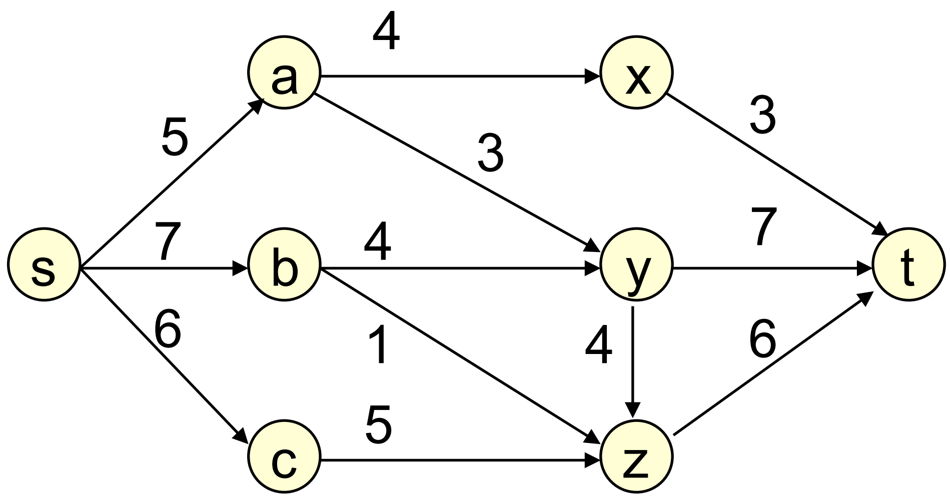
CSE 421
Introduction to Algorithms
Winter 2012

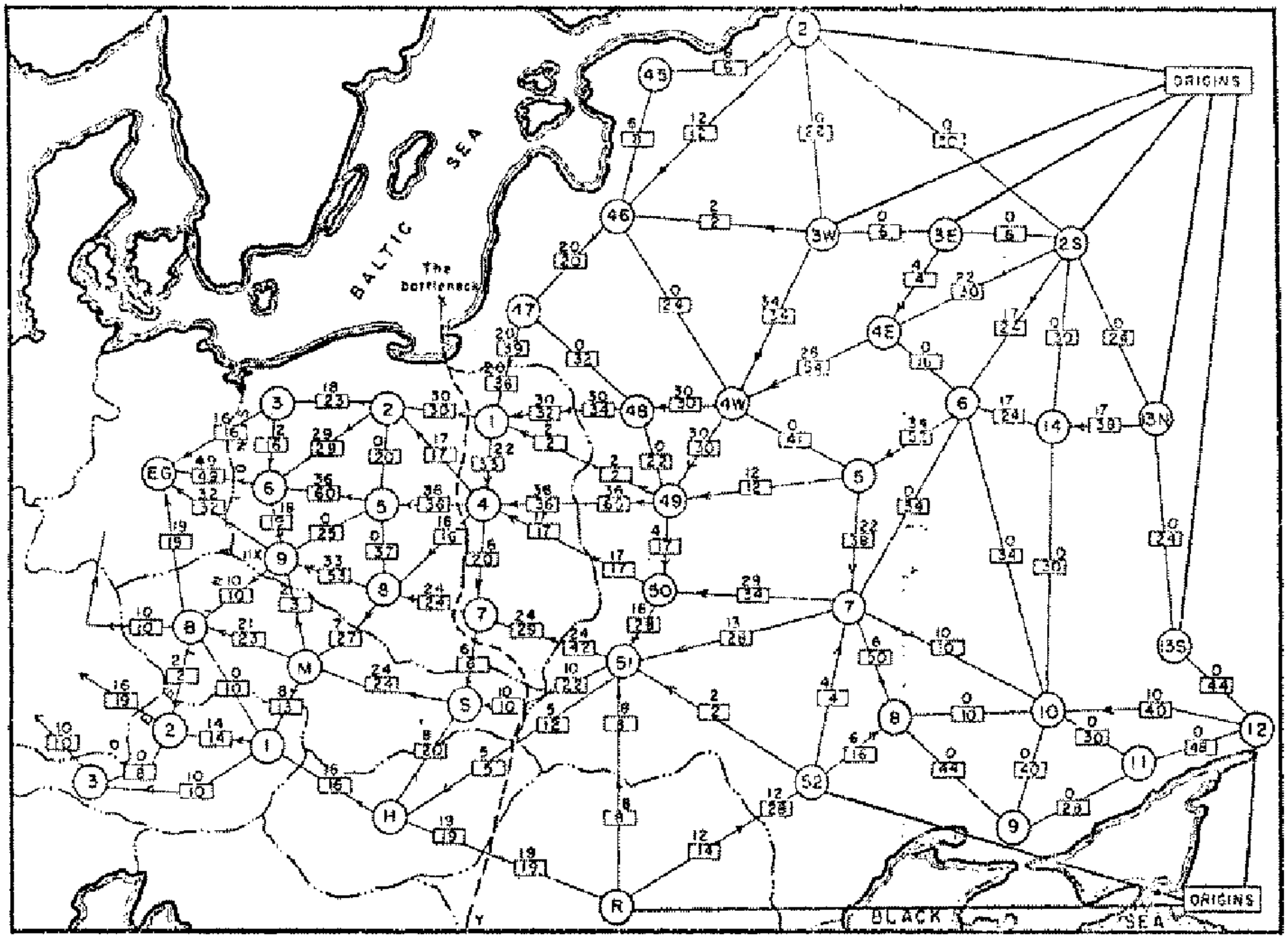
The Network Flow Problem

The Network Flow Problem



How much stuff can flow from s to t?

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems*.
Alexander Schrijver in Math Programming, 91: 3, 2002.

Net Flow: Formal Definition

Given:

A digraph G = (V,E)

Two vertices s,t in V (*source* & *sink*)

A *capacity* $c(u,v) \geq 0$ for each $(u,v) \in E$
(and c(u,v) = 0 for all non-edges (u,v))

Find:

A *flow function* $f: V \times V \rightarrow R$ s.t., for all u,v:

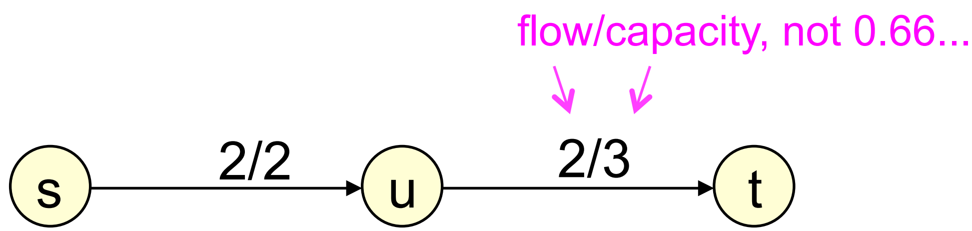
- $f(u,v) \leq c(u,v)$ [Capacity Constraint]
- $f(u,v) = -f(v,u)$ [Skew Symmetry]
- if $u \neq s,t$, $f(u,V) = 0$ [Flow Conservation]

Maximizing total flow $|f| = f(s,V)$

Notation:

