CSE 421
Algorithms

Huffman Codes:
An Optimal Data Compression Method

Compression Example

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits

$2^3 > 6$; 3 bits/char: 300kbits

Why?

Storage, transmission vs 5 Ghz cpu

Compression Example

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100k file, 6 letter alphabet:

File Size:

ASCII, 8 bits/char: 800kbits

$2^3 > 6$; 3 bits/char: 300kbits

better: →

2.52 bits/char 74%*2 +26%*4: 252kbits

Optimal?

E.g.:		Why not:
a	00	00
b	01	01
d	10	10
c	1100	110
e	1101	1101
f	1110	1110

1101110 = cf or ec?

Data Compression

Binary character code ("code")

- each k-bit *source string* maps to unique *code word* (e.g. k=8)
- "compression" alg: concatenate code words for successive k-bit "characters" of source

Fixed/variable length codes

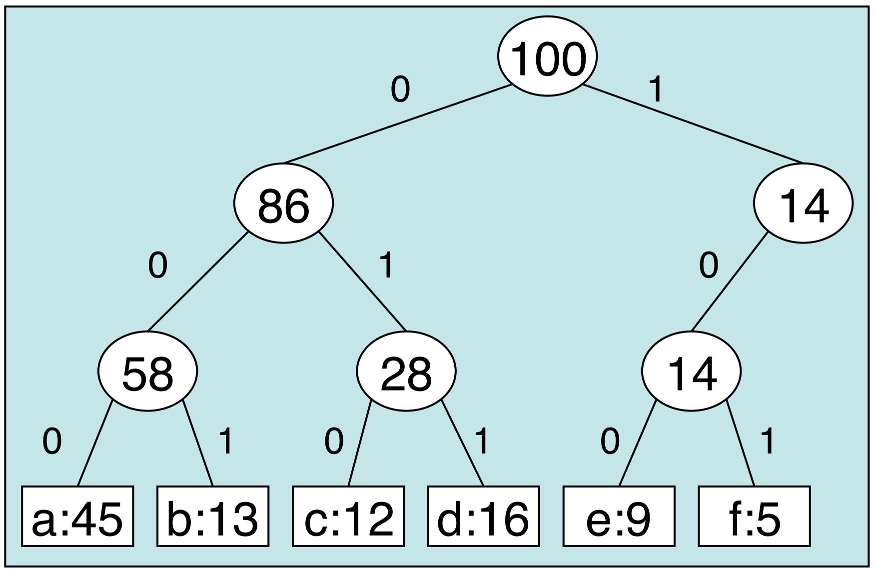
- all code words equal length?

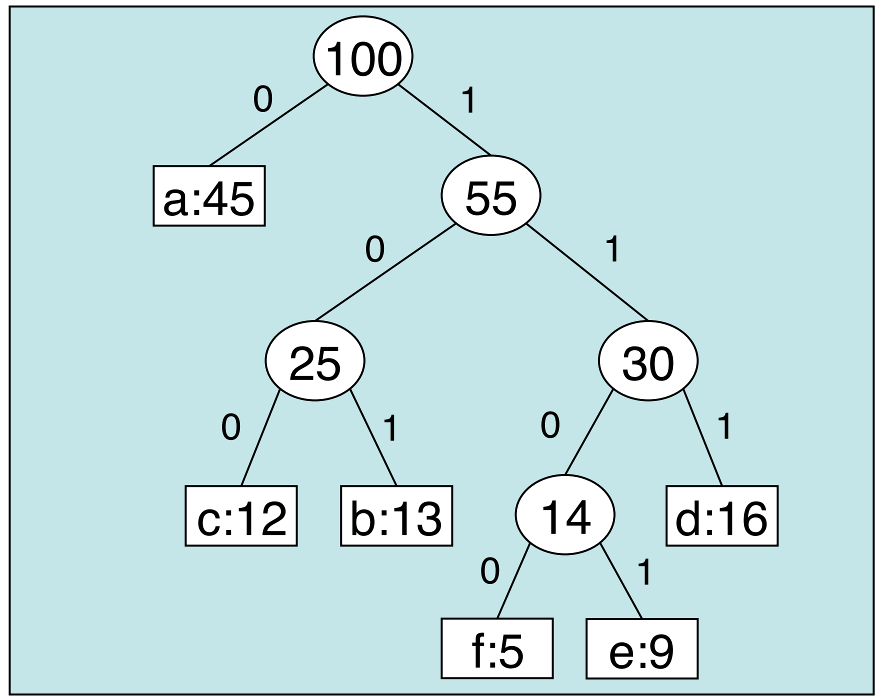
Prefix codes

- no code word is prefix of another (unique decoding)

