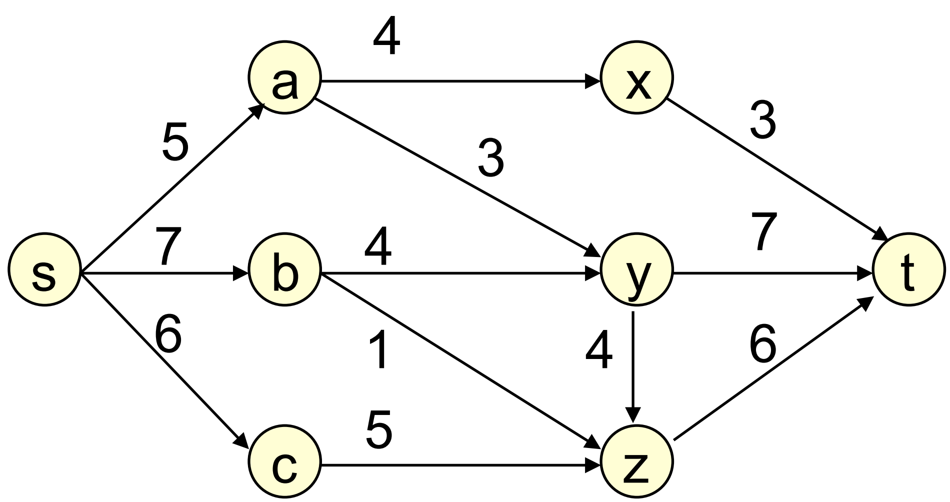
CSE 421
Introduction to Algorithms

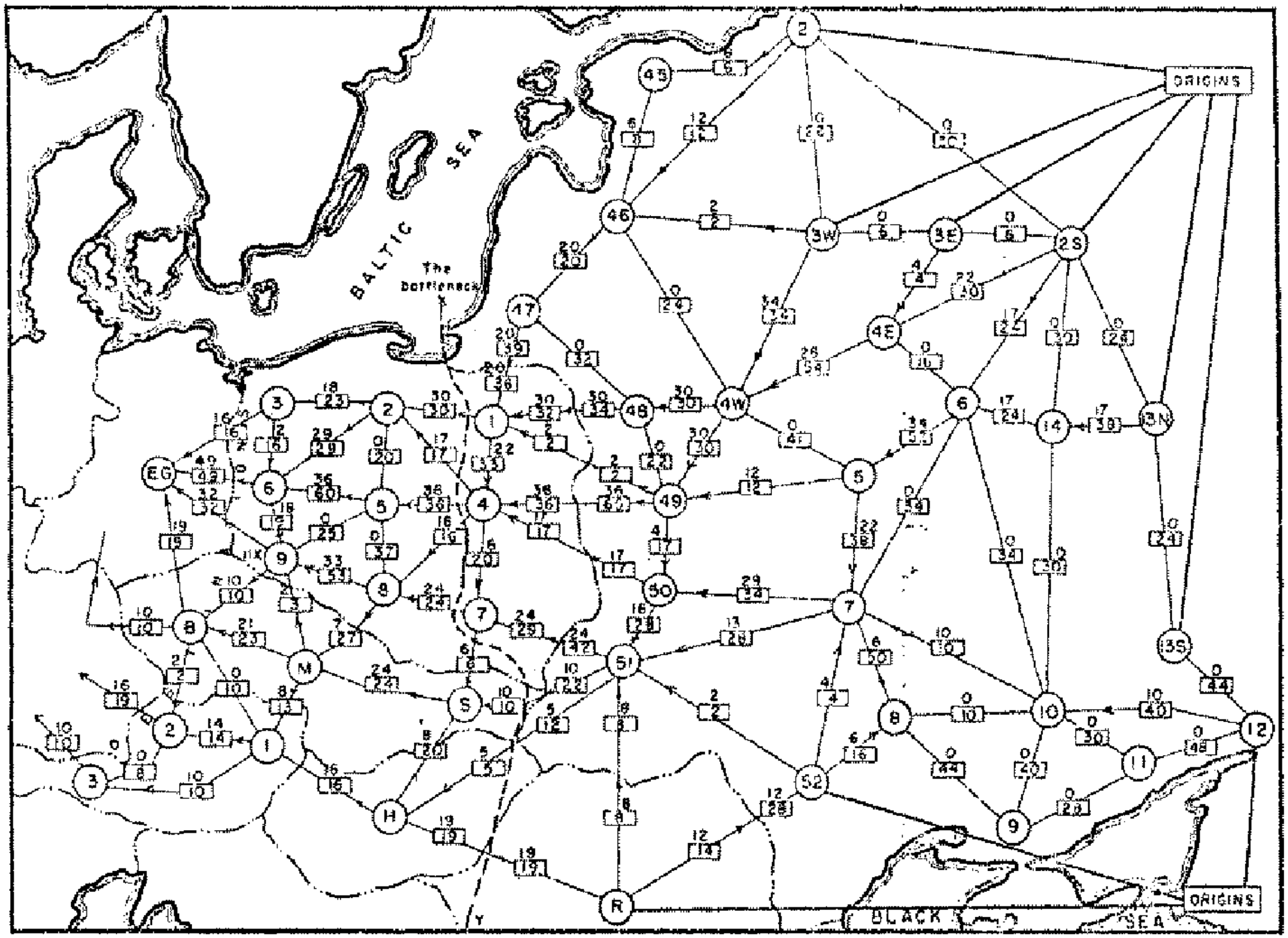
The Network Flow Problem

The Network Flow Problem



How much stuff can flow from *s* to *t*?

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems*.
Alexander Schrijver in Math Programming, 91: 3, 2002.

Net Flow: Formal Definition

Given:

A digraph $G = (V,E)$

Two vertices s,t in V
(s = source, t = sink)

A *capacity* $c(u,v) \geq 0$
for each $(u,v) \in E$
(and $c(u,v) = 0$ for all non-edges (u,v))

Find:

A *flow function* f: $V \times V \rightarrow R$ s.t., for all u,v:

- $f(u,v) \leq c(u,v)$ [Capacity Constraint]
- $f(u,v) = -f(v,u)$ [Skew Symmetry]
- if $u \neq s,t$, $f(u,V) = 0$ [Flow Conservation]

Maximizing total flow $|f| = f(s,V)$

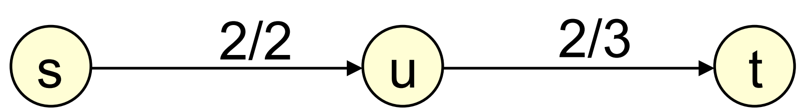
Notation:

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

(technically, not quite the same definition as in the book…)

Example: A Flow Function

"flow"/"capacity", not 0.66...



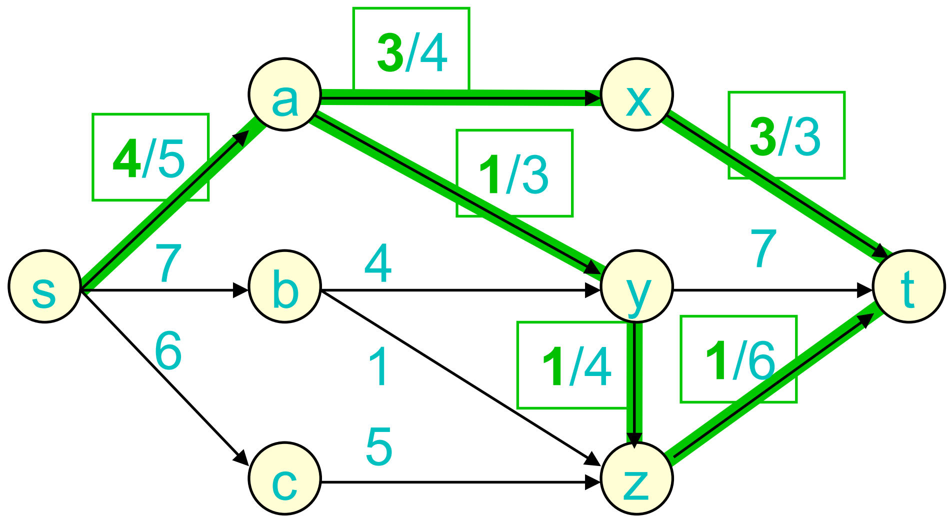
$f(s,u) = f(u,t) = 2$

$f(u,s) = f(t,u) = -2$ *(Why?)*

$f(s,t) = -f(t,s) = 0$ *(In every flow function for this G. Why?)*

$$f(u,V) = \sum_{v \in V} f(u,v) = f(u,s) + f(u,t) = -2 + 2 = 0$$

Example: A Flow Function



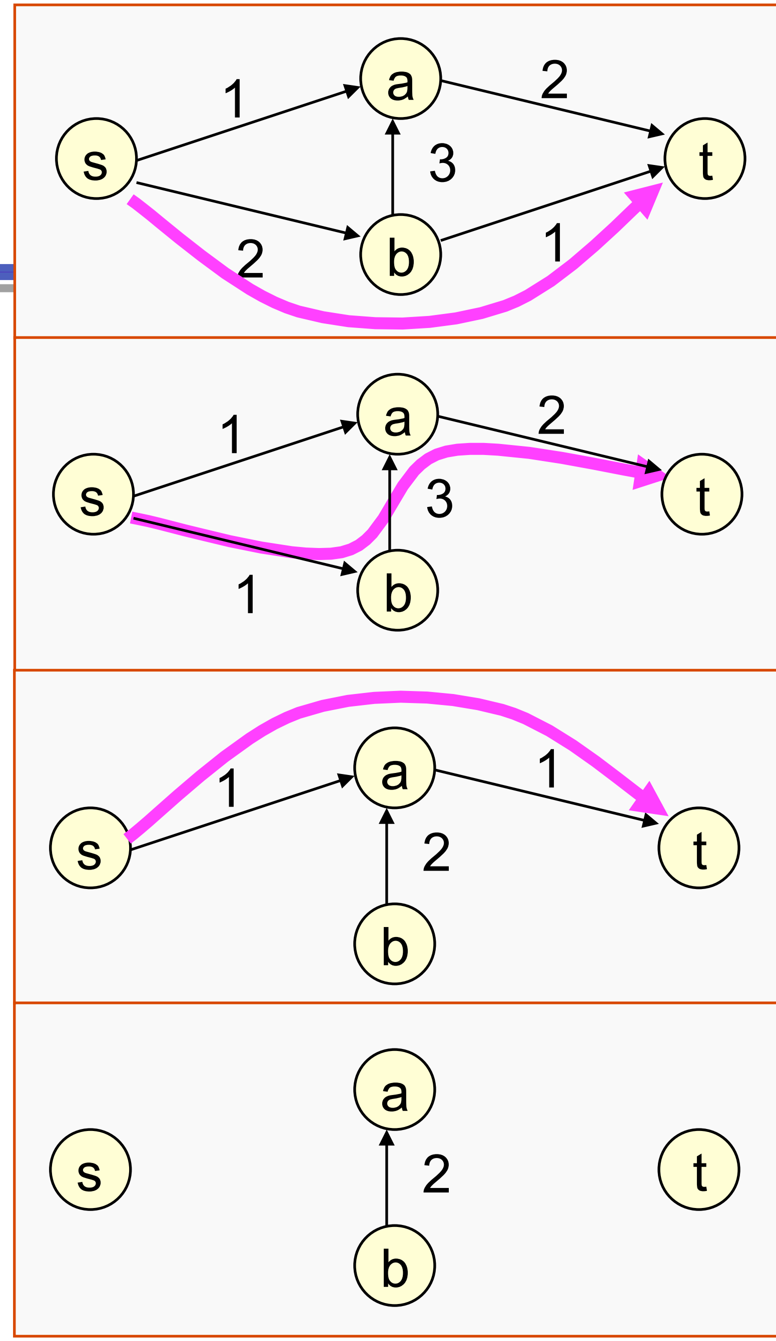
Not shown: $f(u,v)$ if ≤ 0

Note: max flow ≥ 4 since f is a flow, $|f| = 4$

Max Flow via a Greedy Alg?

While there is an $s \rightarrow t$ path in G

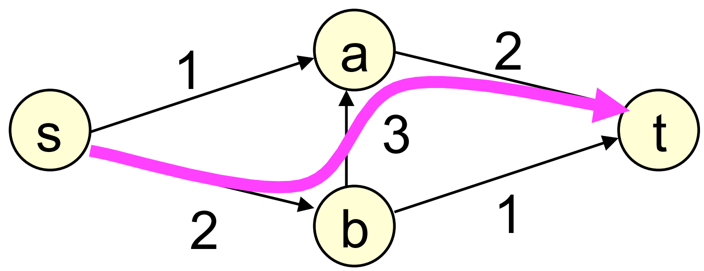
- Pick such a path, p
- Find c_p, the min capacity of any edge in p
- Subtract c_p from all capacities on p
- Delete edges of capacity 0

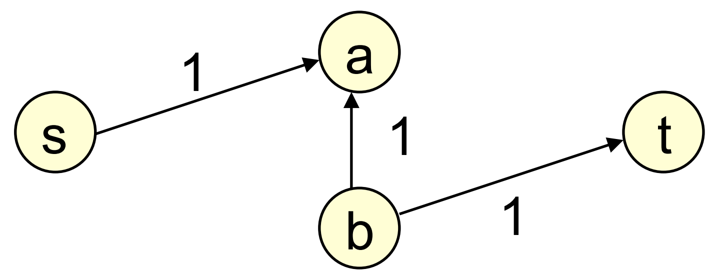


Max Flow via a Greedy Alg?

This does NOT always find a max flow:

If you pick $s \rightarrow b \rightarrow a \rightarrow t$ first,





Flow stuck at 2, but 3 possible (above).

