)_i = \deg(i)v_i - \sum_{j:\{i,j\}\in E} v_j \,.$$

(a) Say why L is a symmetric matrix.

Now, one can associate to the matrix L a *quadratic form* which takes a vector $v \in \mathbb{R}^n$ and outputs $\langle v, Lv\rangle$, where $\langle u, v\rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ denotes the inner product of u and v.

(b) Prove that for any $v \in \mathbb{R}^n$, we have

$$\langle v, Lv\rangle = \sum_{\{i,j\}\in E} (v_i - v_j)^2 \,.$$

Recall that an *eigenvector of* L is a vector $v \in \mathbb{R}^n$ such that $Lv = \lambda v$ for some number $\lambda \geq 0$.

(c) Use part (b) to prove that all the eigenvalues of L are non-negative real numbers.

(d) Show that 0 is an eigenvalue of L with associated eigenvector $(1, 1, \ldots, 1)$.

Parts (c) and (d) together show us that 0 is the smallest eigenvalue of L. By the variational principle (or "min-max" principle) for eigenvalues, we know that the second-smallest eigenvalue λ_2 can be expressed as

$$\lambda_2 = \min_{\substack{v\perp(1,1,\ldots,1)\\ v\neq 0}} \frac{\langle v, Lv\rangle}{\langle v, v\rangle} \,.$$

Now, it's possible that $\lambda_2 = 0$ (in which case the eigenvalue 0 occurs with "multiplicity"). But whether or not this is the case is related to connectivity properties of G.

(e) Show that $\lambda_2 > 0$ if and only if G is a connected graph.

A variational principle holds for higher eigenvalues as well. Let's list the eigenvalues of L as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then it is a fact that

$$\lambda_k = \min_{\substack{S\subseteq\mathbb{R}^n\\ \dim(S)=k}} \max_{0\neq v\in S} \frac{\langle v, Lv\rangle}{\langle v, v\rangle} \,,$$

where the minimum is over all subspaces of dimension k.

(f) Use the preceding fact to show that $\lambda_k = 0$ if and only if G has at least k distinct connected components.