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Example: A Flow Function



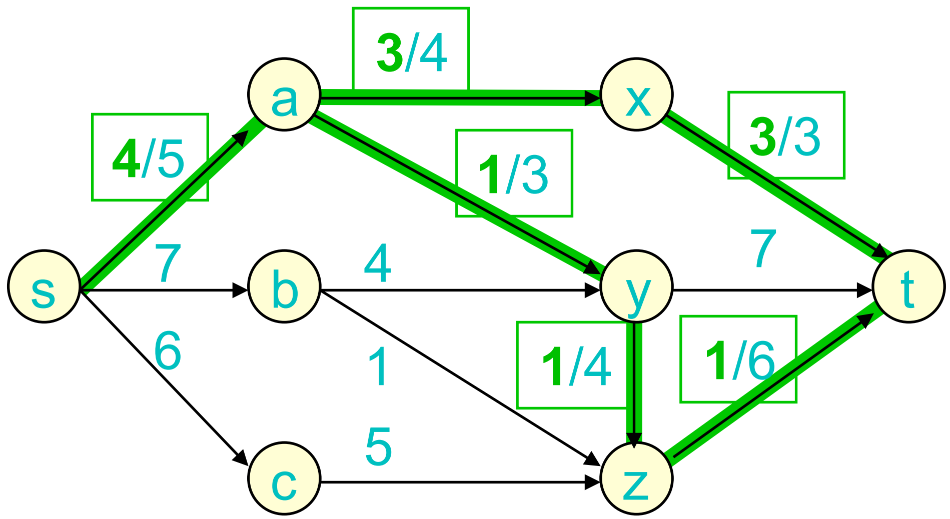
$f(s,u) = f(u,t) = 2$

$f(u,s) = f(t,u) = -2$ *(Why?)*

$f(s,t) = -f(t,s) = 0$ *(In every flow function for this G. Why?)*

$$f(u,V) = \sum_{v \in V} f(u,v) = f(u,s) + f(u,t) = -2 + 2 = 0$$

Example: A Flow Function



Not shown: f(u,v) if ≤ 0

Note: max flow ≥ 4 since f is a flow, |f| = 4

Max Flow via a Greedy Alg?

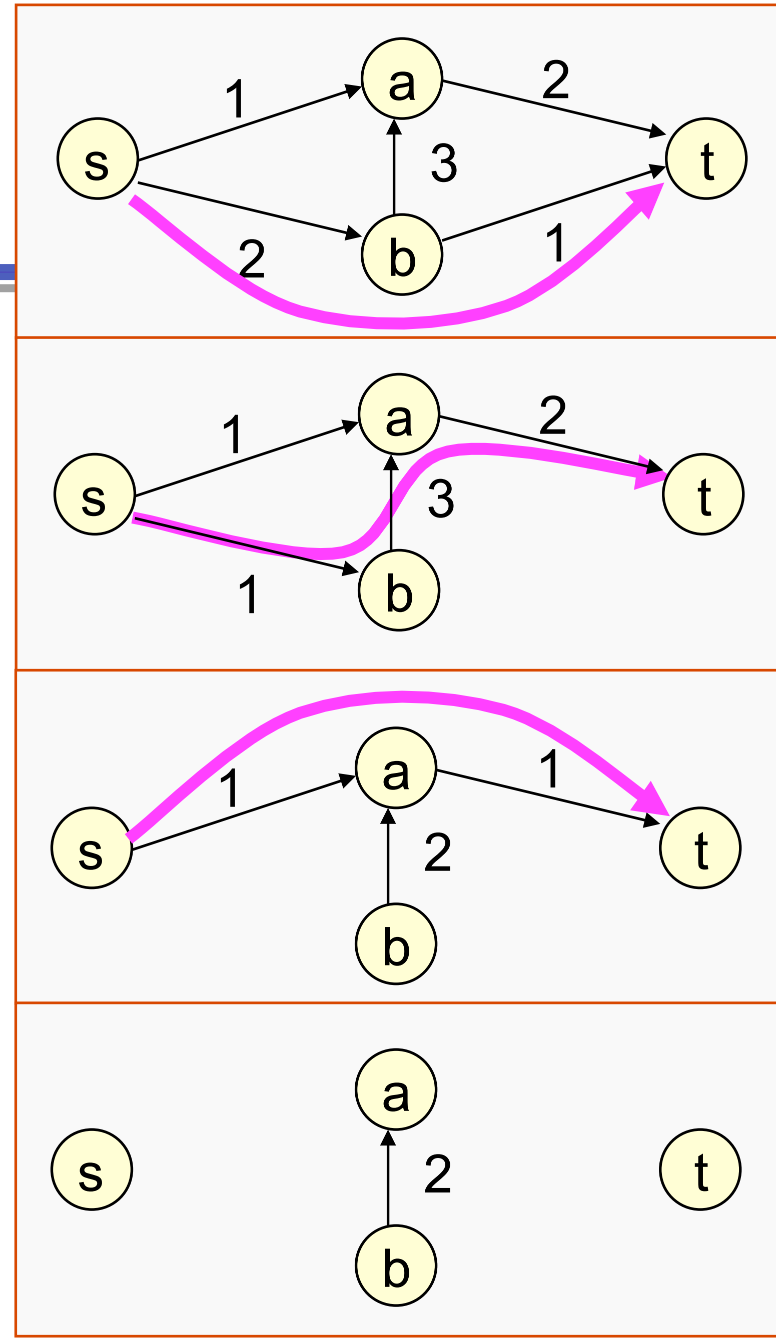
While there is an $s \rightarrow t$ path in G

Pick such a path, p

Find c_p, the min capacity of any edge in p

Subtract c_p from all capacities on p

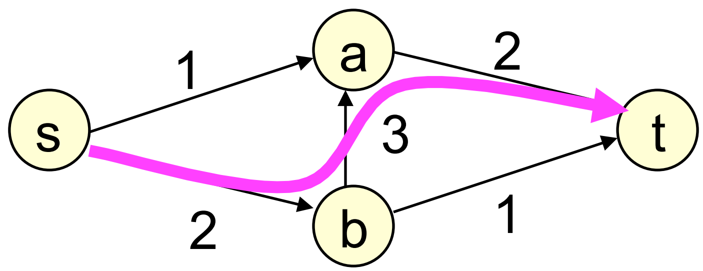
Delete edges of capacity 0

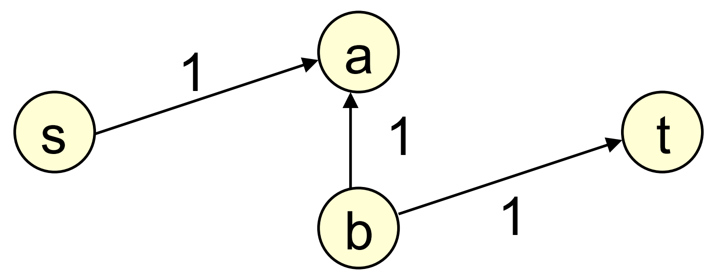


Max Flow via a Greedy Alg?

This does NOT always find a max flow:

If you pick $s \rightarrow b \rightarrow a \rightarrow t$ first,





Flow stuck at 2. But flow 3 possible.