Prefix Codes = Trees

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%





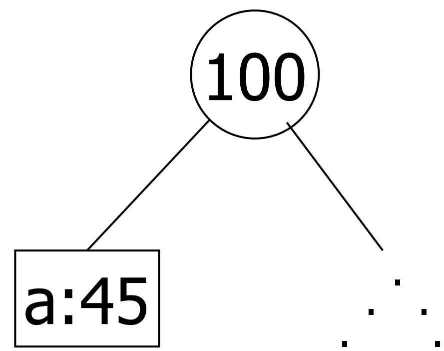
$$\underbrace{1\ 0\ 1}_{f}\ \underbrace{0\ 0\ 0}_{a}\ \underbrace{0\ 0\ 1}_{b}$$

$$\underbrace{1\ 1\ 0\ 0}_{f}\ \underbrace{0}_{a}\ \underbrace{1\ 0\ 1}_{b}$$

Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Put most frequent under root, then recurse …



Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

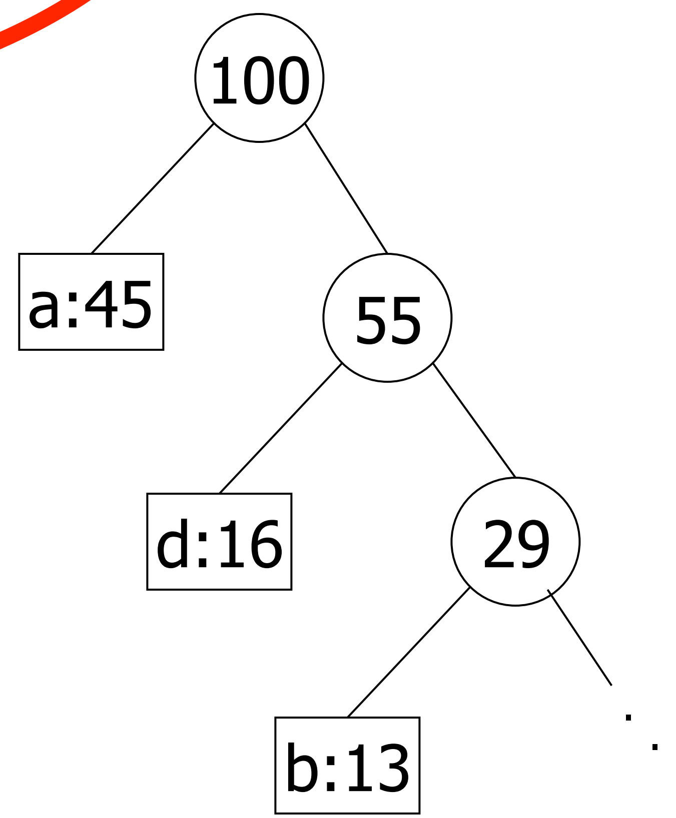
Top down: Put *most* frequent under root, then recurse

Too greedy: unbalanced tree

.45*1 + .16*2 + .13*3 … = 2.34

not too bad, but imagine if all freqs were ~1/6:

(1+2+3+4+5+5)/6=3.33



Greedy Idea #2

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

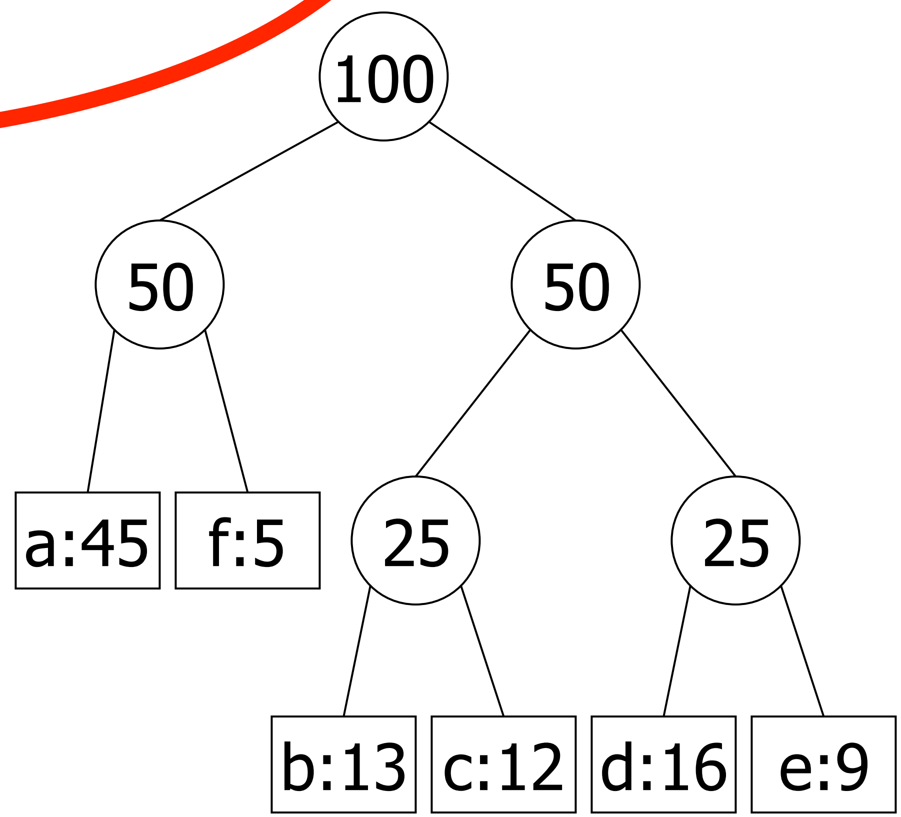
Top down: Divide letters into 2 groups, with ~50% weight in each; recurse

(Shannon-Fano code)

Again, not terrible

2*.5+3*.5 = 2.5

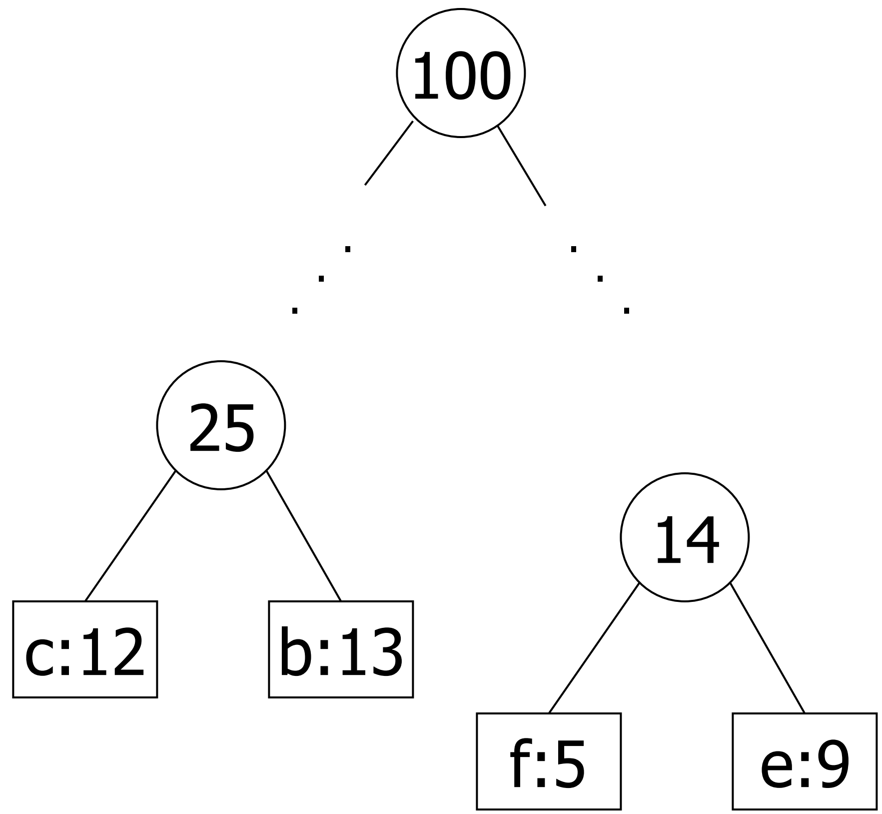
But this tree can easily be improved! (How?)