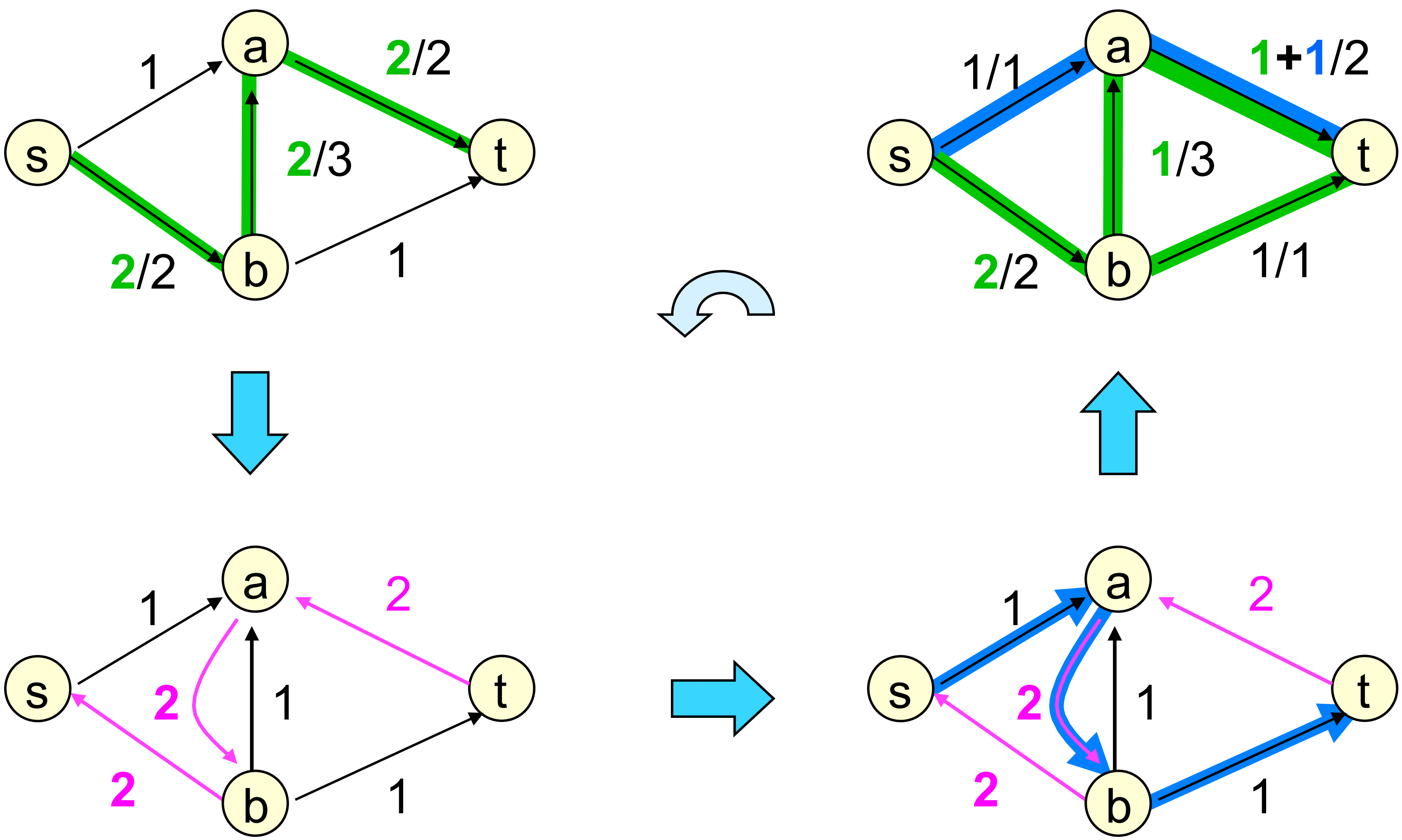
A Brief History of Flow

#	Year	Discoverer(s)	Bound
1	1951	Dantzig	$O(n^2mC)$
2	1955	Ford & Fulkerson	$O(nmC)$
3	1970	Dinitz; Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2m)$
5	1972	Edmonds & Karp; Dinitz	$O(m^2 \log C)$
6	1973	Dinitz;Gabow	$O(nm \log C)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2 \text{sqrt}(m))$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg &Tarjan	$O(nm \log (n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log C)$
13	1987	Ahuja et al.	$O(nm \log(n \text{ sqrt}(\log C)/(m+2))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\varepsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\varepsilon} n)$
19	1994	King et al.	$O(nm(\log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log C)$; $O(n^{2/3} m \log(n^2/m) \log C)$
…	…	…	…

n = # of vertices
m= # of edges
C = Max capacity

Source: Goldberg & Rao, FOCS '97

Greed Revisited

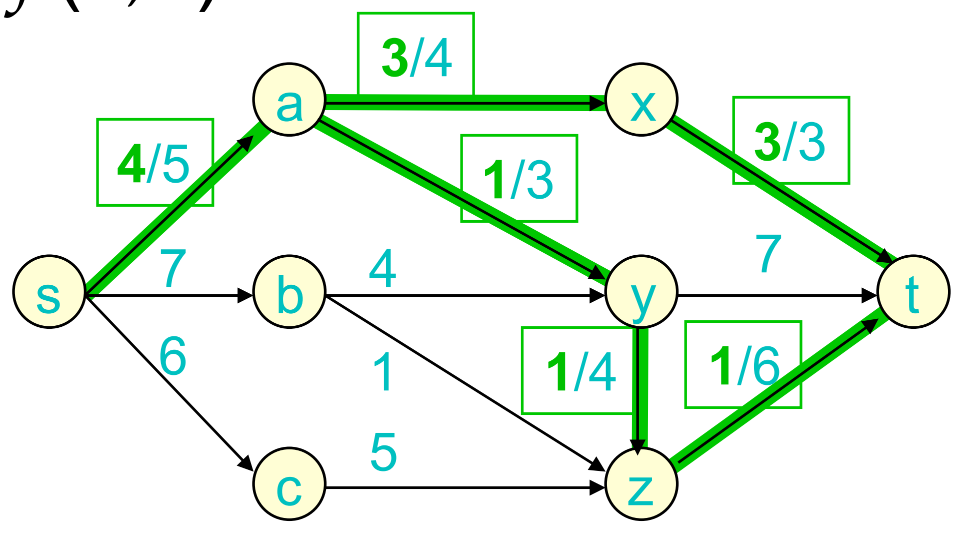


Residual Capacity

The *residual capacity* (w.r.t. f) of (u,v) is

$c_f(u,v) = c(u,v) - f(u,v)$

E.g.:



$c_f(s,b) = 7;$

$c_f(a,x) = 1;$

$c_f(x,a) = 3;$

$c_f(x,t) = 0$ (*a* saturated edge)

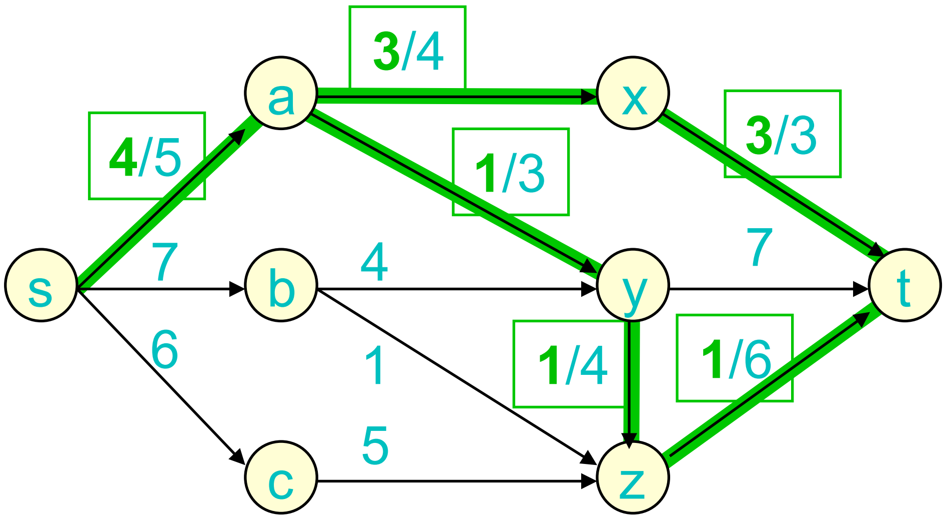
Residual Networks & Augmenting Paths

The *residual network* (w.r.t. f) is the graph $G_f = (V, E_f)$, where

$$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$$

An *augmenting path* (w.r.t. f) is a simple $s \to t$ path in G_f

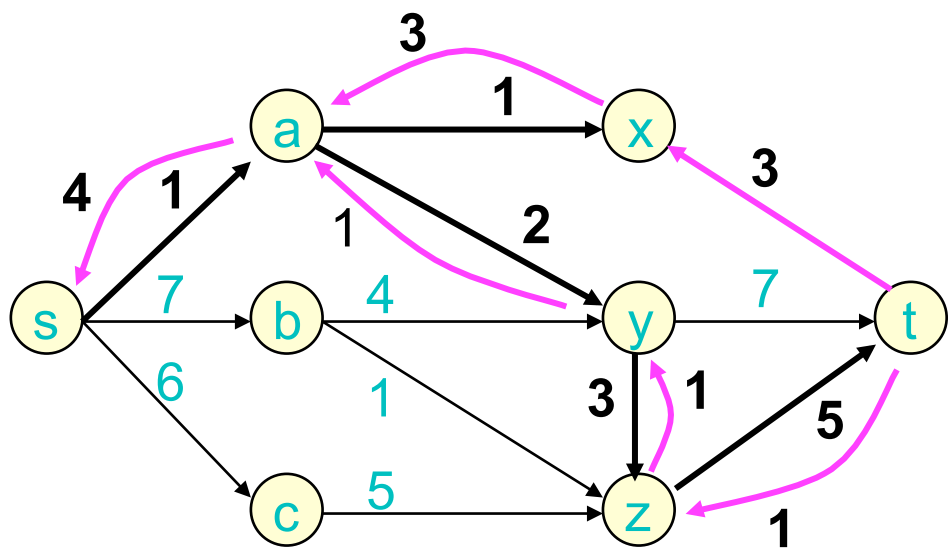
A Residual Network



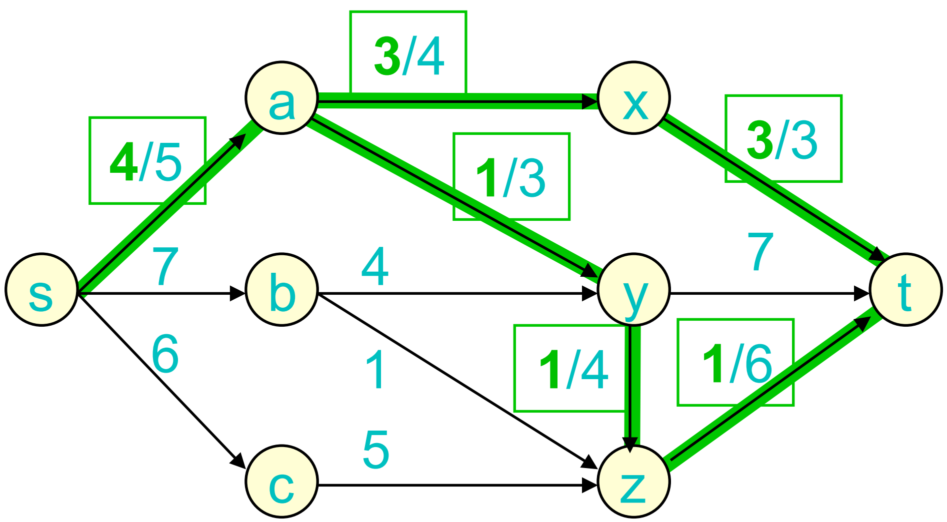
residual network: the graph

$G_f = (V,E_f)$, where

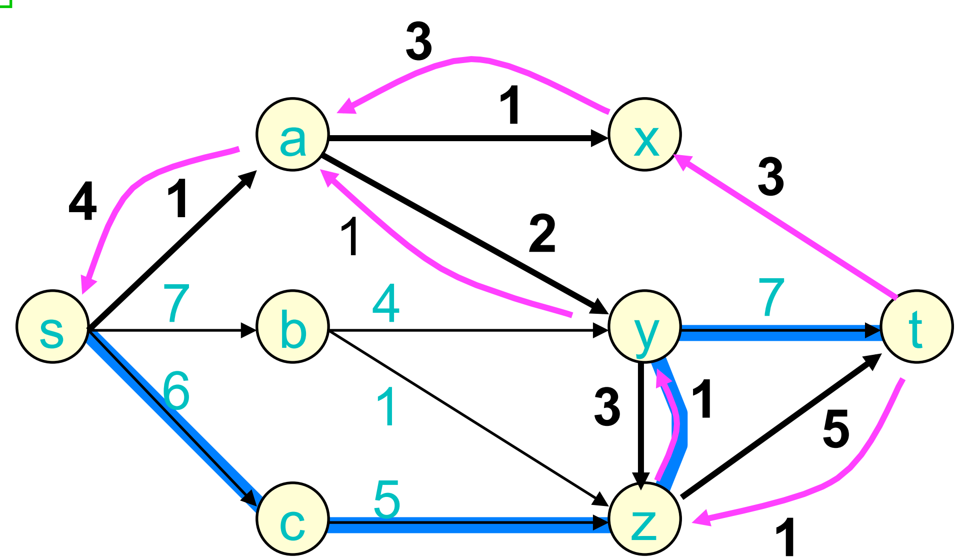
$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$



An Augmenting Path



augmenting path:
a simple $s \rightarrow t$ path in G_f

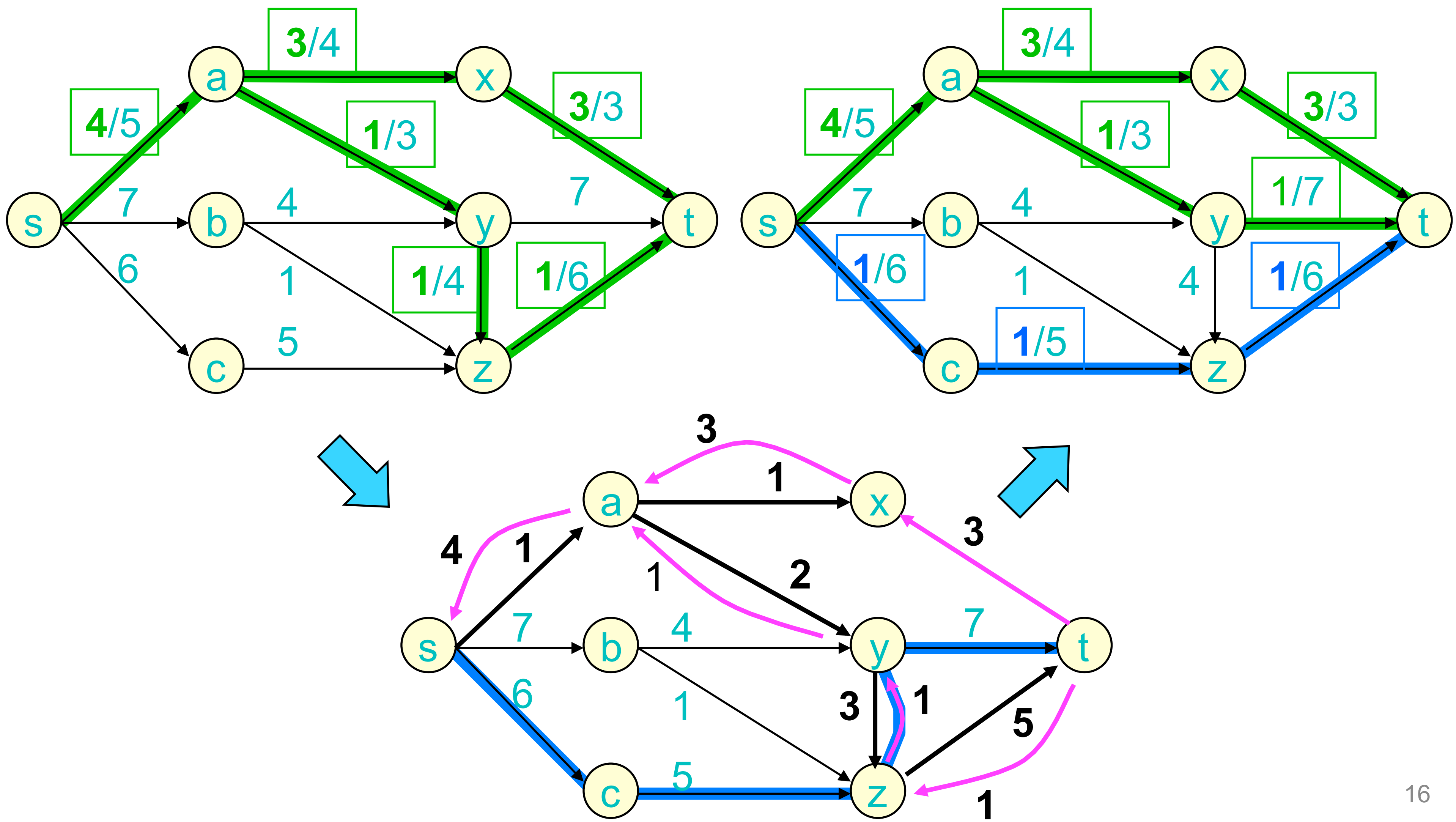


Lemma 1

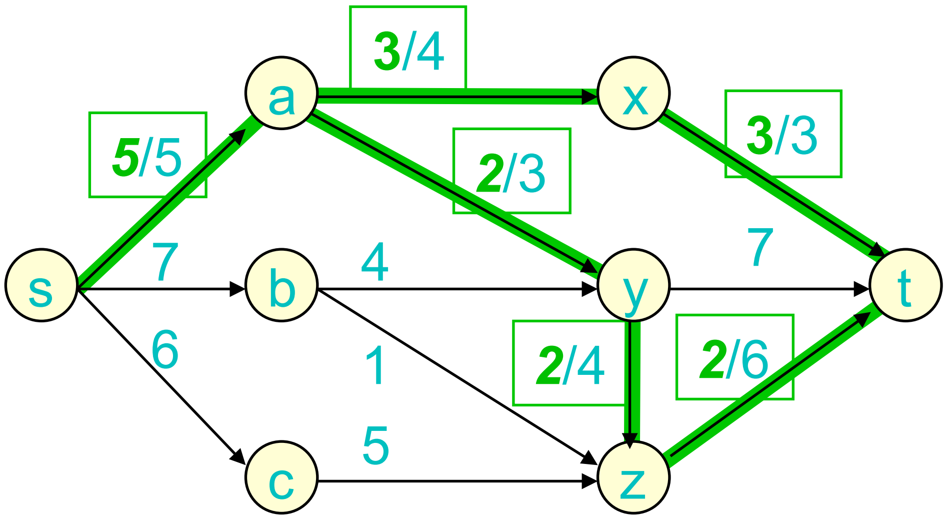
If f admits an augmenting path p, then f is not maximal.