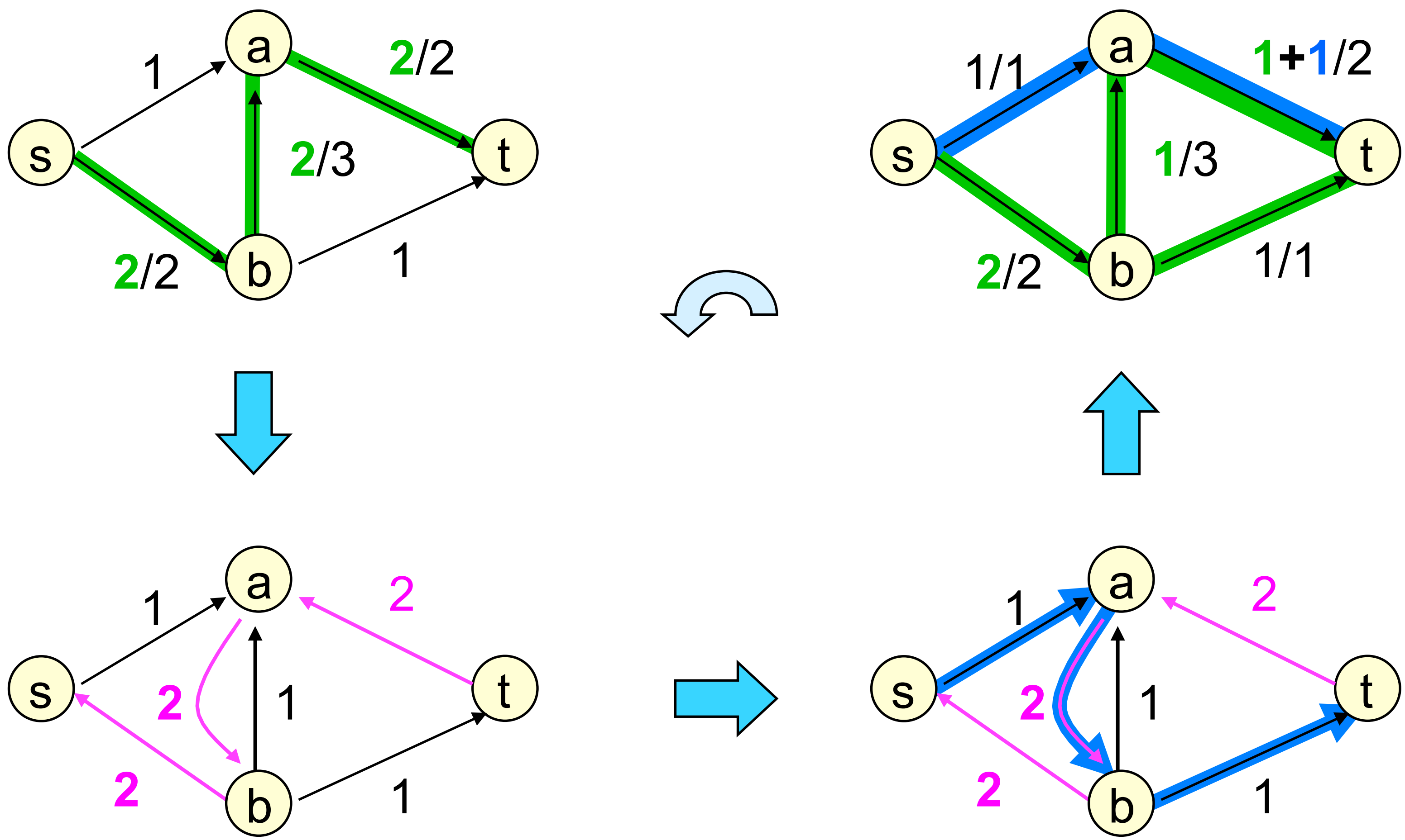
A Brief History of Flow

n = # of vertices
m= # of edges
U = Max capacity

Source: Goldberg & Rao, FOCS '97

#	Year	Discoverer(s)	Bound
1	1951	Dantzig	$O(n^2mU)$
2	1955	Ford & Fulkerson	$O(nmU)$
3	1970	Dinitz; Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2m)$
5	1972	Edmonds & Karp; Dinitz	$O(m^2 \log U)$
6	1973	Dinitz;Gabow	$O(nm \log U)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2 \text{sqrt}(m))$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg &Tarjan	$O(nm \log (n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log U)$
13	1987	Ahuja et al.	$O(nm \log(n \text{ sqrt}(\log U)/(m+2))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\varepsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\varepsilon} n)$
19	1994	King et al.	$O(nm(\log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log U)$; $O(n^{2/3} m \log(n^2/m) \log U)$
…	…	…	…

Greed Revisited



Residual Capacity

The *residual capacity* (w.r.t. f) of (u,v) is

$c_f(u,v) = c(u,v) - f(u,v)$

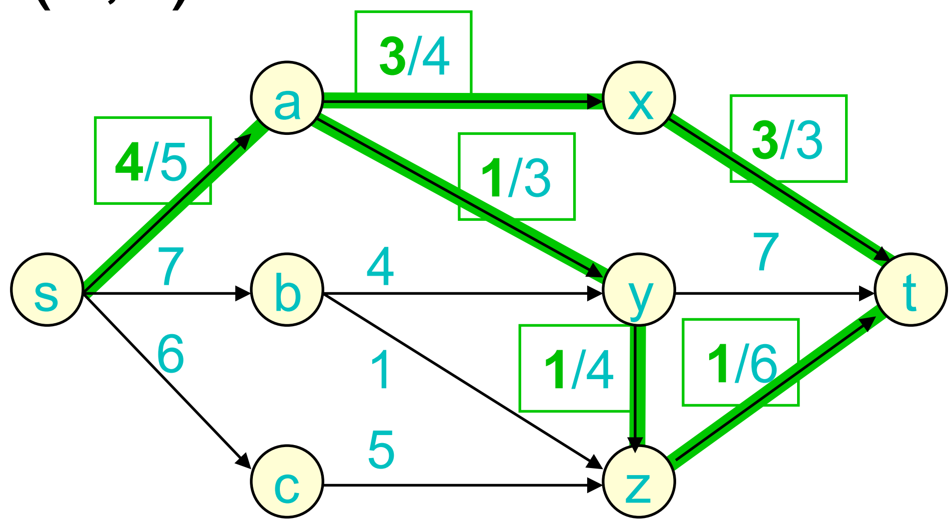
E.g.:

$c_f(s,b) = 7;$

$c_f(a,x) = 1;$

$c_f(x,a) = 3;$

$c_f(x,t) = 0$ (a saturated edge)



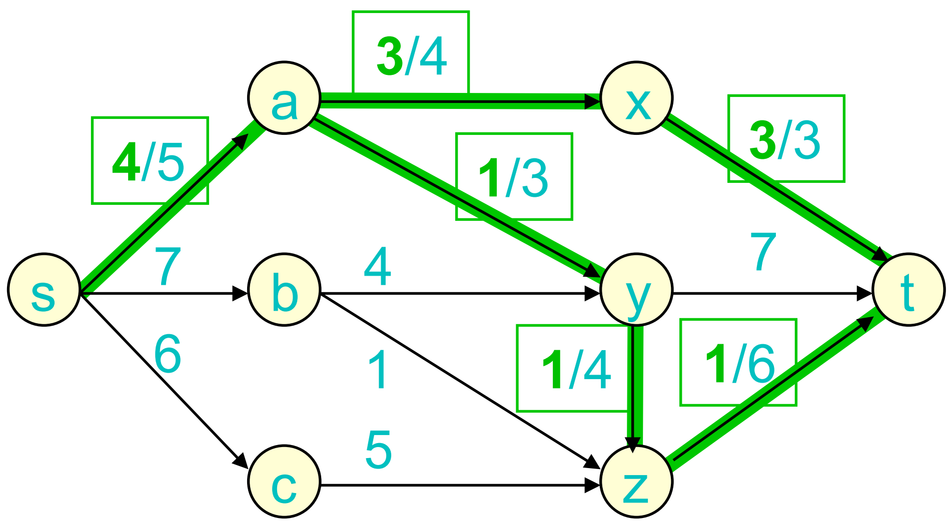
Residual Networks & Augmenting Paths

The *residual network* (w.r.t. f) is the graph $G_f = (V,E_f)$, where

$$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$$

An *augmenting path* (w.r.t. f) is a simple $s \to t$ path in G_f.