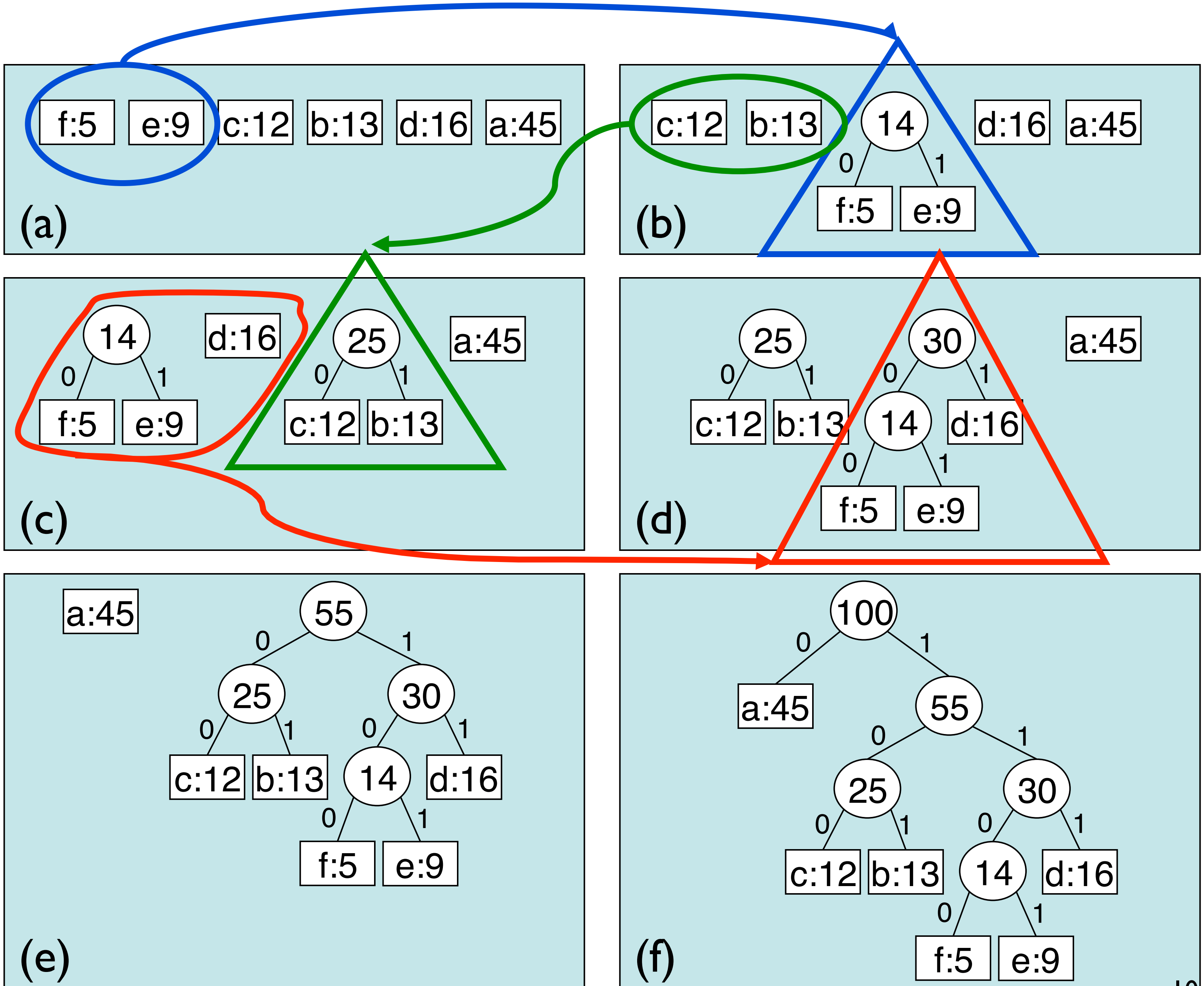


Greedy idea #3

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

Bottom up: Group *least* frequent letters near bottom





$.45*1 + .41*3 + .14*4 = 2.24$ bits per char

Huffman's Algorithm (1952)