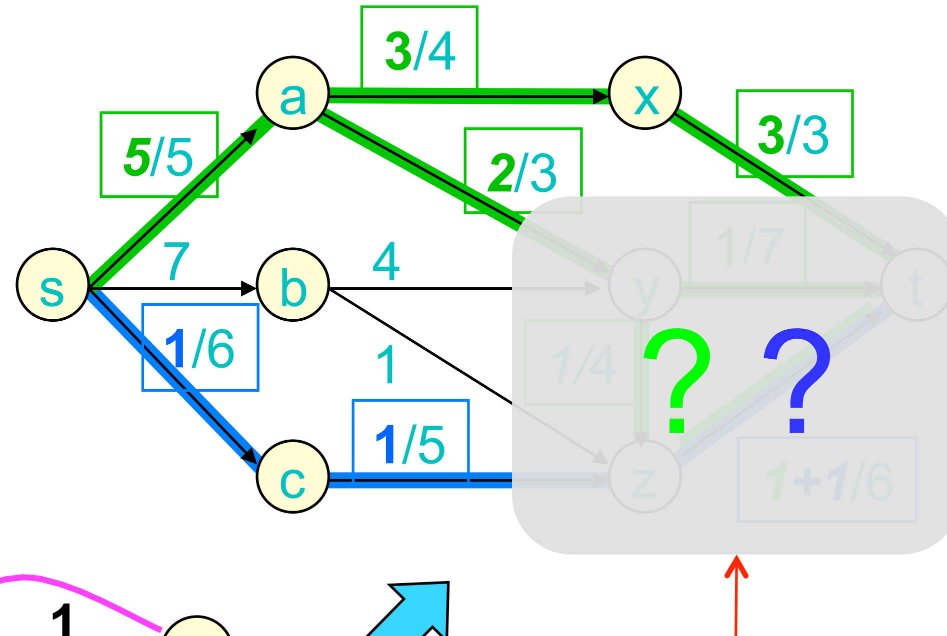
Proof: “obvious” -- augment along p by c_p, the min residual capacity of p’s edges.

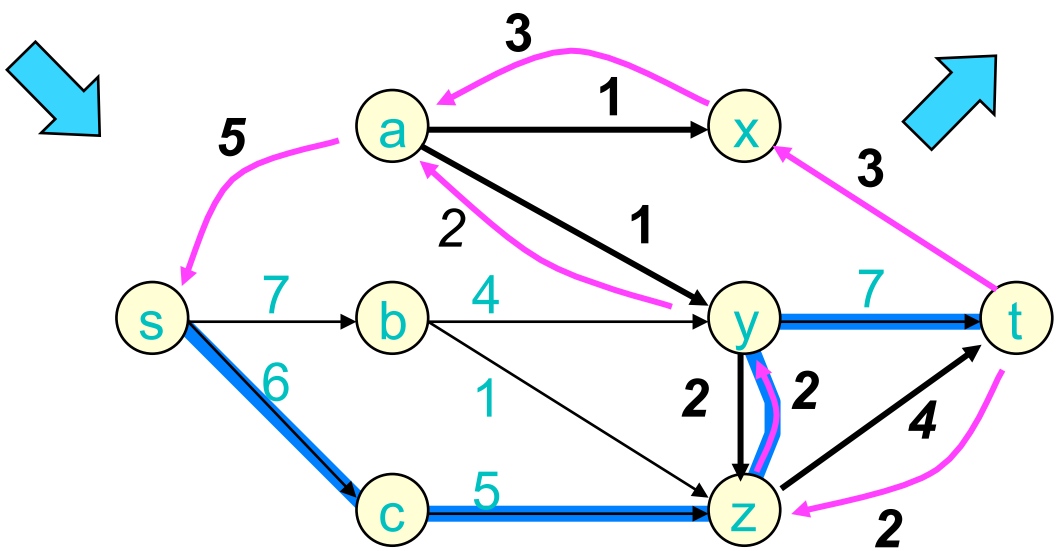
Augmenting A Flow



Augmenting A Flow







new green,
same blue;
what is result?

Lemma 1’:
Augmented Flows are Flows

If f is a flow & p an augmenting path of capacity c_p, then f' is also a valid flow, where

$$\boxed{f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}}$$

Proof:

a) Flow conservation – easy
b) Skew symmetry – easy
c) Capacity constraints – pretty easy; next slides

Lma 1': Augmented Flows are Flows

$$f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}$$

f a flow & p an aug path of cap c_p, then f' also a valid flow.

Proof (Capacity constraints):

(u,v), (v,u) not on path: no change

(u,v) on path:

$$\begin{aligned} f'(u,v) &= f(u,v) + c_p \\ &\le f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v) \end{aligned}$$

$$\begin{aligned} f'(v,u) &= f(v,u) - c_p \\ &< f(v,u) \\ &\le c(v,u) \end{aligned}$$

QED

Residual Capacity:

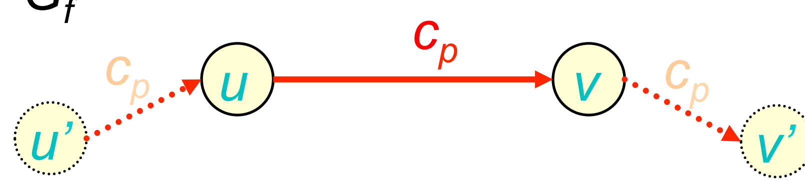
$0 < c_p \le c_f(u,v) = c(u,v) - f(u,v)$

Cap Constraints:

$-c(v,u) \le f(u,v) \le c(u,v)$

Lemma 1' Example – Case 1

G_f

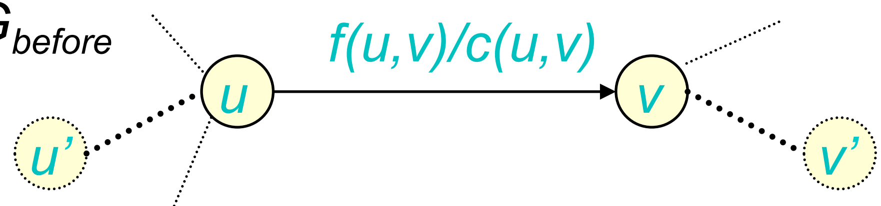


Let *(u,v)* be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

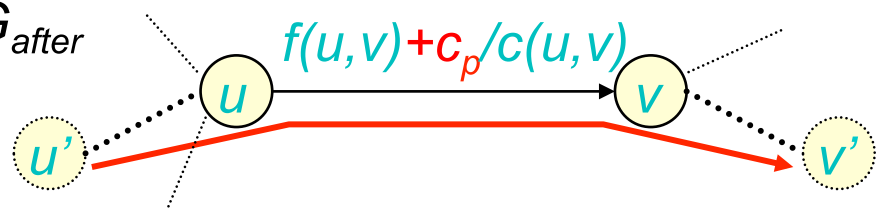
Case 1: $f(u,v) \geq 0$:

G_{before}



Add forward flow

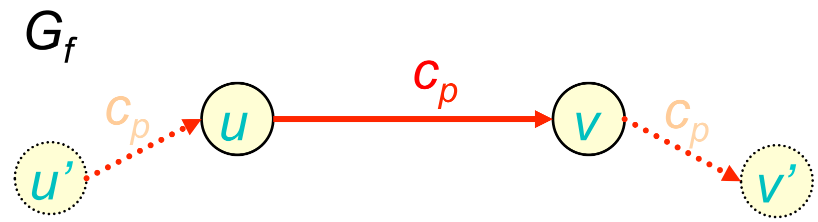
G_{after}



Lemma 1' Example – Case 2

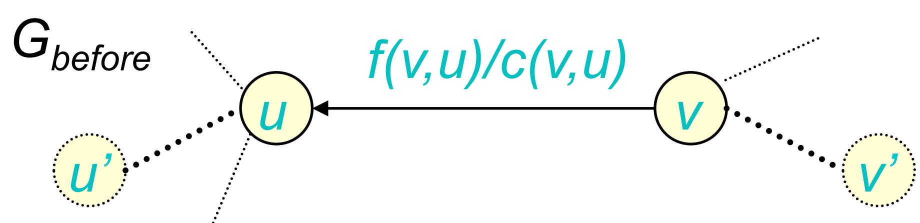
Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

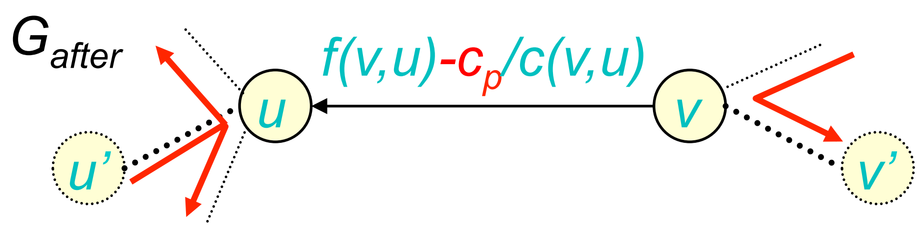


Case 2: $f(u,v) \leq -c_p$:

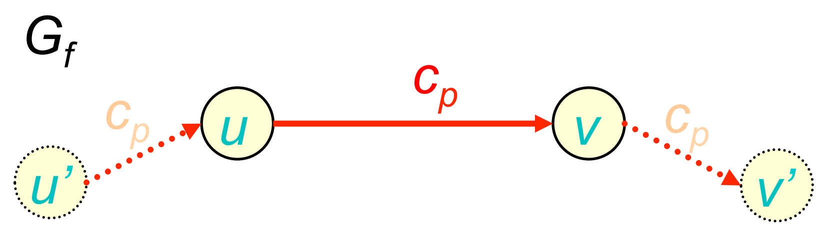
$f(v,u) = -f(u,v) \geq c_p$



Cancel/redirect reverse flow



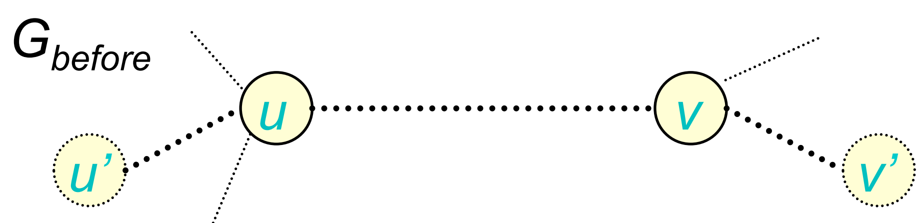
Lemma 1' Example – Case 3



Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

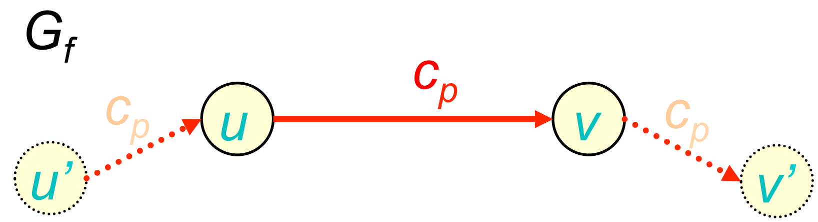
Case 3: $-c_p < f(u,v) < 0$:



???

[E.g., $c_p = 8$, $f(u,v) = -5$]

Lemma 1' Example – Case 3

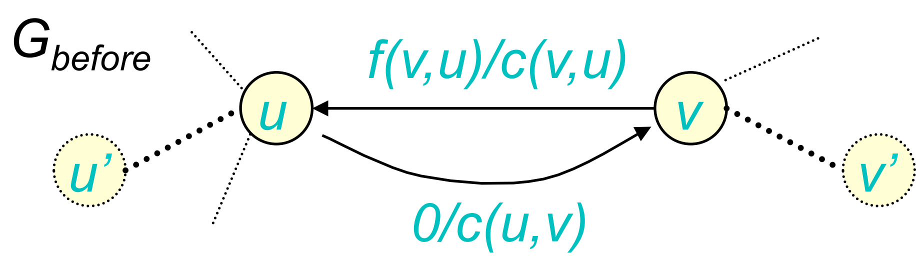


Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

Case 3: $-c_p < f(u,v) < 0$

$c_p > f(v,u) > 0$:



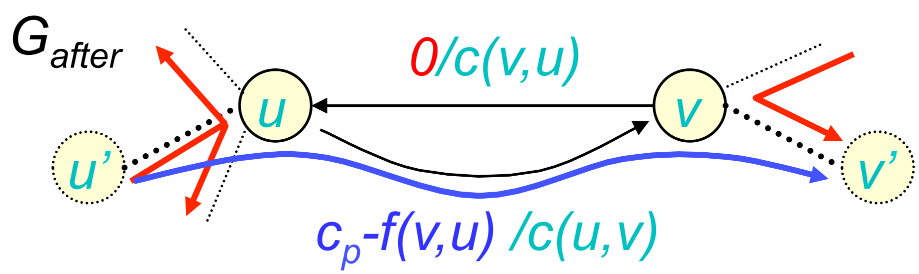
Both:

cancel/redirect

reverse flow

and

add forward flow



Ford-Fulkerson Method

While G_f has an augmenting path, augment

Questions:

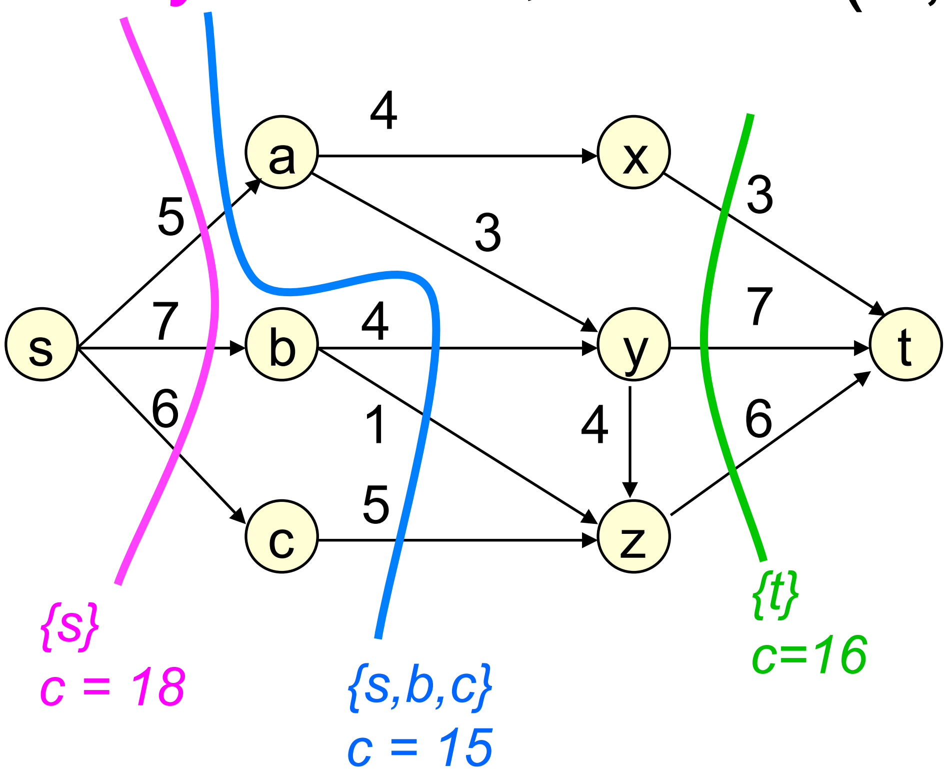
- » Does it halt?
- » Does it find a maximum flow?
- » How fast?