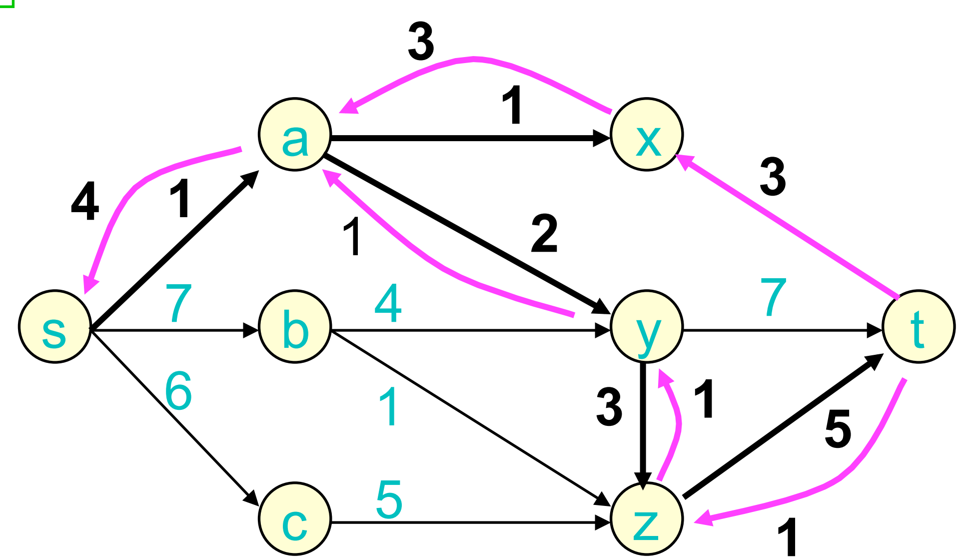
A Residual Network



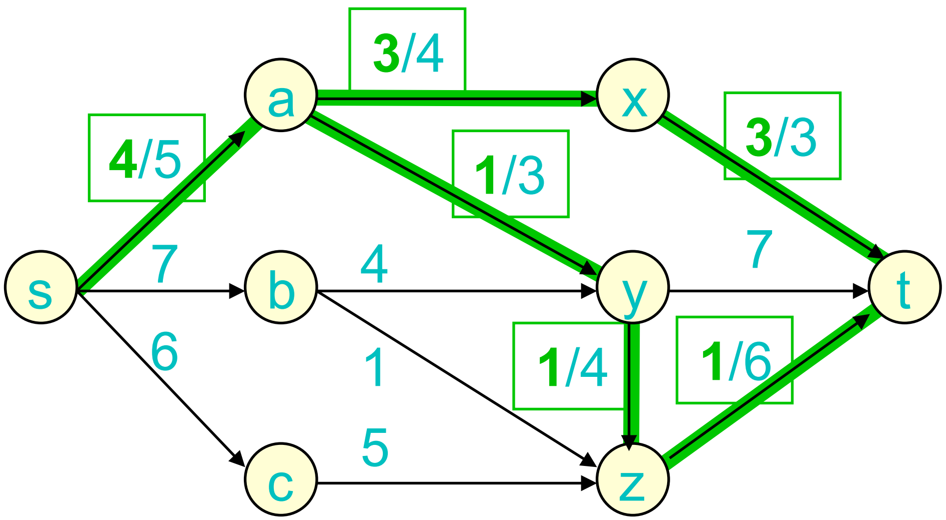
residual network: the graph

$G_f = (V, E_f)$, where

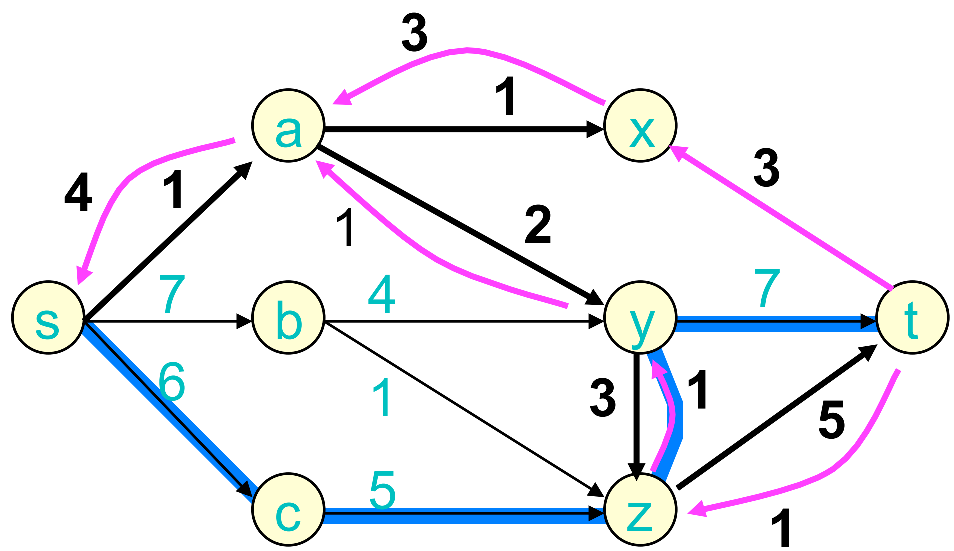
$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$



An Augmenting Path



augmenting path:
a simple s → t path in G_f.