Algorithm:

insert node for each letter into priority queue by freq

while queue length > 1 do

  remove smallest 2; call them x, y

  make new node z from them, with f(z) = f(x) + f(y)

  insert z into queue

Analysis: O(n) heap ops: O(n log n)

Goal: Minimize $B(T) = \sum_{c \in C} \mathrm{freq}(c) * \mathrm{depth}(c)$

T = Tree
C = alphabet

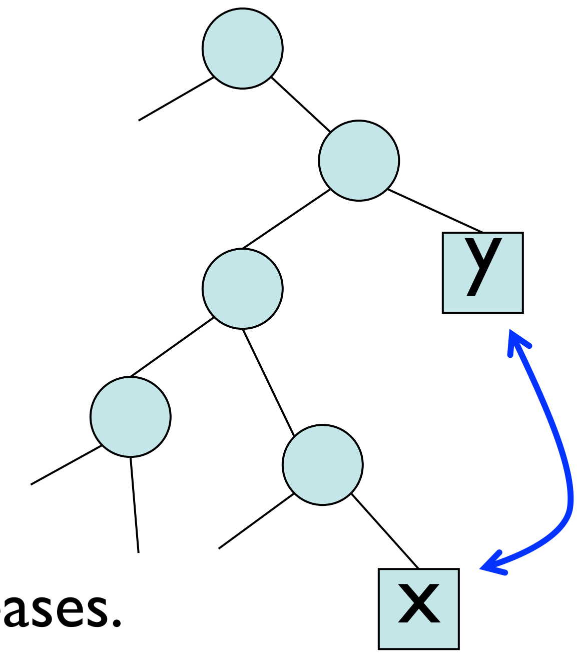
Correctness: ???

Correctness Strategy

Optimal solution may not be unique, so cannot prove that greedy gives the *only* possible answer.

Instead, show that greedy's solution is as good as any.

How: an exchange argument



Defn: A pair of leaves x,y is an inversion if

depth(x) ≥ depth(y)

and

freq(x) ≥ freq(y)

Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

$$\overbrace{(d(x)*f(x) + d(y)*f(y))}^{\text{before}} - \overbrace{(d(x)*f(y) + d(y)*f(x))}^{\text{after}} =$$

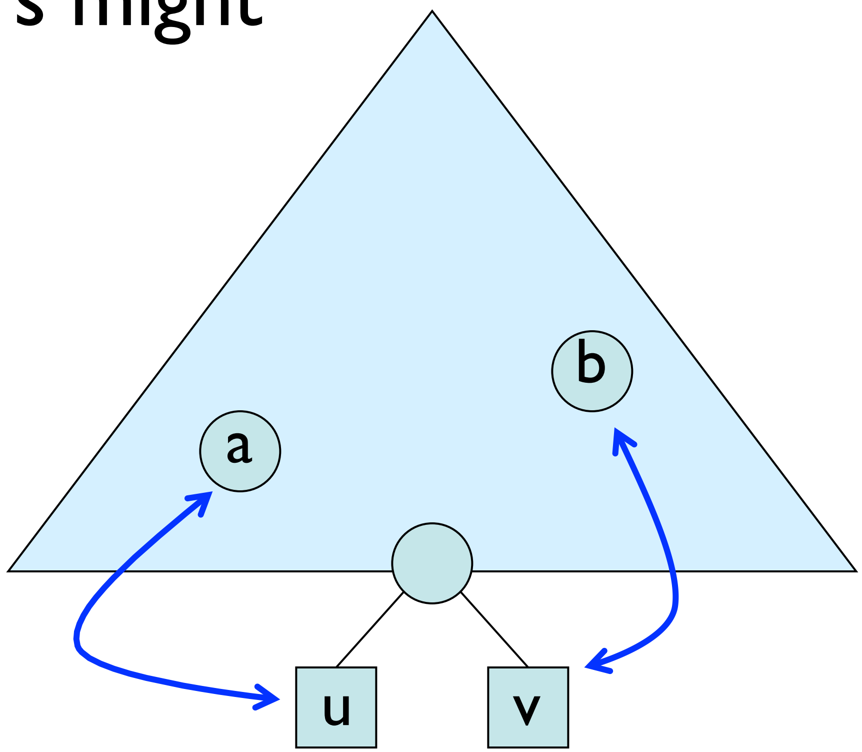
$$(d(x) - d(y)) * (f(x) - f(y)) \geq 0$$

I.e., non-negative cost savings.