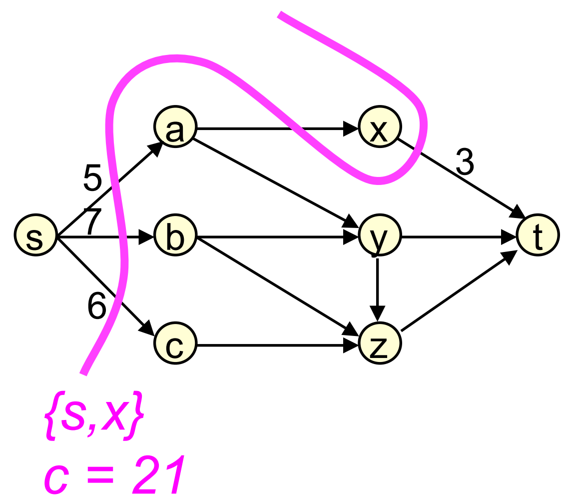
Cuts

A partition S, T of V is a *cut* if $s \in S$, $t \in T$.

Capacity of cut S, T is $\quad c(S,T) = \sum_{\substack{u \in S \\ v \in T}} c(u,v)$

sum of caps
of edges
from S *to* T





Lemma 2

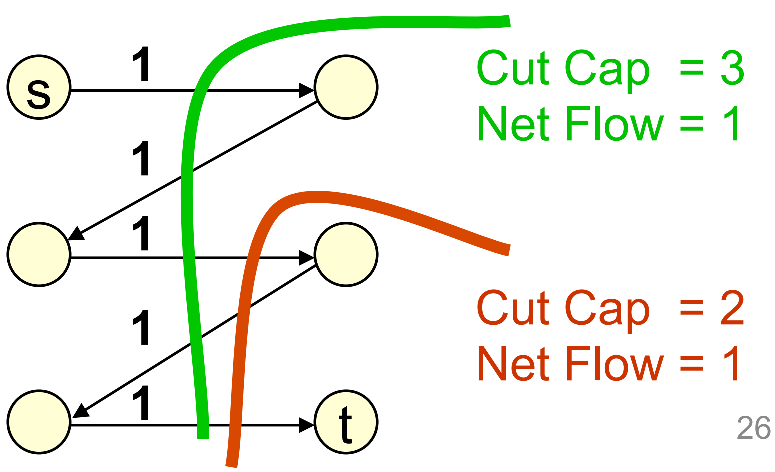
For any flow f and any cut S, T,

the net flow across the cut equals the total flow, i.e., $|f| = f(S,T)$, and

the net flow across the cut cannot exceed the capacity of the cut, i.e. $f(S,T) \leq c(S,T)$

Corollary:

Max flow ≤ Min cut



Lemma 2

For any flow *f* and any cut *S, T,*

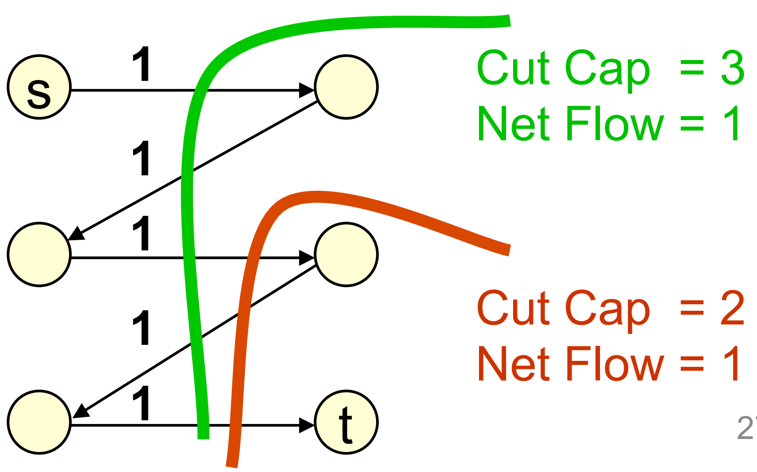
net flow across cut = total flow ≤ cut capacity

Proof:

Track a flow unit. Starts at *s*, ends at *t*.

crosses cut an odd # of times; net = 1.

Last crossing uses a forward edge totaled in *C(S,T)*



Max Flow / Min Cut Theorem

For any flow f, the following are equivalent

(1) $|f| = c(S,T)$ for some cut S,T (a min cut)

(2) f is a maximum flow

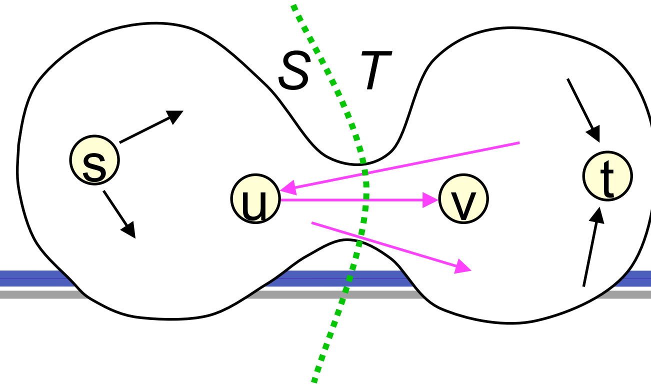
(3) f admits no augmenting path

Proof:

(1) $\Rightarrow$ (2): corollary to lemma 2

(2) $\Rightarrow$ (3): contrapositive of lemma 1

$(3) \Rightarrow (1)$
(no aug) $\Rightarrow$ (cut)



Idea: where's the bottleneck

$S = \{\, u \mid \exists$ an augmenting path wrt f from s to $u \,\}$

$T = V - S;\;\; s \in S,\, t \in T$

For any (u,v) in $S \times T$, $\exists$ an augmenting path from s to u, but not to v.

$\therefore$ (u,v) has 0 residual capacity:

$(u,v) \in E \Rightarrow$ saturated $\qquad f(u,v) = c(u,v)$

$(v,u) \in E \Rightarrow$ no flow $\qquad f(u,v) = 0 = -f(v,u)$

This is true for every edge crossing the cut, i.e.

$$|f| = f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) =$$

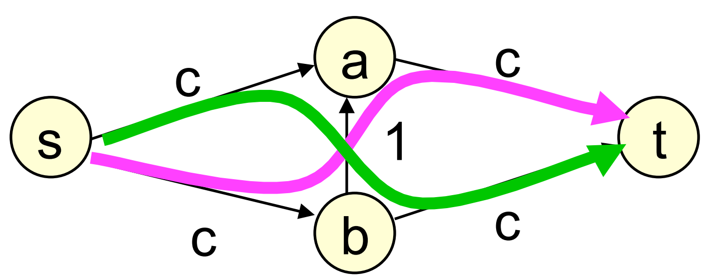
$$\sum_{u \in S, v \in T, (u,v) \in E} f(u,v) = \sum_{u \in S, v \in T, (u,v) \in E} c(u,v) = c(S,T)$$

Corollaries & Facts

If Ford-Fulkerson terminates, then it's found a max flow.

It will terminate if *c(e)* integer or rational (but may not if they're irrational).

However, may take exponential time, even with integer capacities:



$c = 10^{99}$, say

How to Make it Faster

Several ways. Three important ones:

Edmonds-Karp '70; Dinitz '70

1st "strongly" poly time alg. (next) $T = O(nm^2)$

"Scaling" [Edmonds-Karp, '72; Dinitz '72]

do *largest* edges first; see text, and below.
if C = max capacity, $T = O(m^2 \log C)$

Preflow-Push [Goldberg, Tarjan '86]

see text $T = O(n^3)$

Edmonds-Karp-Dinitz ‘70 Algorithm

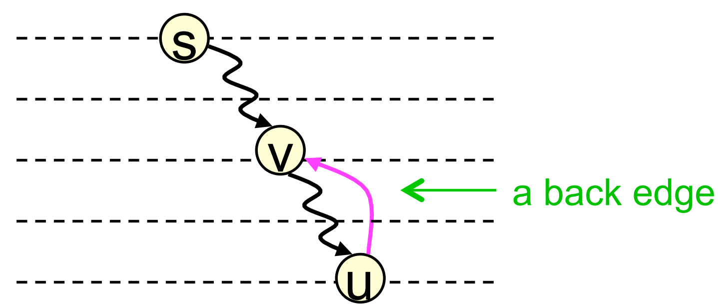
Use a shortest augmenting path
(via Breadth First Search in residual graph)

Time: $O(n\ m^2)$

BFS/Shortest Path Lemmas

Distance from s is never reduced by:

- Deleting an edge
  proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  proof: BFS is unchanged, since v visited before (u,v) examined

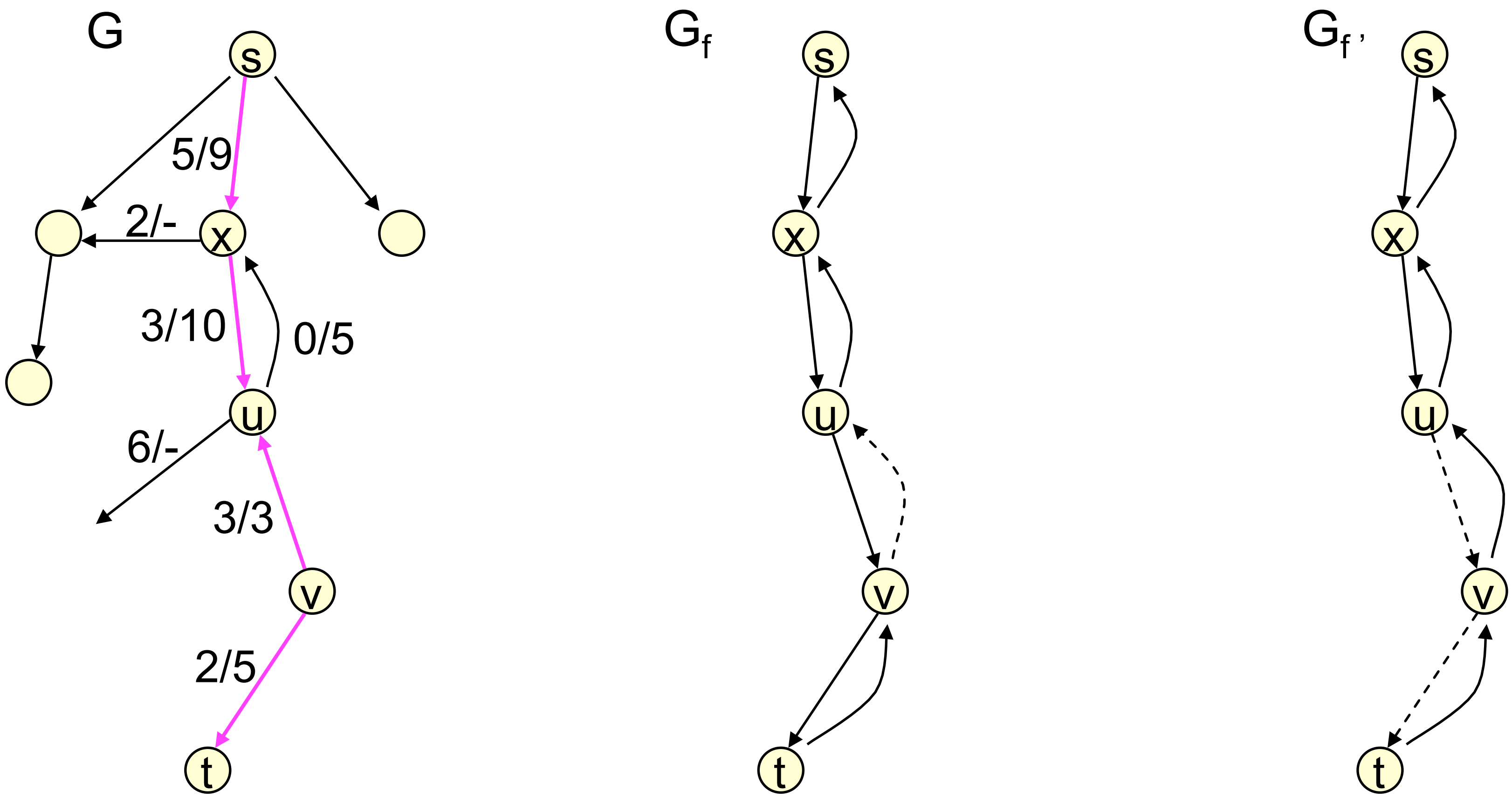


Lemma 3

Let f be a flow, G_f the residual graph, and p a shortest augmenting path. Then no vertex is closer to s in the new residual graph G_{f+p} after augmentation along p.

Proof: Augmentation only deletes edges, adds back edges

Augmentation vs BFS



Theorem 2

The Edmonds-Karp-Dinitiz Algorithm performs O(mn) flow augmentations

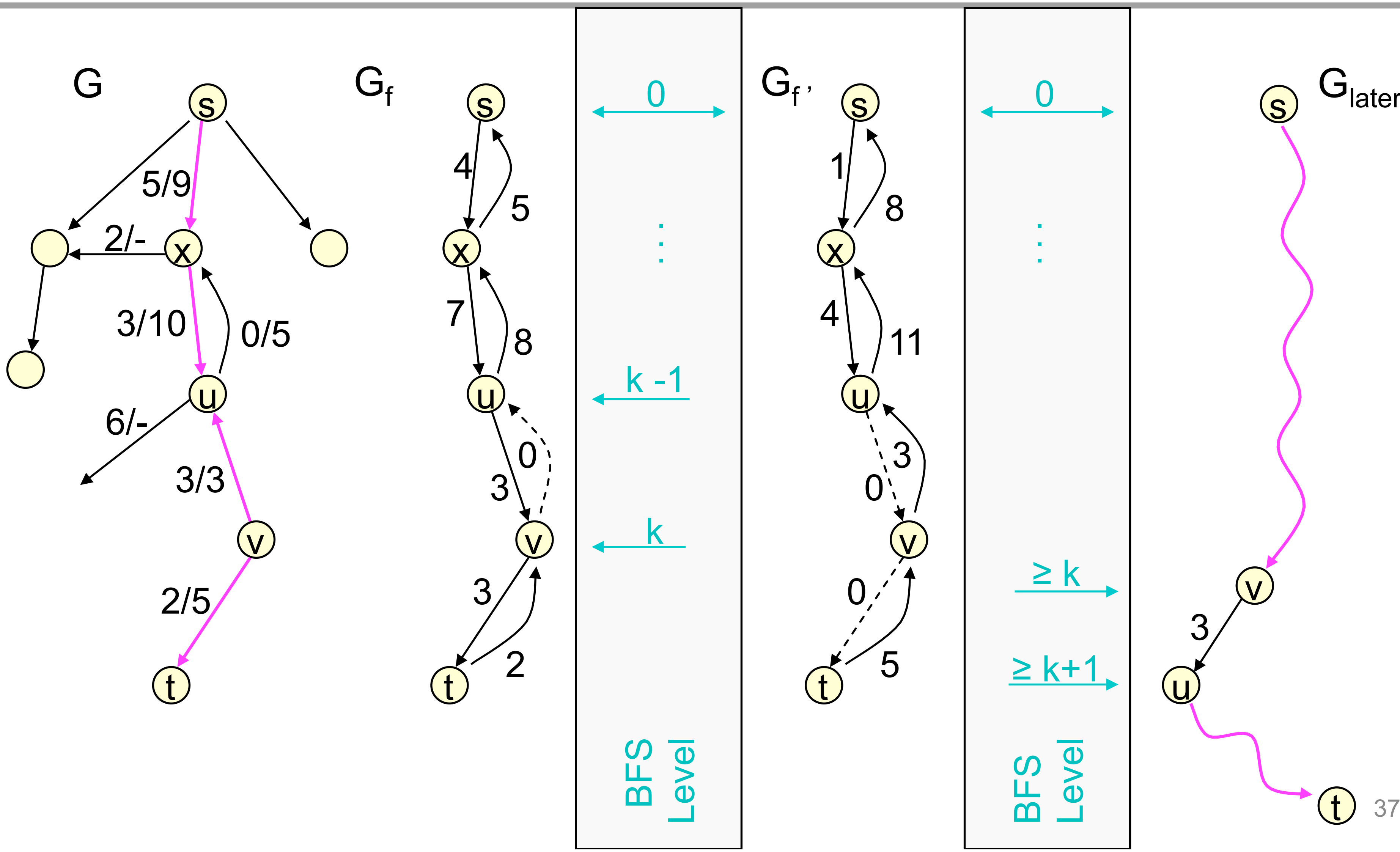
Proof:

{*u*,*v*} is critical on augmenting path *p* if it's closest to *s* having min residual capacity.