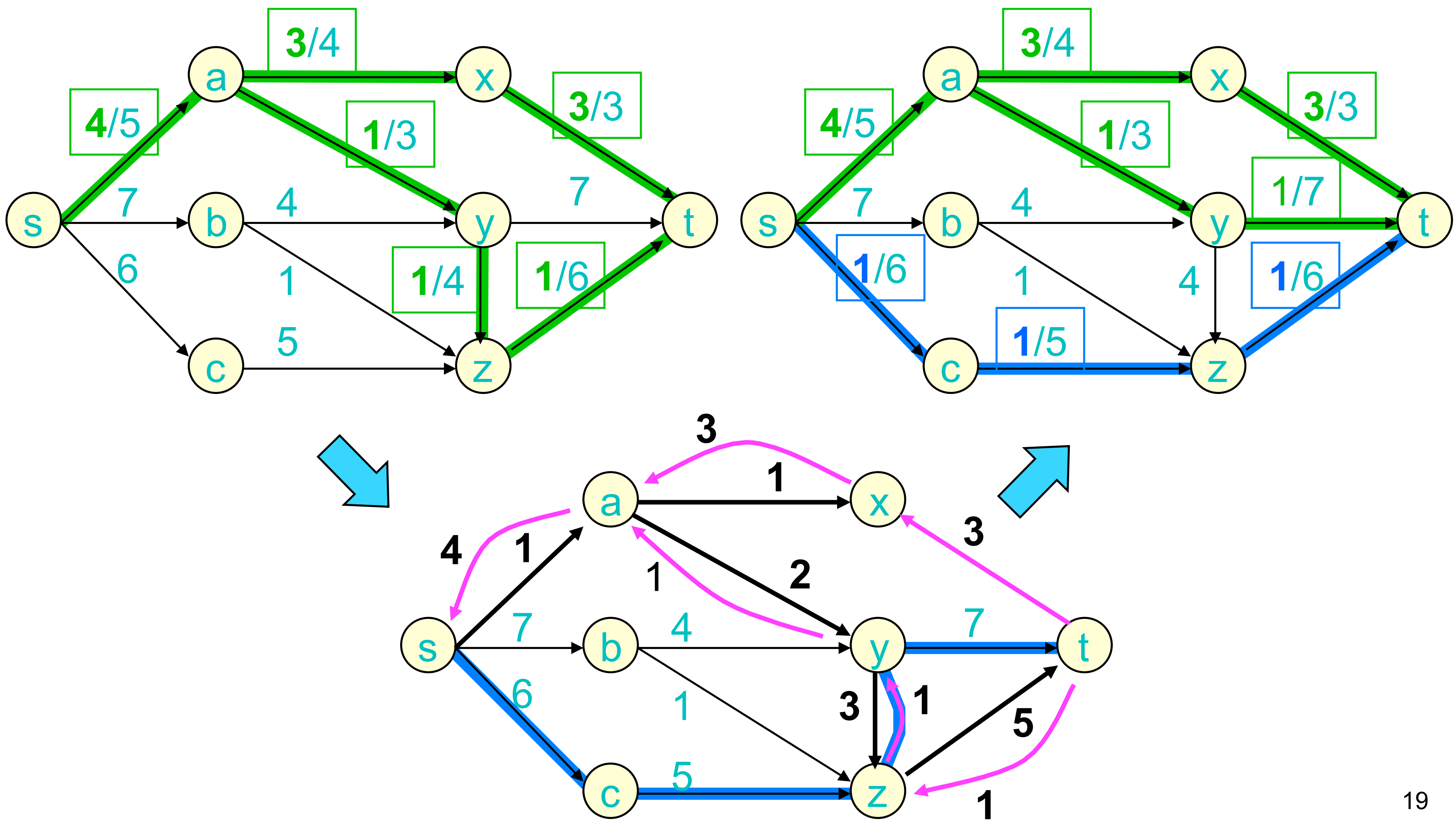


Lemma 1

If f admits an augmenting path p, then f is not maximal.

Proof: “obvious” -- augment along p by c_p, the min residual capacity of p’s edges.

Augmenting A Flow



Lemma 1’:
Augmented Flows are Flows

If f is a flow & p an augmenting path of capacity c_p, then f' is also a valid flow, where

$$\boxed{f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}}$$

Proof:

a) Flow conservation – easy

b) Skew symmetry – easy

c) Capacity constraints – pretty easy

Lma 1': Augmented Flows are Flows

$$f'(u,v) = \begin{cases} f(u,v) + c_p, & \text{if } (u,v) \text{ in path } p \\ f(u,v) - c_p, & \text{if } (v,u) \text{ in path } p \\ f(u,v), & \text{otherwise} \end{cases}$$

f a flow & p an aug path of cap c_p, then f' also a valid flow.

Proof (Capacity constraints):

(u,v), (v,u) not on path: no change

(u,v) on path:

$$\begin{aligned} f'(u,v) &= f(u,v) + c_p \\ &\le f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v) \end{aligned}$$

$$\begin{aligned} f'(v,u) &= f(v,u) - c_p \\ &< f(v,u) \\ &\le c(v,u) \end{aligned}$$

Residual Capacity:

$0 < c_p \le c_f(u,v) = c(u,v) - f(u,v)$

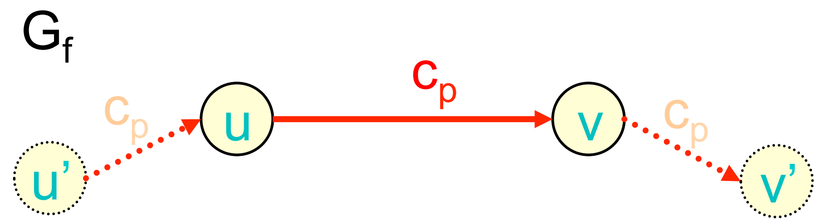
Cap Constraints:

$-c(v,u) \le f(u,v) \le c(u,v)$

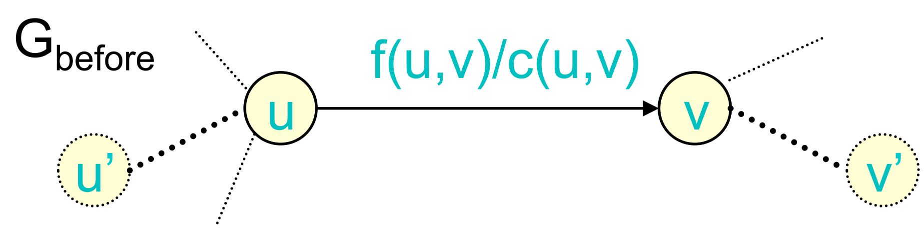
Lemma 1' Example – Case 1

Let (u,v) be any edge in augmenting path. Note

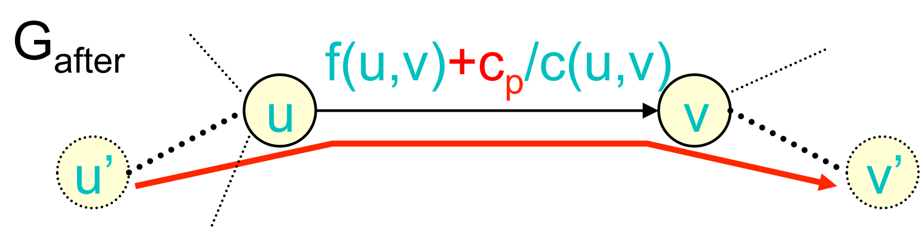
$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$



Case 1: $f(u,v) \geq 0$:



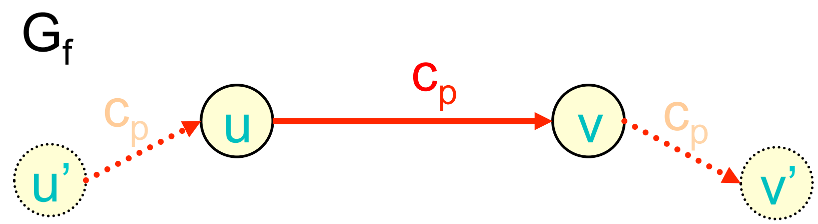
Add forward flow



Lemma 1' Example – Case 2

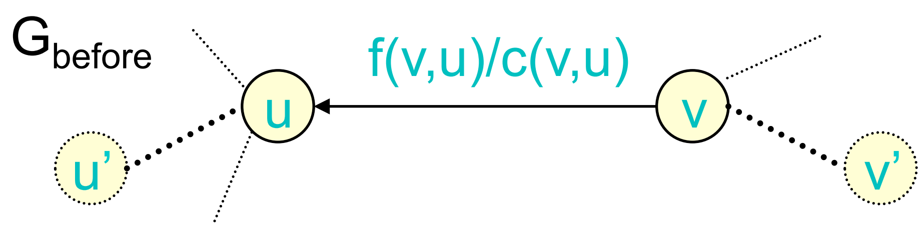
Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

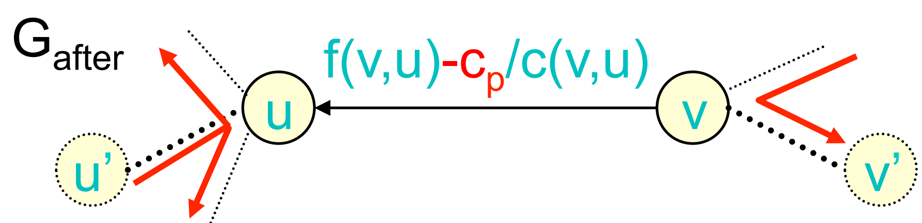


Case 2: $f(u,v) \leq -c_p$:

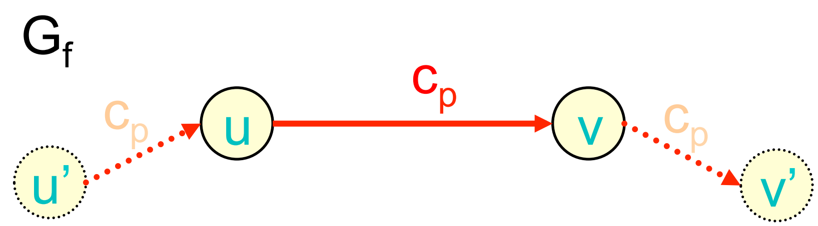
$f(v,u) = -f(u,v) \geq c_p$



Cancel/redirect
reverse flow



Lemma 1' Example – Case 3

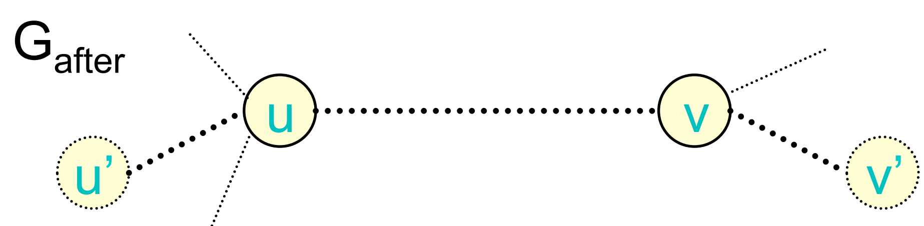


Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) - f(u,v) \geq c_p > 0$

Case 3: $-c_p < f(u,v) < 0$:

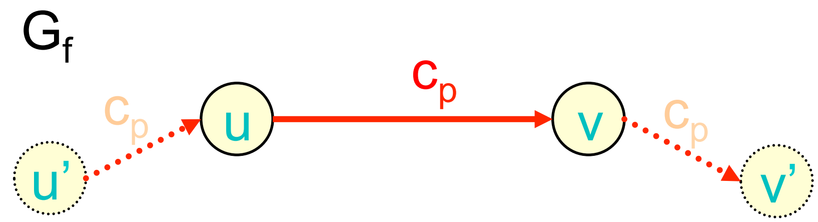
???



Lemma 1' Example – Case 3

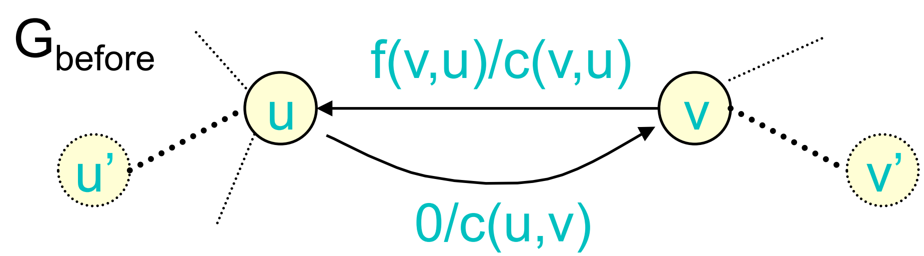
Let (u,v) be any edge in augmenting path. Note

$c_f(u,v) = c(u,v) – f(u,v) \geq c_p > 0$



Case 3: $-c_p < f(u,v) < 0$

$c_p > f(v,u) > 0$:

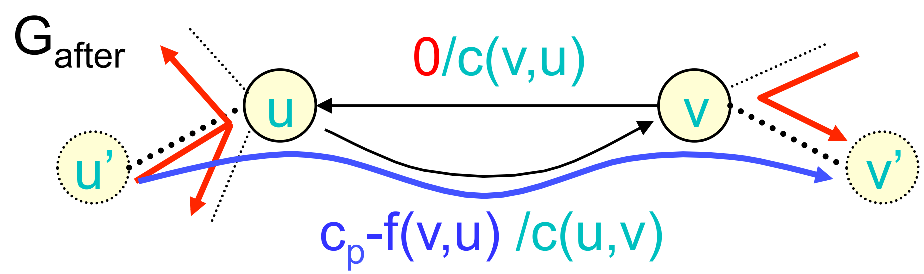


Both:

cancel/redirect
reverse flow

and

add forward flow



Ford-Fulkerson Method

While G_f has an augmenting path, augment

Questions:

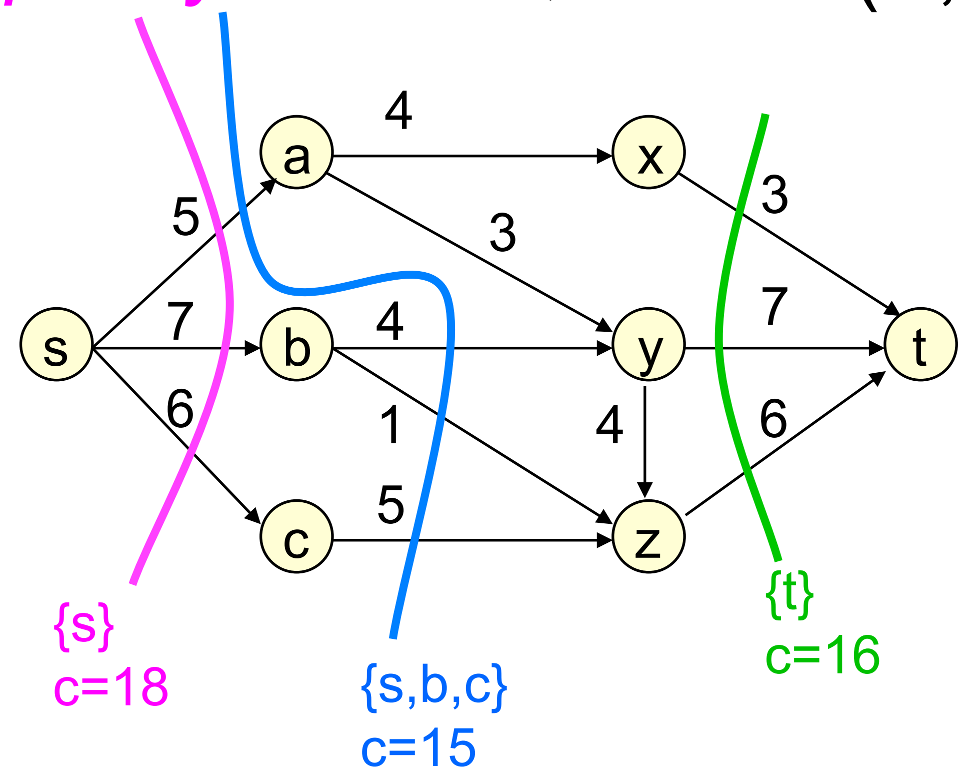
- » Does it halt?
- » Does it find a maximum flow?
- » How fast?

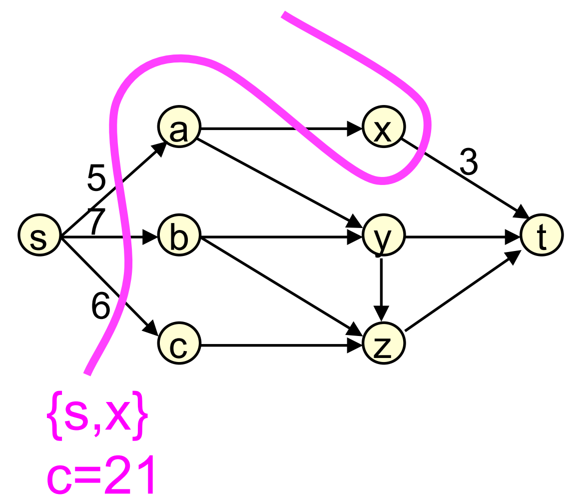
Cuts

A partition S,T of V is a *cut* if s ∈ S, t ∈ T.