Lemma 1:
"Greedy Choice Property"

The 2 least frequent letters might as well be siblings

- Let a be least freq, b 2^{nd}
- Let u, v be siblings at max depth, $f(u) \leq f(v)$ (why must they exist?)
- Then (a,u) and (b,v) are inversions. Swap them.



Why Important? Algorithm is not wrong to join them.

Lemma 2

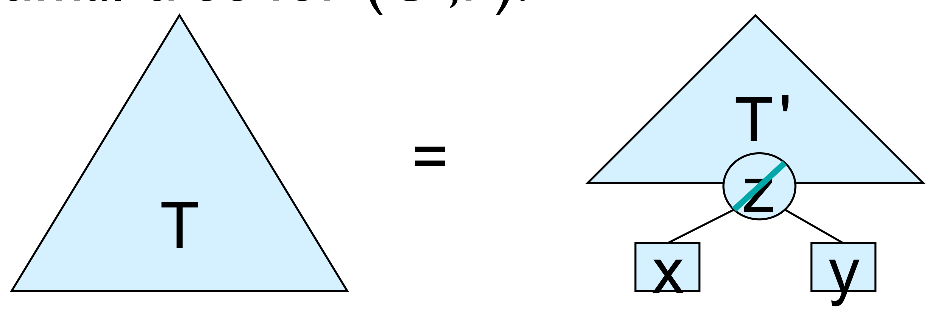
Let (C, f) be a problem instance: C an n-letter alphabet with letter frequencies f(c) for c in C.

For any x, y in C, z not in C, let C' be the (n-1) letter alphabet C - {x,y} $\cup$ {z} and for all c in C' define

$$f'(c) = \begin{cases} f(c), & \text{if } c \neq x,y,z \\ f(x) + f(y), & \text{if } c = z \end{cases}$$

Let T' be an optimal tree for (C',f').

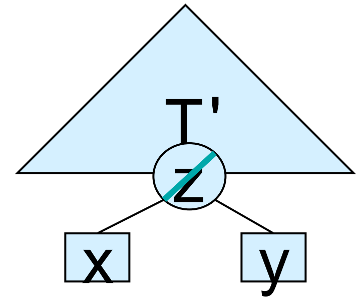
Then



is optimal for (C,f) among all trees having x,y as siblings

Why Important? Algorithm is not wrong to treat x:y as z.

Proof:



$$B(T) = \sum_{c \in C} d_T(c) \cdot f(c)$$

$$\begin{aligned} B(T) - B(T') &= d_T(x) \cdot (f(x) + f(y)) - d_{T'}(z) \cdot f'(z) \\ &= (d_{T'}(z) + 1) \cdot f'(z) - d_{T'}(z) \cdot f'(z) \\ &= f'(z) \end{aligned}$$

Suppose $\hat{T}$ (having x & y as siblings) is better than T, i.e.

$B(\hat{T}) < B(T)$. Collapse x & y to z, forming $\hat{T}'$; as above:

$$B(\hat{T}) - B(\hat{T}') = f'(z)$$

Then:

$$B(\hat{T}') = B(\hat{T}) - f'(z) < B(T) - f'(z) = B(T')$$

Contradicting optimality of T'

Theorem: Huffman gives optimal codes

Proof: induction on |C|

- Basis: n=1,2 – immediate
- Induction: n>2
  - Let x,y be least frequent
  - Form C′, f′, & z, as above
  - By induction, T′ is opt for (C′,f′)
  - By lemma 2, T′ →T is opt for (C,f) among trees with x,y as siblings
  - By lemma 1, some opt tree has x, y as siblings
  - Therefore, T is optimal.