Won't be critical again until farther from *s*.

So each edge critical at most *n* times.

Augmentation vs BFS Level

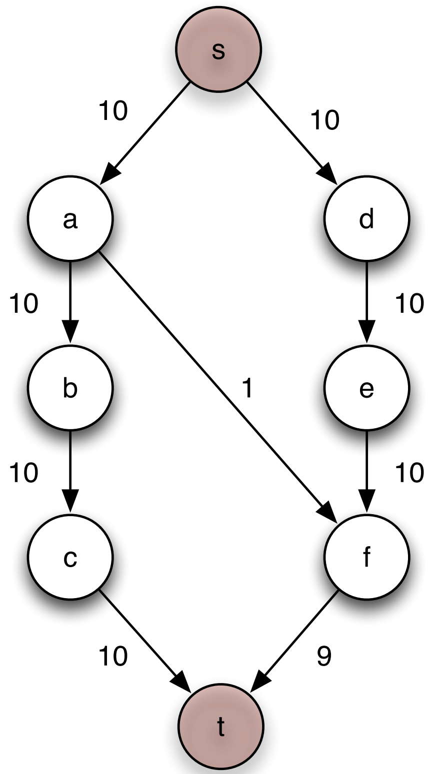


Corollary

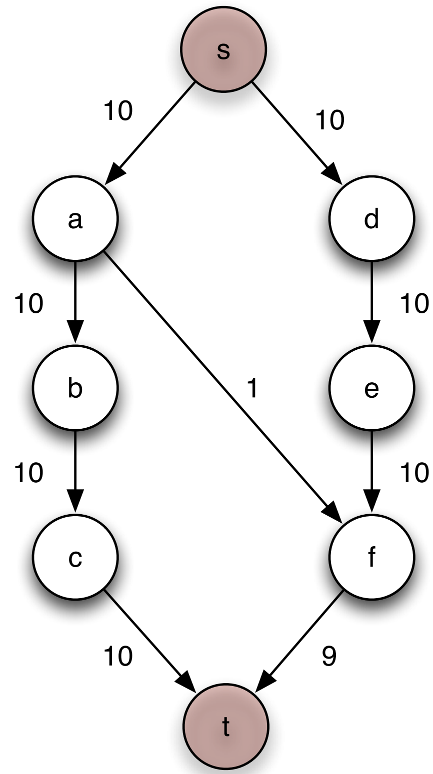
Edmonds-Karp-Dinitz runs in $O(nm^2)$

Example

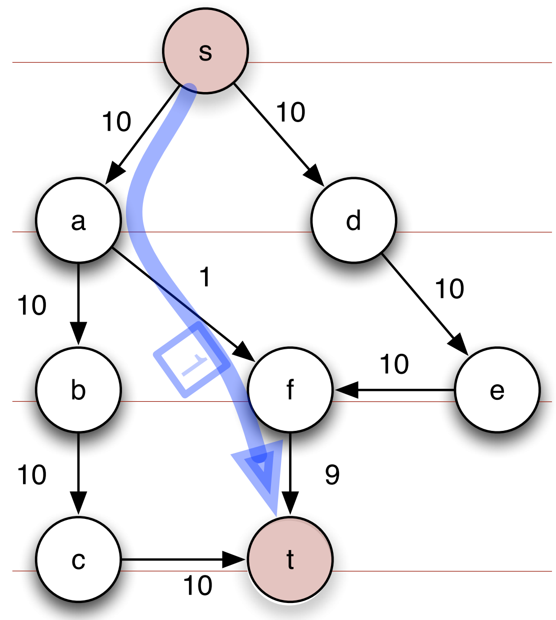
See “Edmonds-Karp-Dinitz Example” on course web page