Capacity of cut S,T is $$c(S,T) = \sum_{\substack{u \in S \\ v \in T}} c(u,v)$$

sum of caps of edges *<u>from</u> S <u>to</u> T*



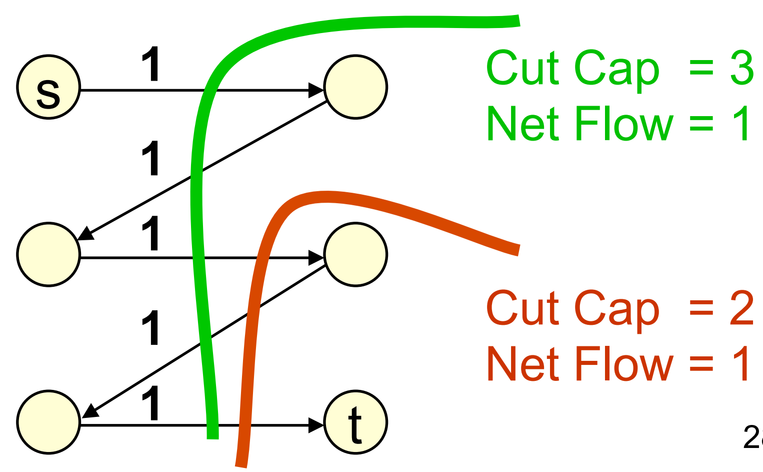


Lemma 2

For any flow f and any cut S,T,

- the net flow across the cut equals the total flow, i.e., $|f| = f(S,T)$, and
- the net flow across the cut cannot exceed the capacity of the cut, i.e. $f(S,T) \le c(S,T)$

Corollary:
Max flow ≤ Min cut



Lemma 2

For any flow f and any cut S,T,

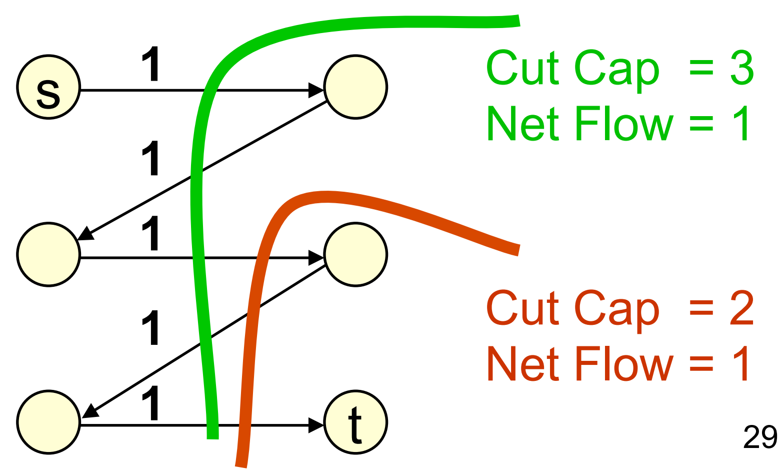
net flow across cut = total flow ≤ cut capacity

Proof:

Track a flow unit. Starts at s, ends at t.

crosses cut an odd # of times; net = 1.

Last crossing uses a forward edge totaled in C(S,T)



Max Flow / Min Cut Theorem

For any flow f, the following are equivalent

(1) $|f| = c(S,T)$ for some cut S,T (a min cut)

(2) f is a maximum flow

(3) f admits no augmenting path

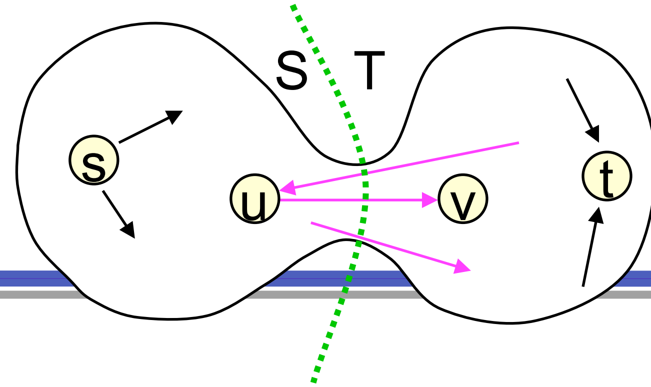
Proof:

(1) $\Rightarrow$ (2): corollary to lemma 2

(2) $\Rightarrow$ (3): contrapositive of lemma 1

$(3) \Rightarrow (1)$

(no aug) $\Rightarrow$ (cut)



Idea: where's bottleneck

$S = \{ u \mid \exists$ an augmenting path wrt f from s to u $\}$

$T = V - S$; $s \in S$, $t \in T$

For any (u,v) in $S \times T$, $\exists$ an augmenting path from s to u, but not to v.

$\therefore$ (u,v) has 0 residual capacity:

$(u,v) \in E \Rightarrow$ saturated	$f(u,v) = c(u,v)$
$(v,u) \in E \Rightarrow$ no flow	$f(u,v) = 0 = -f(v,u)$

This is true for every edge crossing the cut, i.e.

$$|f| = f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) =$$

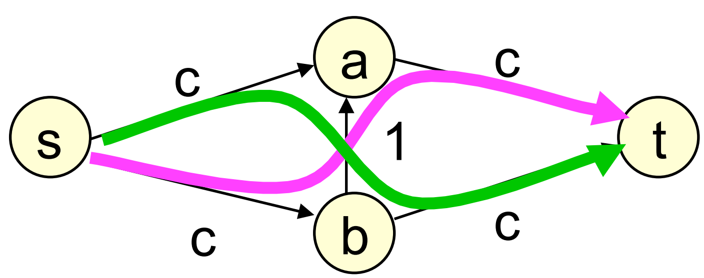
$$\sum_{u \in S, v \in T, (u,v) \in E} f(u,v) = \sum_{u \in S, v \in T, (u,v) \in E} c(u,v) = c(S,T)$$

Corollaries & Facts

If Ford-Fulkerson terminates, then it’s found a max flow.

It will terminate if c(e) integer or rational (but may not if they’re irrational).

However, may take exponential time, even with integer capacities:



c = 10^{99}, say

How to Make it Faster

Many ways. Three important ones:

"Scaling" – do big edges first; see text.
if C = max capacity, T = $O(m^2 \log C)$

Preflow-Push – see text. T = $O(n^3)$

Edmonds-Karp (next) T = $O(nm^2)$

Edmonds-Karp Algorithm

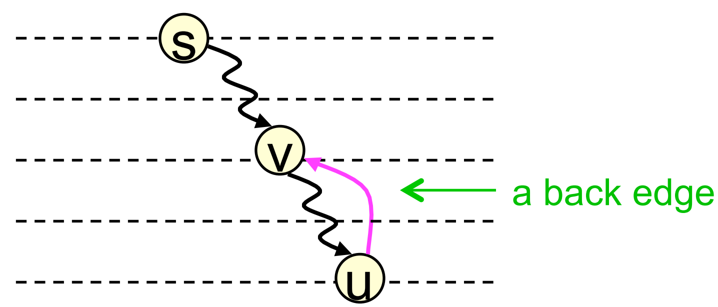
Use a shortest augmenting path
(via Breadth First Search in residual graph)

Time: $O(n\ m^2)$

BFS/Shortest Path Lemmas

Distance from s is never reduced by:

- Deleting an edge
  proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  proof: BFS is unchanged, since v visited before (u,v) examined