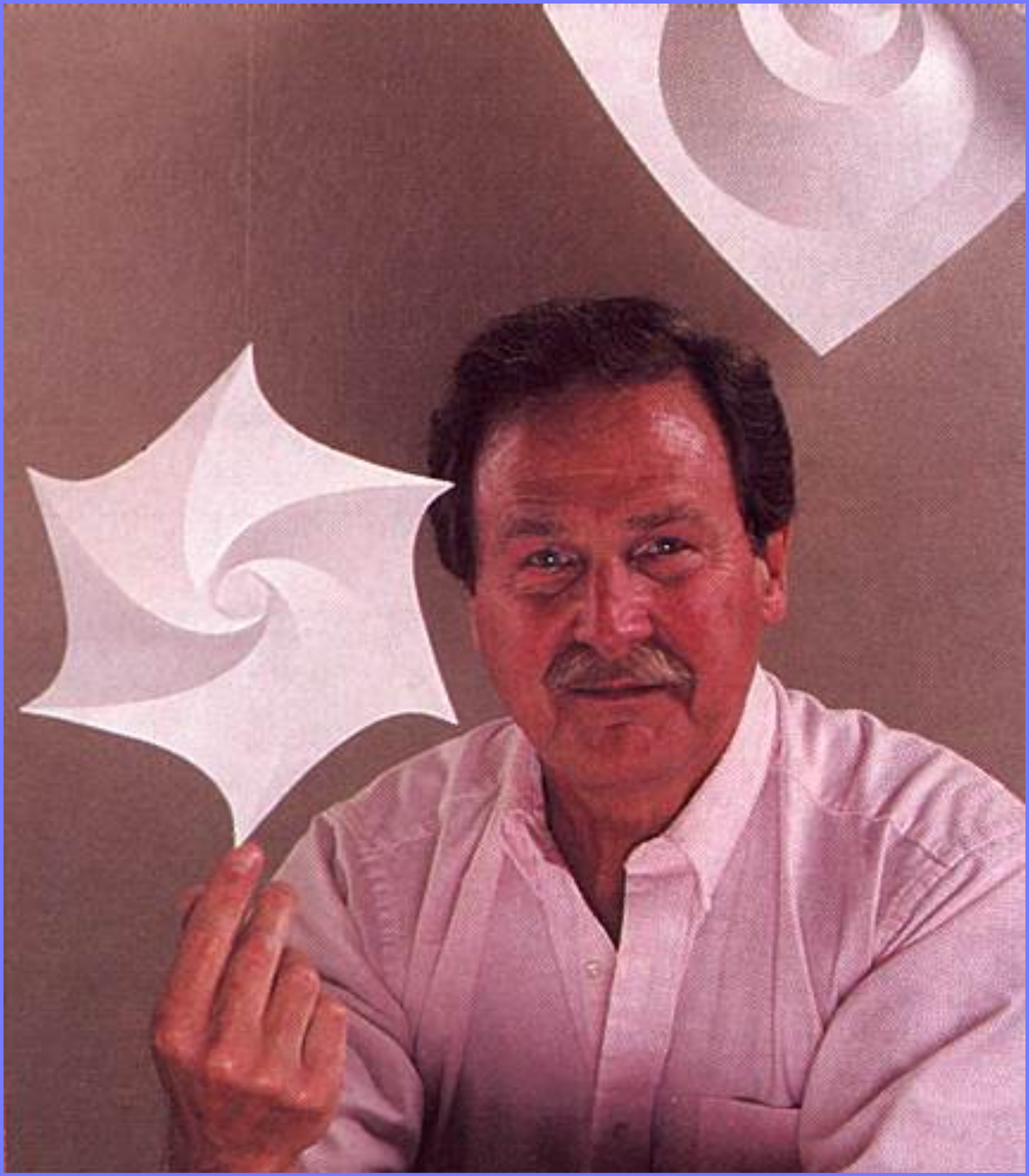
Data Compression

Huffman is optimal.

BUT still might do better!

- Huffman encodes fixed length blocks. What if we vary them?
- Huffman uses one encoding throughout a file. What if characteristics change?
- What if data has structure? E.g. raster images, video,…
- Huffman is lossless. Necessary?

LZW, MPEG, …



David A. Huffman, 1925-1999