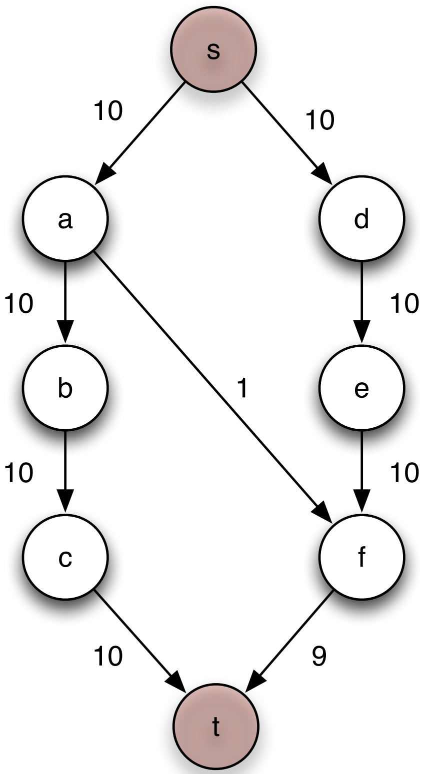
G_0: the flow problem



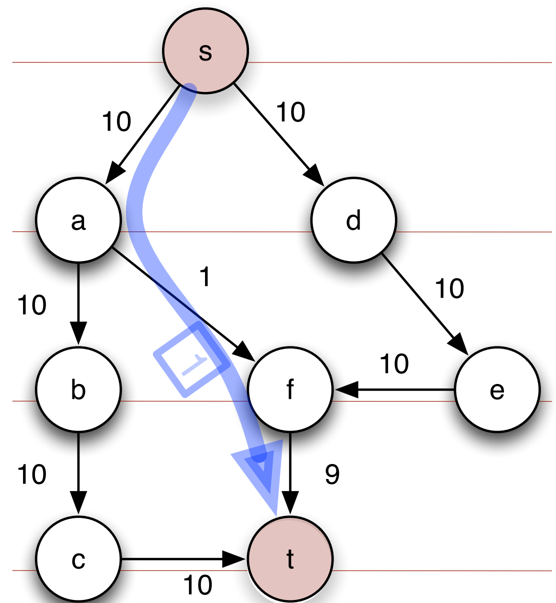
G_0: the flow problem



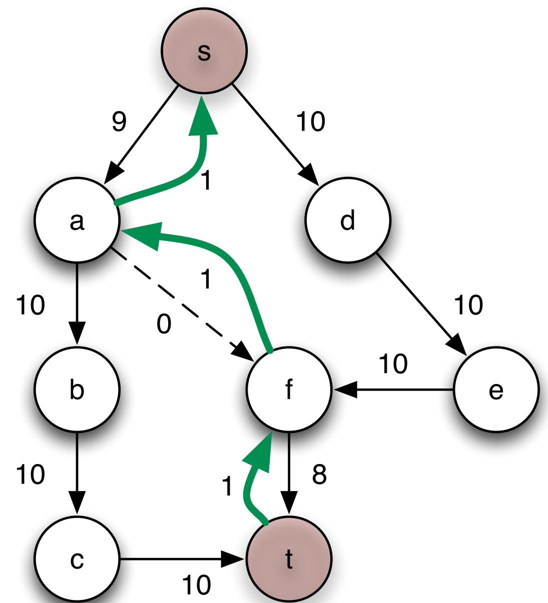
G_0: BFS layering + Aug Path



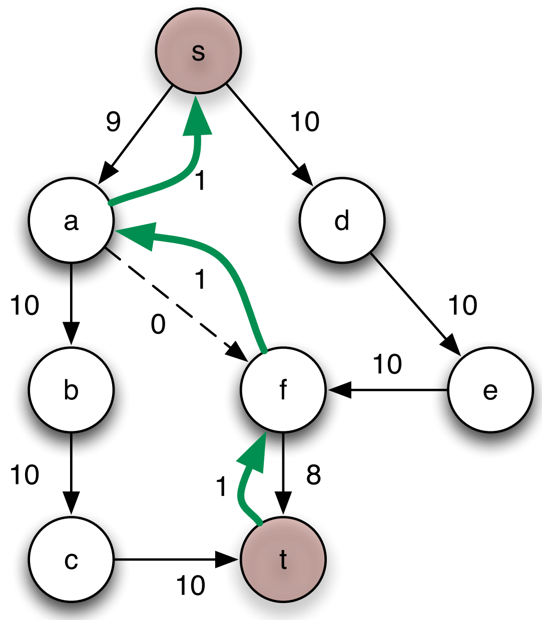
G_0: the flow problem



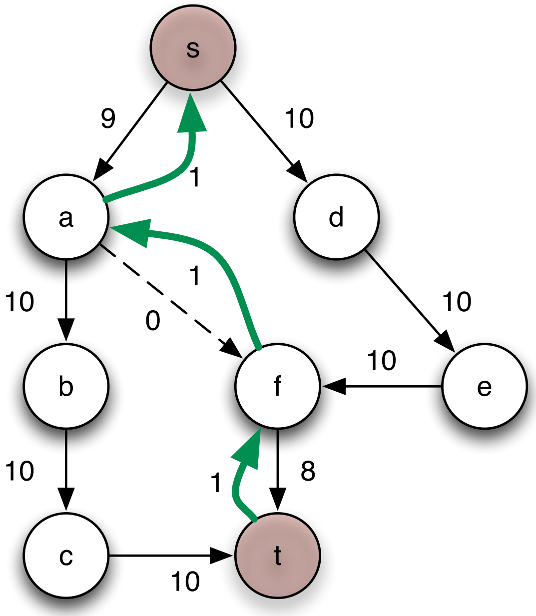
G_0: BFS layering + Aug Path



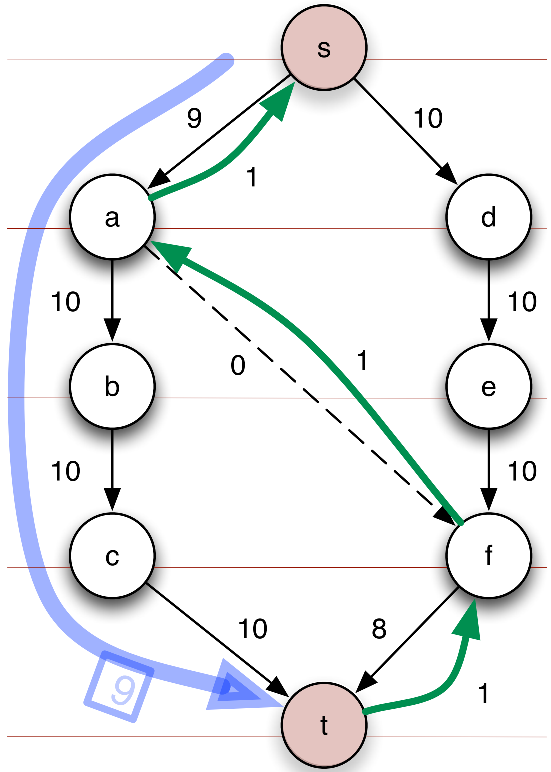
G_1: 1st Residual Graph



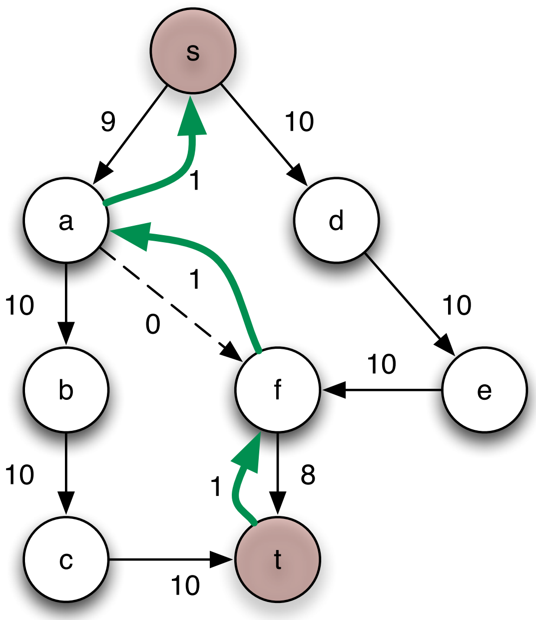
G_1: 1st Residual Graph



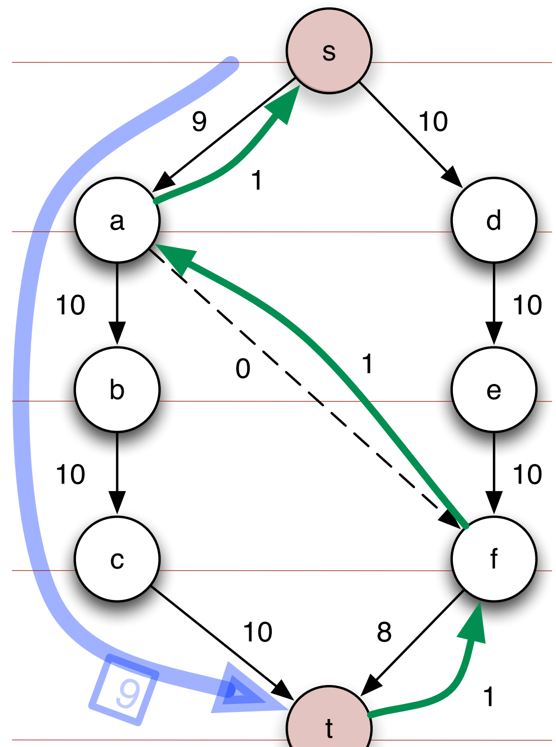
G$_1$: 1st Residual Graph



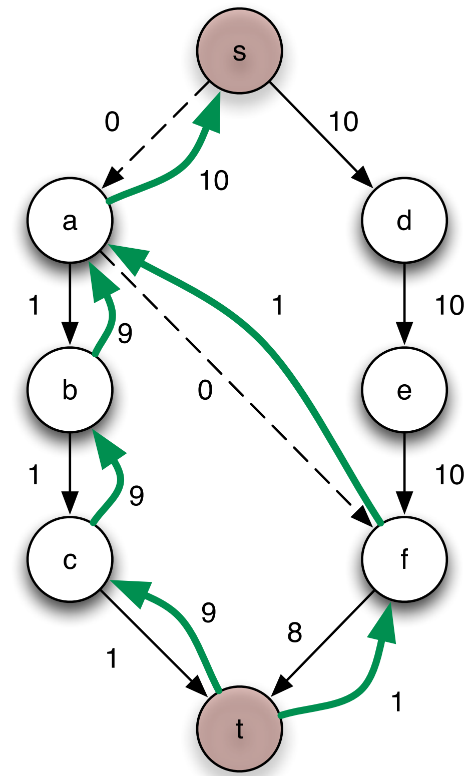
G$_1$: BFS layering + Aug Path



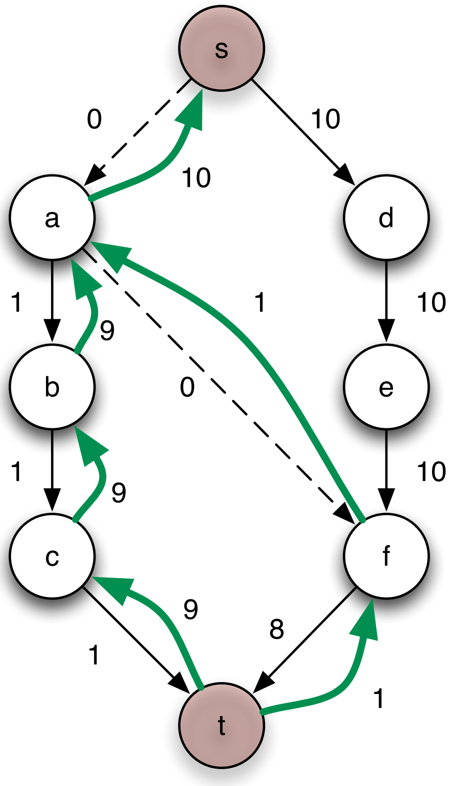
G$_1$: 1st Residual Graph



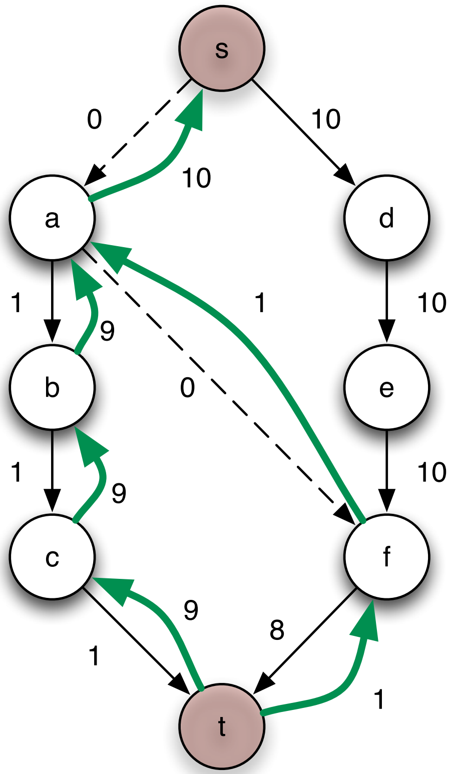
G$_1$: BFS layering + Aug Path



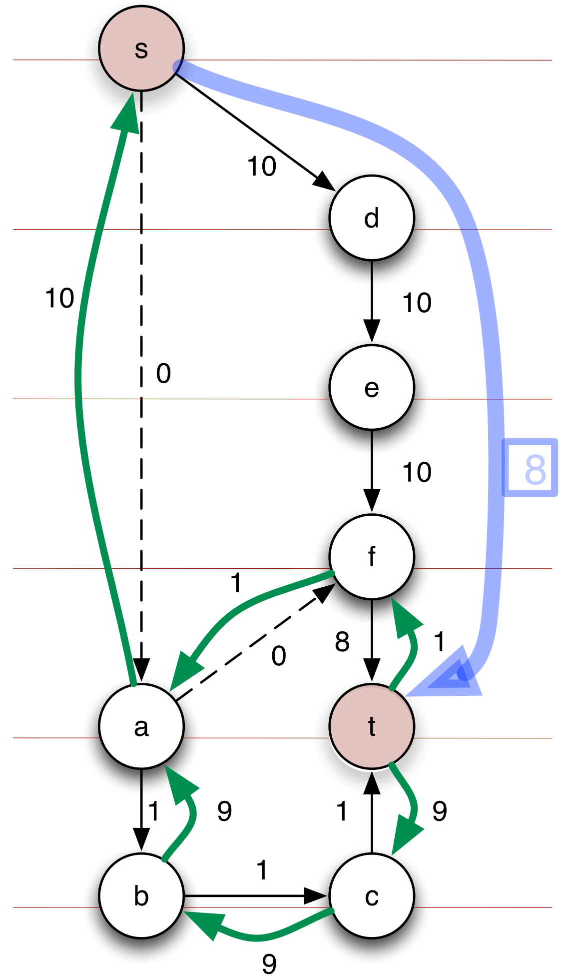
G$_2$: 2nd Residual Graph



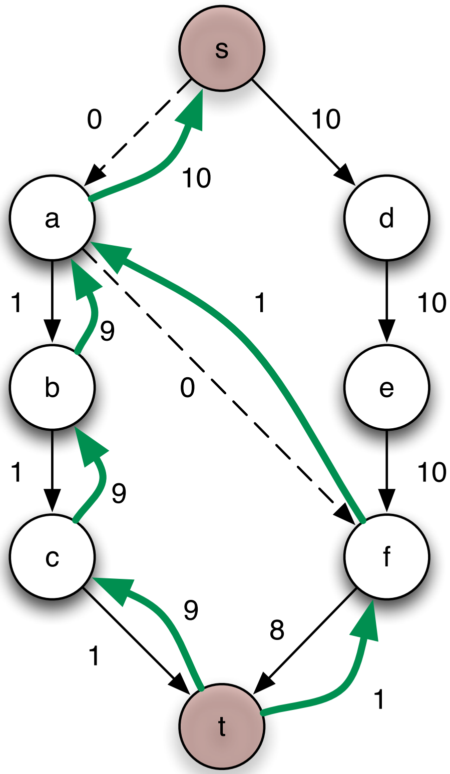
G_2: 2nd Residual Graph



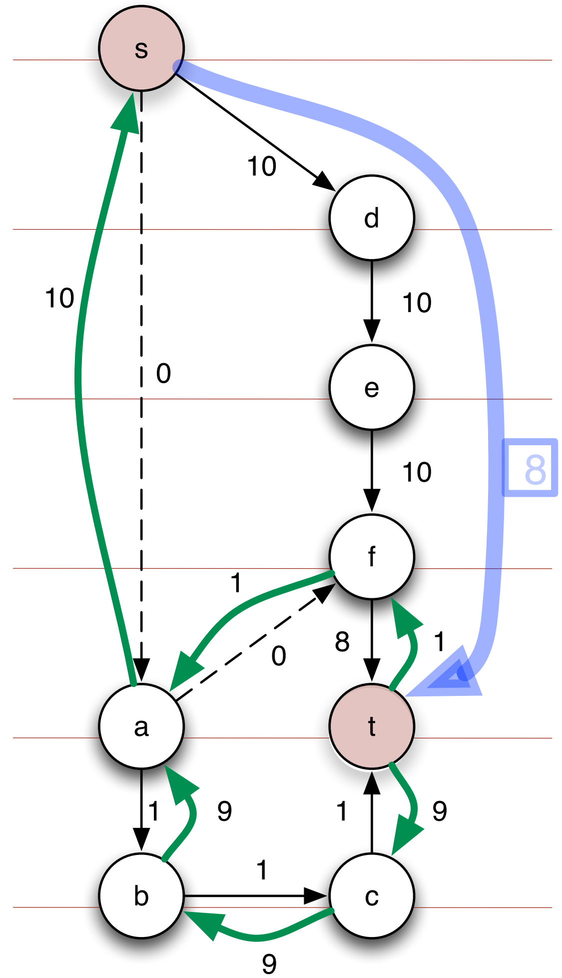
G_2: 2nd Residual Graph



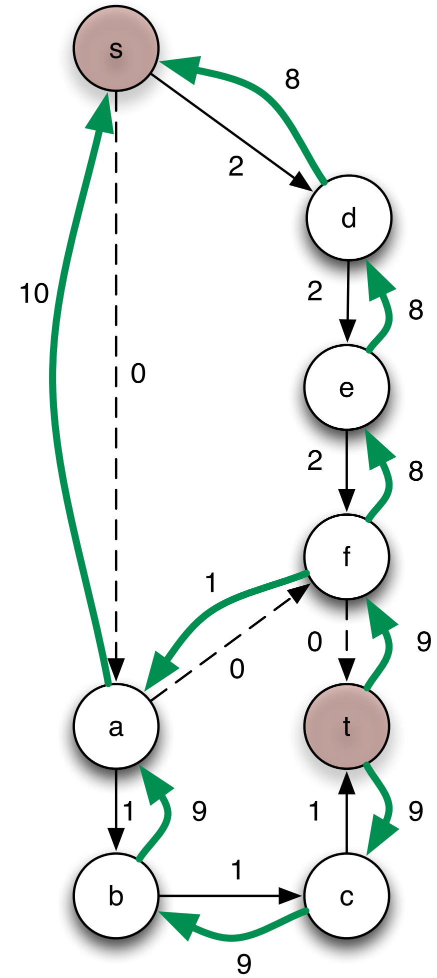
G_2: BFS layering + Aug Path



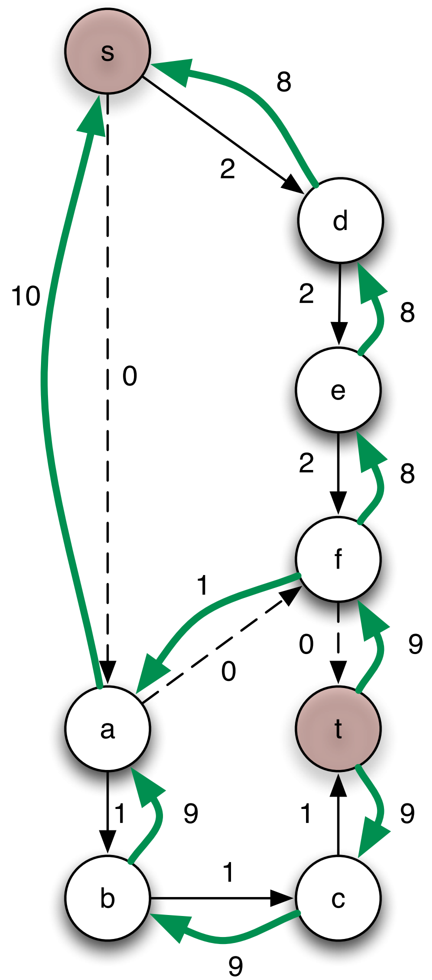
G$_2$: 2nd Residual Graph



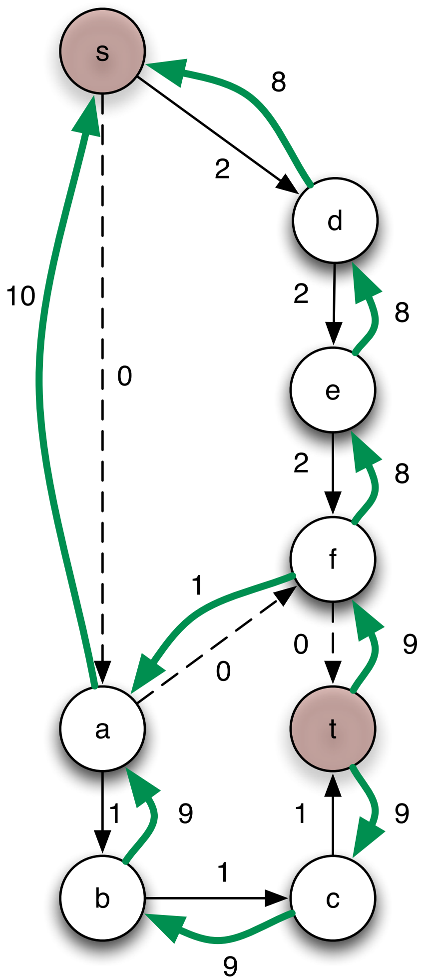
G$_2$: BFS layering + Aug Path



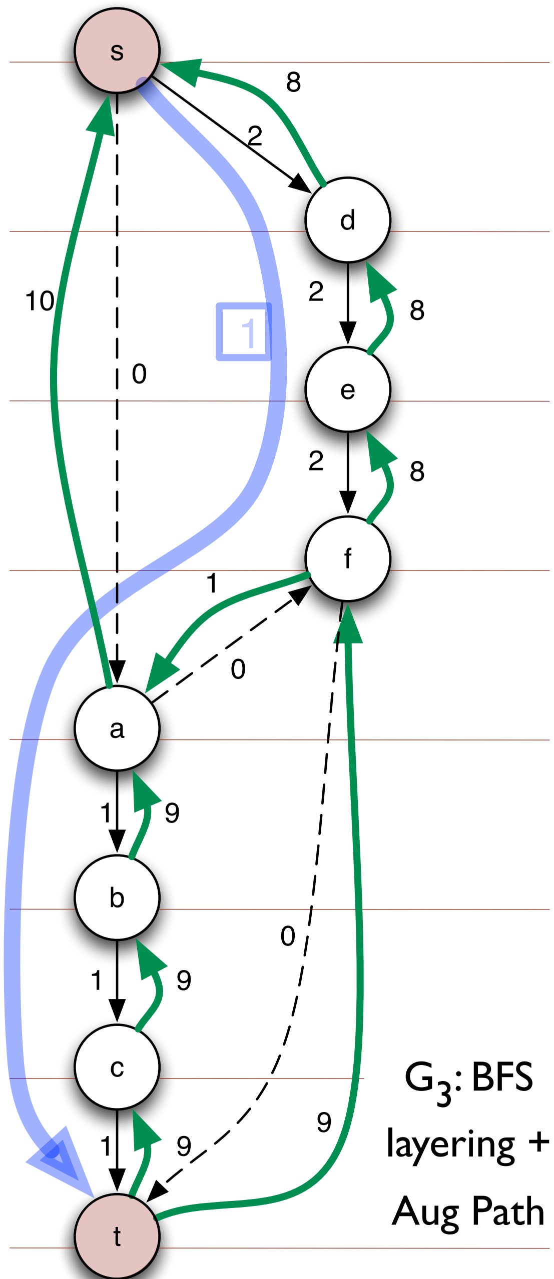
G$_3$: 3rd Residual Graph



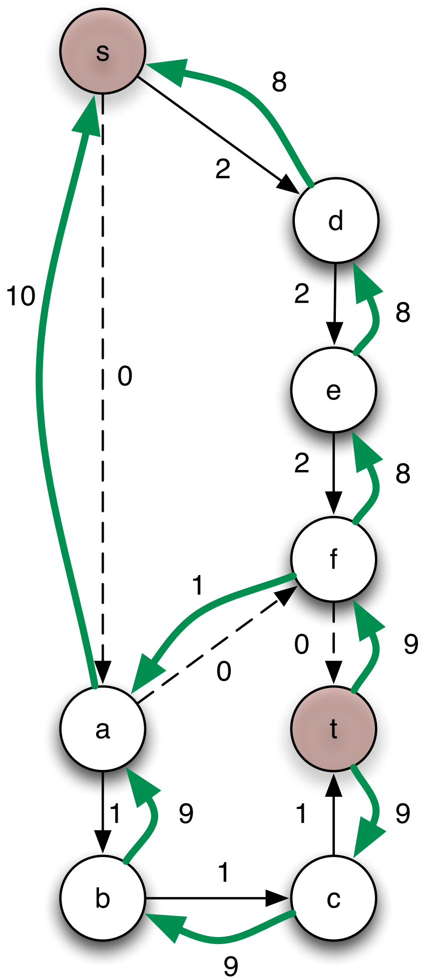
G_3: 3rd Residual Graph



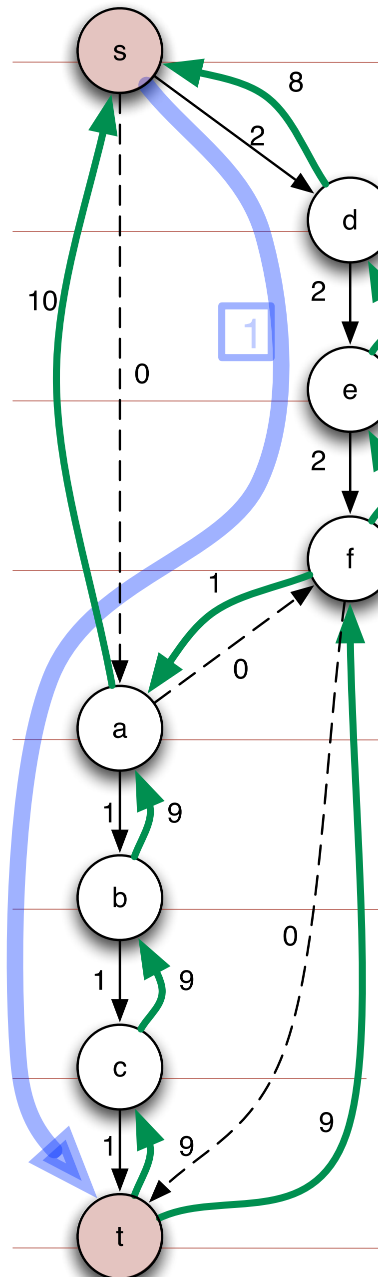
G$_3$: 3rd Residual Graph



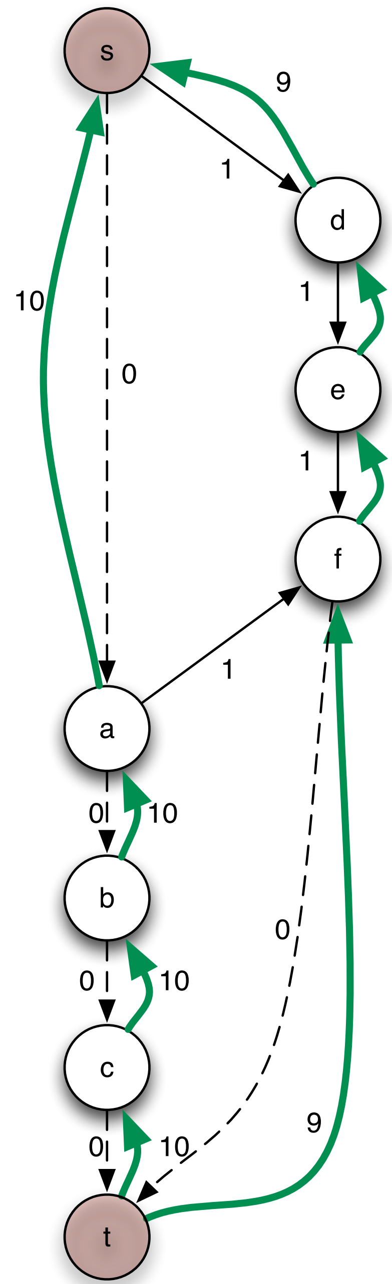
G$_3$: BFS layering + Aug Path



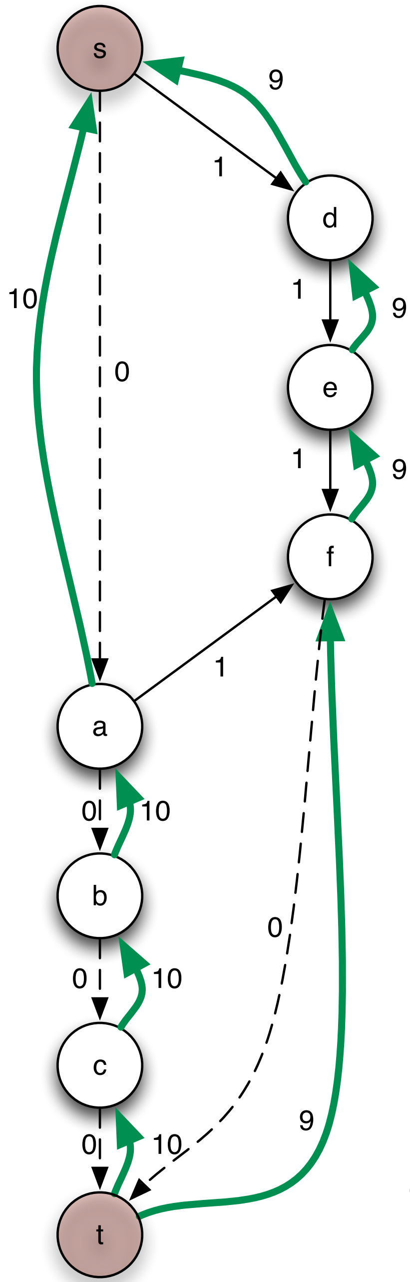
G_3: 3rd Residual Graph



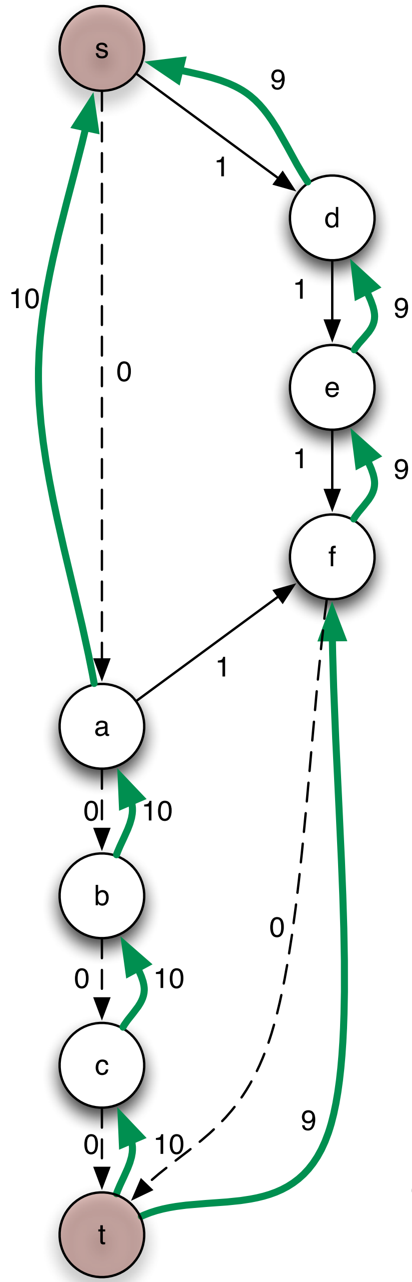
G_3: BFS layering + Aug Path



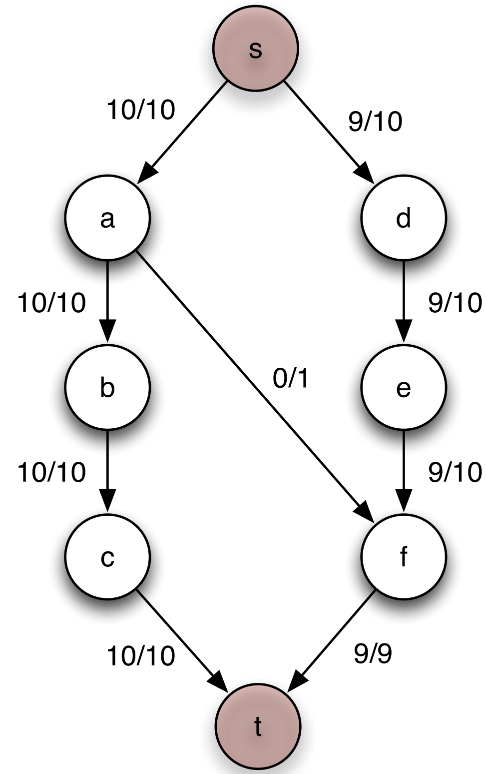
G_4: 4th residual graph



G_4: 4^{th} residual graph



G_4: 4^{th} residual graph



G_5:The Max Flow (19)

Flow Applications

Applications of Max Flow

Many!

Most look nothing like flow, at least superficially, but are deeply connected

Several interesting examples in 7.5-7.13

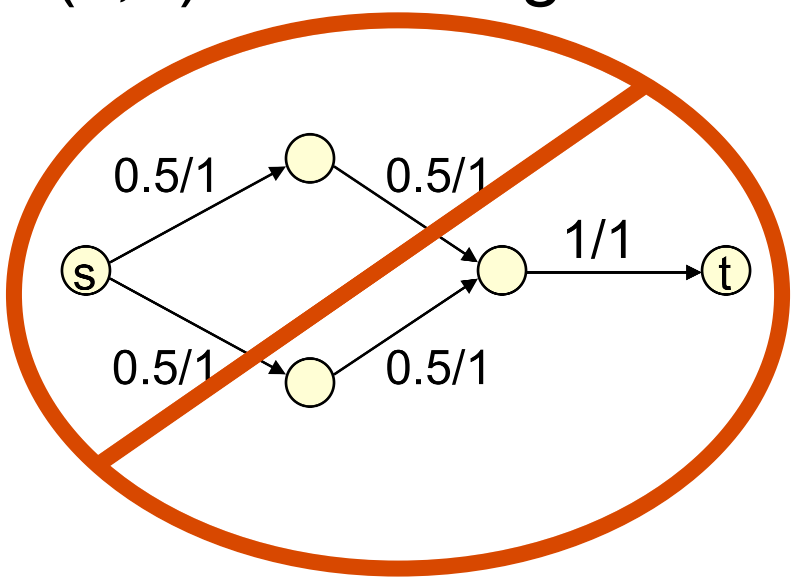
(7.8-7.11, 7.13 are optional, but interesting. Airline scheduling and image segmentation are especially recommended.)

A few more in following slides

Flow Integrality Theorem

Useful facts: If all capacities are integers

- Some max flow has an integer value
- Ford-Fulkerson method finds a max flow in which $f(u,v)$ is an integer for all edges (u,v)



A valid flow,
but unnecessary

7.6: Disjoint Paths

Given a digraph with designated nodes *s,t*, are there *k* edge-disjoint paths from *s* to *t* ?

You might try depth-first search; you might fail…

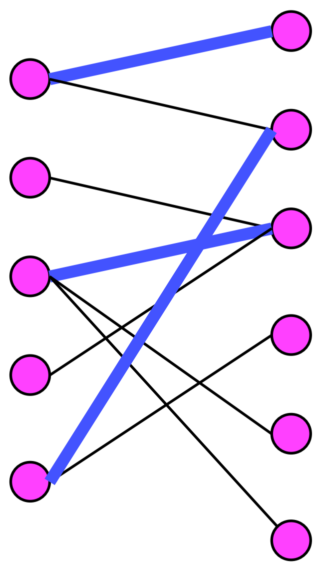
You might instead try "Is max flow ≥ *k*?" Success!

Max-flow/min-cut also implies max number of edge disjoint paths = min number of edges whose removal separates *s* from *t*.

Many variants: node-disjoint, undirected, …

See 7.6

7.5: Bipartite Maximum Matching



Bipartite Graphs:

$G = (V,E)$

$V = L \cup R \;\; (L \cap R = \emptyset)$

$E \subseteq L \times R$

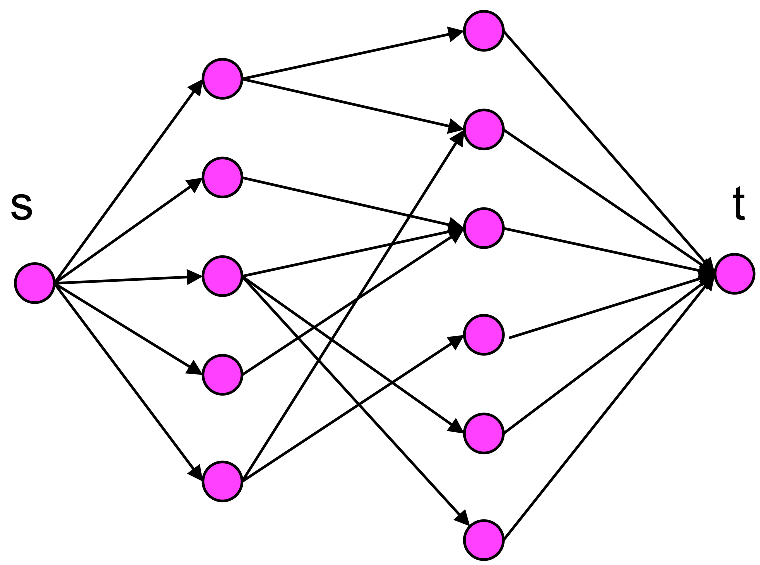
Matching:

A set of edges $M \subseteq E$ such that no two edges touch a common vertex

Problem:

Find a max size matching M

Reducing Matching to Flow



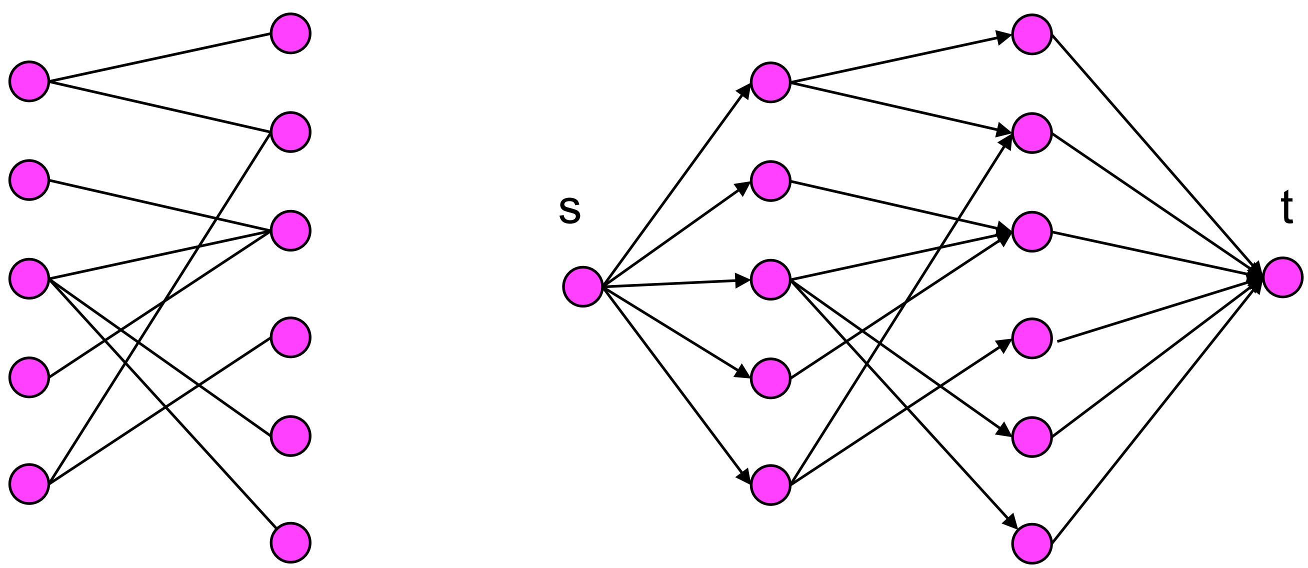
Given bipartite *G*, build flow network *N* as follows:

- Add source *s*, sink *t*
- Add edges $s \rightarrow L$
- Add edges $R \rightarrow t$
- All edge capacities 1

Theorem:
Max flow iff
max matching

Reducing Matching to Flow

Theorem: Max matching size = max flow value



$M \rightarrow f$? Easy – send flow only through M

$f \rightarrow M$? Flow Integrality Thm, + cap constraints

Notes on Matching

Max Flow Algorithm is probably overly general here

But most direct matching algorithms use "augmenting path"-type ideas similar to that in max flow – See text (& homework?)

Time $mn^{1/2}$ possible via Edmonds-Karp

7.12 Baseball Elimination

Some slides by Kevin Wayne

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- » Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.
- » $w_i + g_i < w_j \Rightarrow$ team i eliminated.
- » Only reason sports writers appear to be aware of.
- » Sufficient, but not necessary!

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- » Philly can win 83, but still eliminated . . .
- » If Atlanta loses a game, then some other team wins one.

Remark. Depends on *both* how many games already won and left to play, *and* on which opponents.

Baseball Elimination

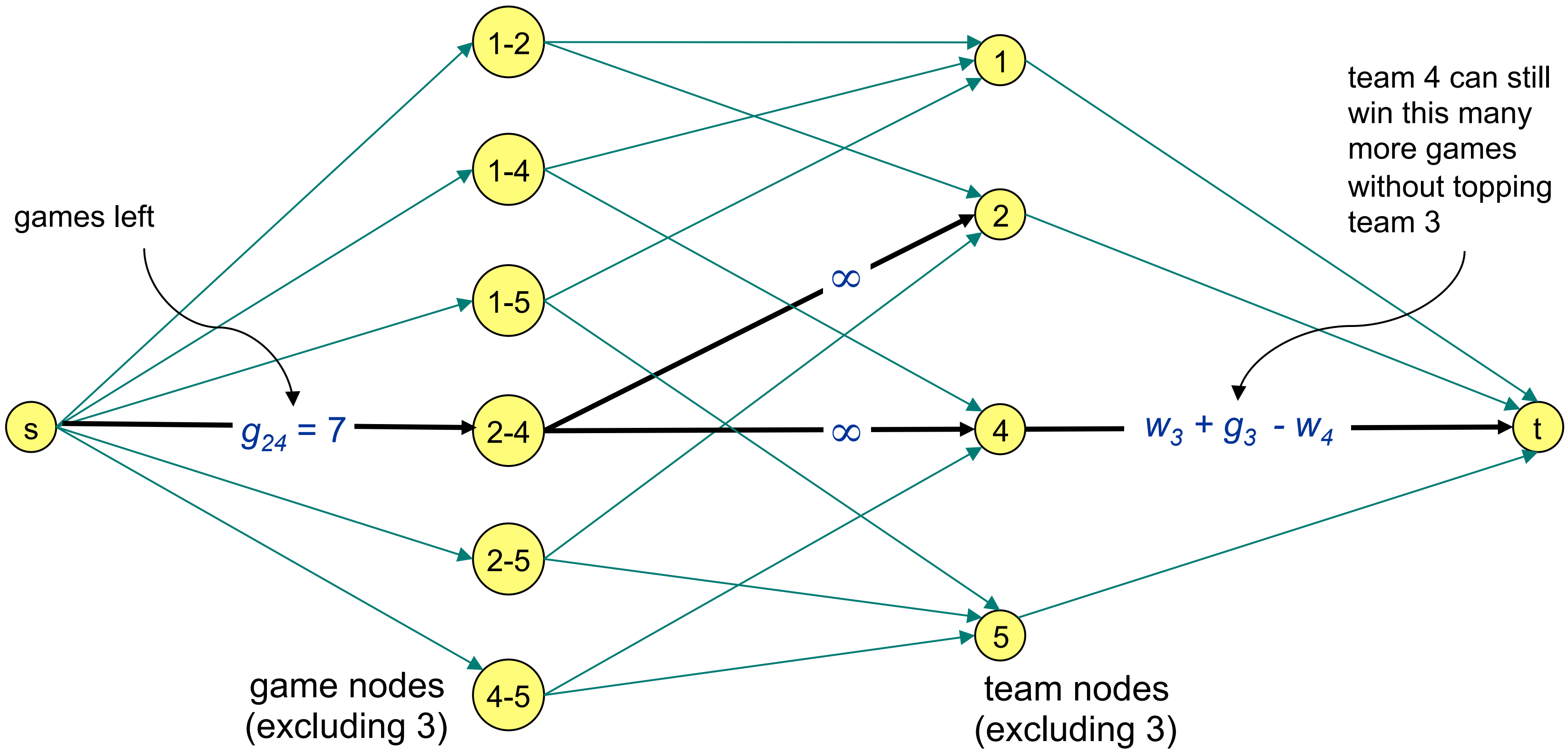
Baseball elimination problem.

- Set of teams S.
- Distinguished team $s \in S$.
- Team x has won w_x games already.
- Teams x and y play each other g_{xy} additional times.
- Is there any outcome of the remaining games in which team s finishes with the most (or tied for the most) wins?

Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

- Assume team 3 wins all remaining games $\Rightarrow$ $w_3 + g_3$ wins.
- Divvy remaining games so that all teams have $\leq$ $w_3 + g_3$ wins.

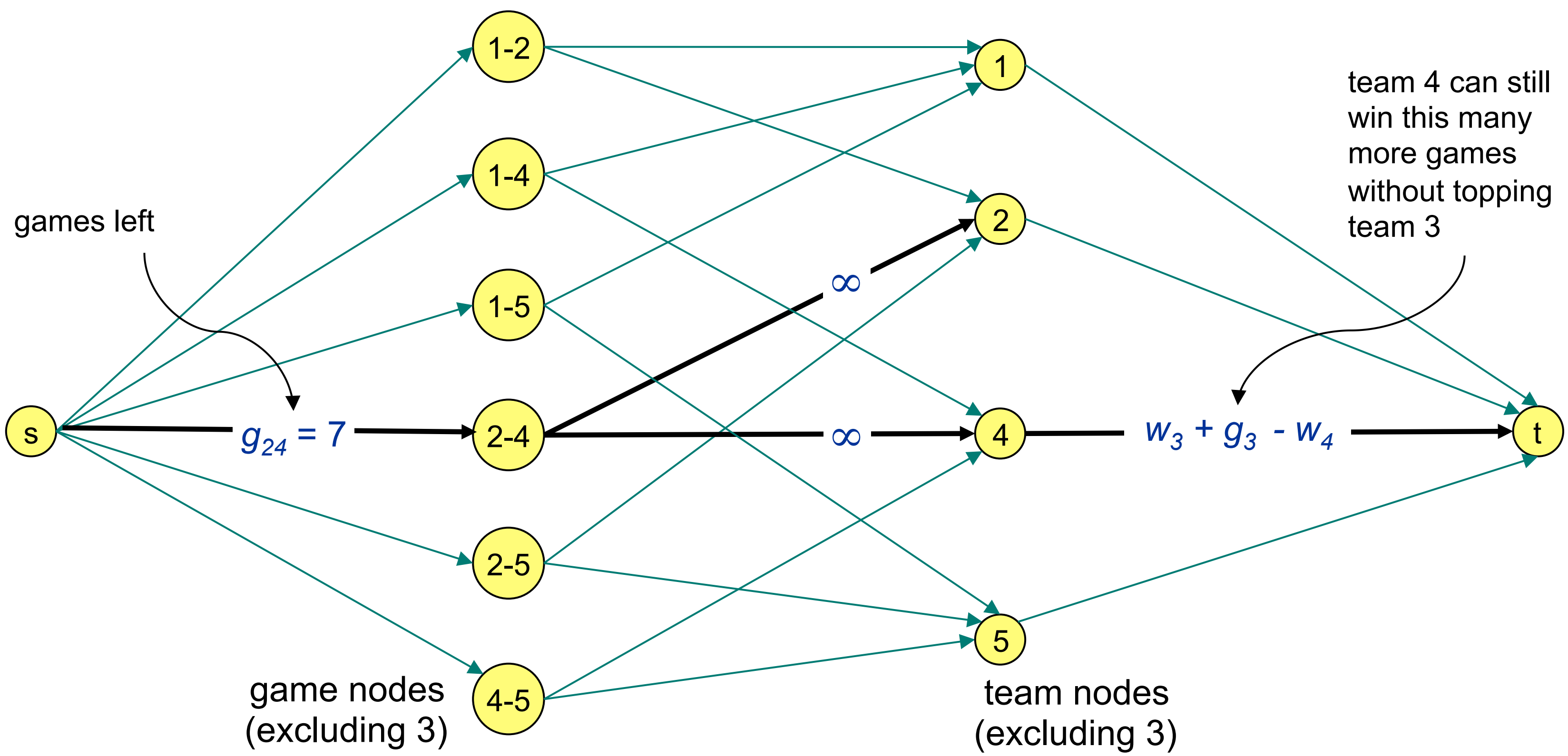


Baseball Elimination: As Max Flow

Integrality ⇒ each remaining *x : y* game added to # wins for *x* or *y*.

Capacity on *(x, t)* edges ensure no team wins too many games.

In max flow, unsaturated source edge = unplayed game; if played, (either) winner would push ahead of team 3



Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams have a chance of finishing the season with most wins?

Detroit could finish season with 49 + 27 = 76 wins.

Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams could finish the season with most wins?

- Detroit could finish season with 49 + 27 = 76 wins.

Certificate of elimination. R = {NY, Bal, Bos, Tor}

- Have already won w(R) = 278 games.
- Must win at least r(R) = 27 more.
- Average team in R wins at least 305/4 > 76 games.

Baseball Elimination: Explanation for Sports Writers

Certificate of elimination

$$T \subseteq S, \quad w(T) := \overbrace{\sum_{i \in T} w_i}^{\#\text{ wins}}, \quad g(T) := \overbrace{\sum_{\{x,y\} \subseteq T} g_{xy}}^{\#\text{ remaining games}},$$

If $\overbrace{\dfrac{w(T) + g(T)}{|T|}}^{\text{LB on avg \# games won}} > w_z + g_z$ then z eliminated (by subset T).

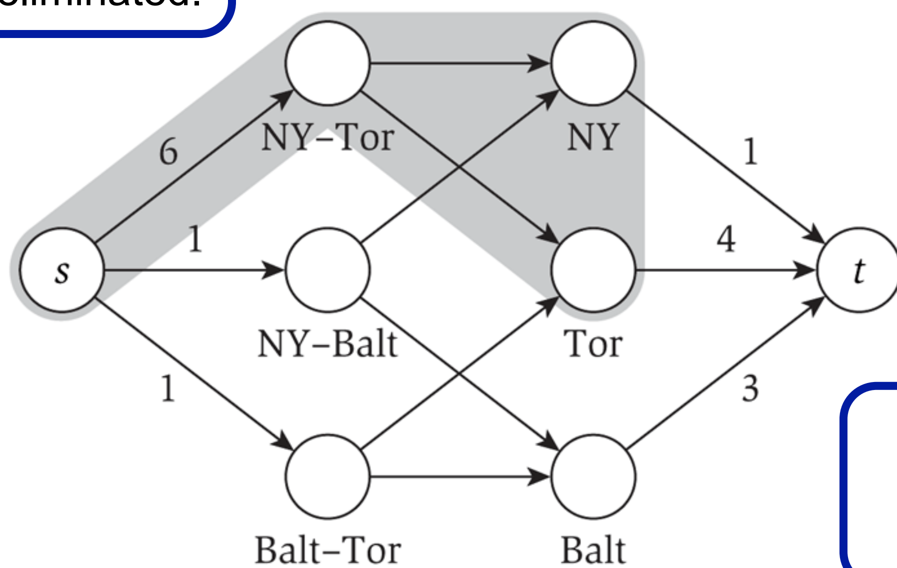
Theorem. [Hoffman-Rivlin 1967] Team z is eliminated iff there exists a subset T^* that eliminates z.

Proof idea. Let T^* = teams on source side of min cut.

	w	*l*	*g*	NY	Balt	Tor	Bos
NY	90		11	-	1	6	4
Baltimore	88		6	1	-	1	4
Toronto	87		10	6	1	-	4
Boston	79		12	4	4	4	-

$g^* = 1+6+1 = 8$

(90 + 87 + 6) / 2 > 91,
so the set T = {NY, Tor}
proves Boston is eliminated.



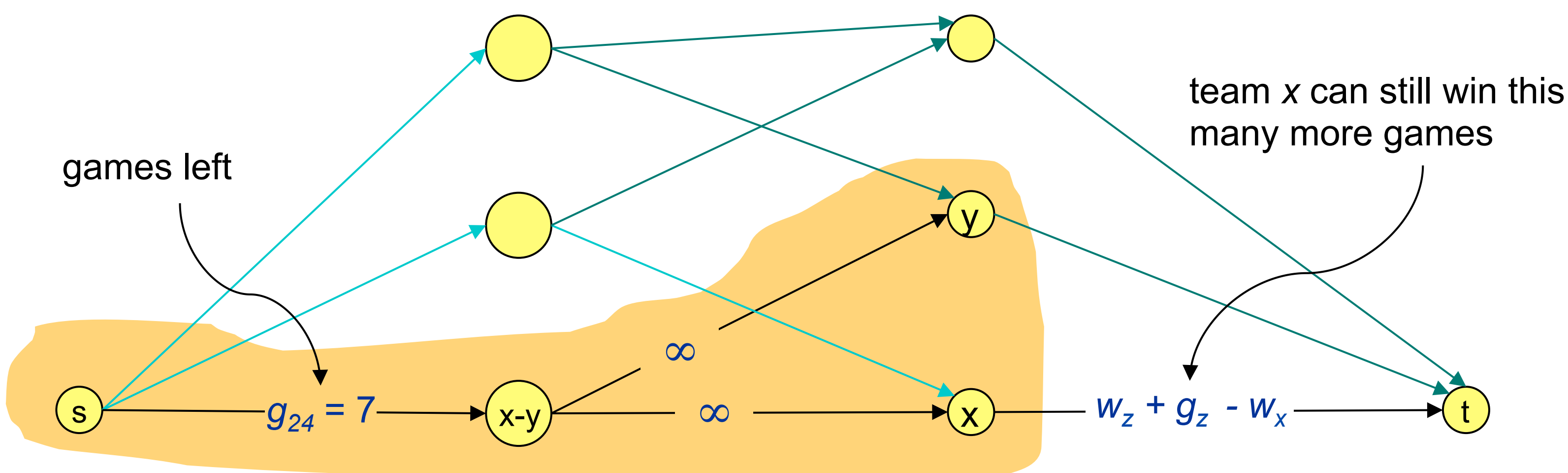
Note: *T* = {NY,Tor, Balt} is
NOT a certificate, since
(90+88+87+8)/3 = 91

Fig 7.21 Min cut $\Rightarrow$ no flow of value g^*, so Boston eliminated.

Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

- Use max flow formulation, and consider min cut (A, B).
- Define T^* = team nodes on source side of min cut.
- Observe $x\text{-}y \in A$ iff both $x \in T^*$ and $y \in T^*$.
  - infinite capacity edges ensure if $x\text{-}y \in A$ then $x \in A$ and $y \in A$
  - if $x \in A$ and $y \in A$ but $x\text{-}y \notin T^*$, then adding $x\text{-}y$ to A decreases capacity of cut



Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

Use max flow formulation, and consider min cut (*A, B*).

Define *T** = team nodes on source side of min cut.

Observe *x-y* $\in$ *A* iff both *x* $\in$ *T** and *y* $\in$ *T**.

$$
\begin{aligned}
g(S - \{z\}) &> cap(A,\ B) \\
&= \overbrace{g(S-\{z\}) - g(T^*)}^{\text{capacity of game edges leaving A}} + \overbrace{\sum_{x \in T^*} (w_z + g_z - w_x)}^{\text{capacity of team edges leaving A}} \\
&= g(S-\{z\}) - g(T^*) - w(T^*) + |T^*|(w_z + g_z)
\end{aligned}
$$

Rearranging:

$$
w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}
$$

Matching & Baseball: Key Points

Can (sometimes) take problems that seemingly have *nothing* to do with flow & reduce them to a flow problem

How? Build a clever network; map allocation of *stuff* in original problem (match edges; wins) to allocation of *flow* in network. Clever edge capacities constrain solution to mimic original problem in some way. Integrality useful.

Matching & Baseball: Key Points

Furthermore, in the baseball example, min cut can be translated into a succinct *certificate* or *proof* of some property that is much more transparent than "see, I ran max-flow and it says flow must be less than g^*".

These examples suggest why max flow is so important – *it's a very general tool used in many other algorithms*.