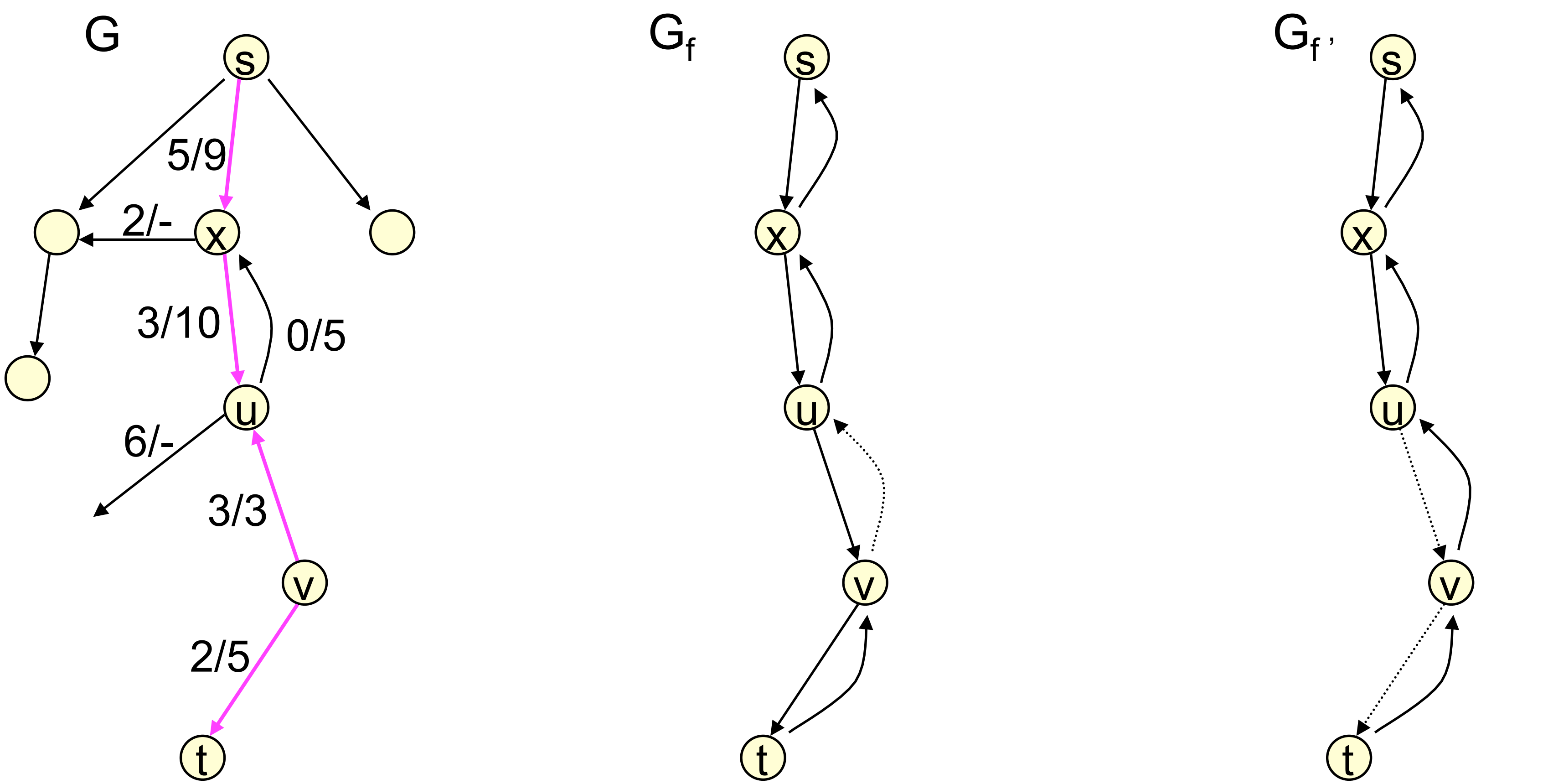


Lemma 3

Let f be a flow, G_f the residual graph, and p a shortest augmenting path. Then no vertex is closer to s after augmentation along p.

Proof: Augmentation only deletes edges, adds back edges

Augmentation vs BFS



Theorem 2

The Edmonds-Karp Algorithm performs O(mn) flow augmentations

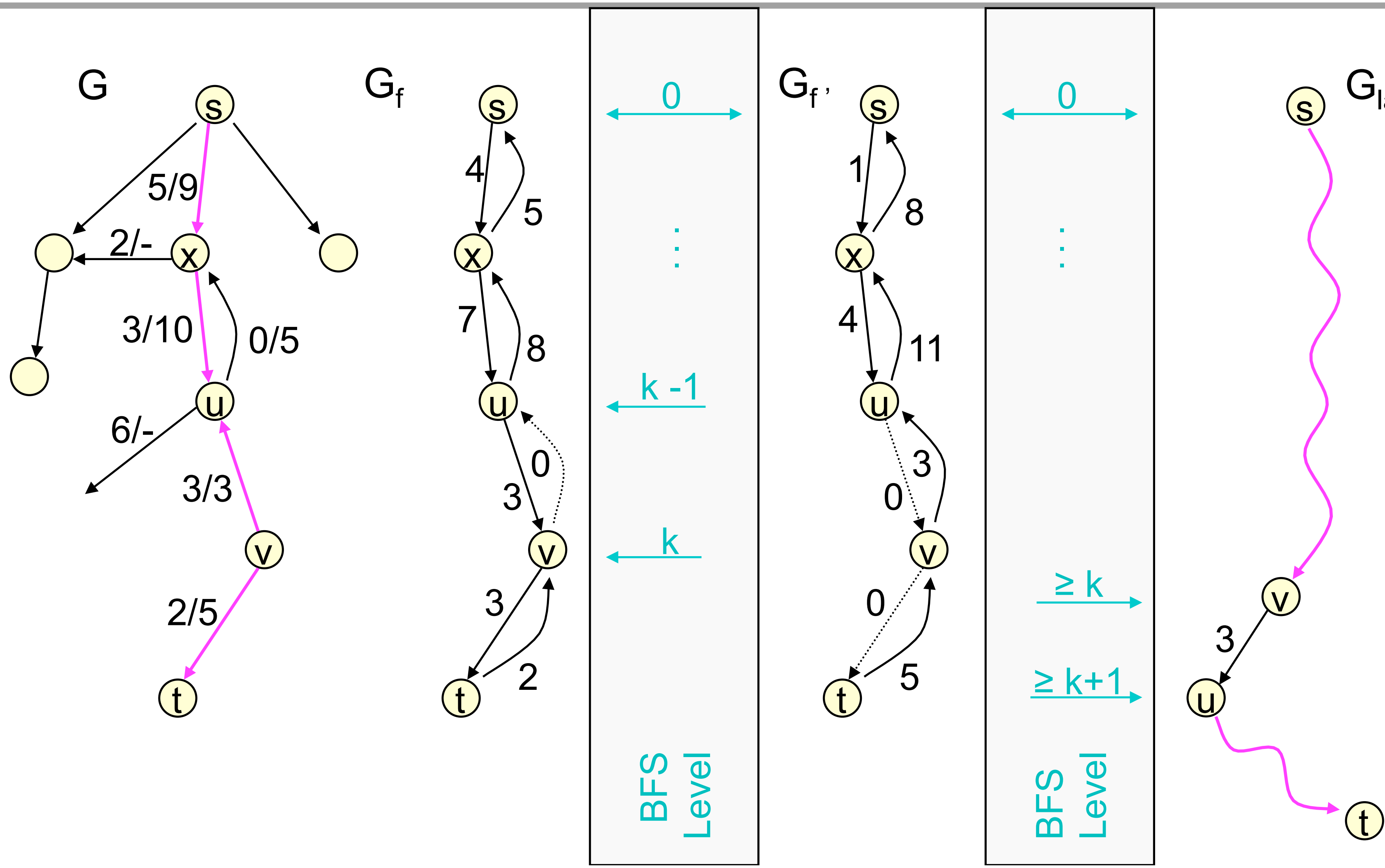
Proof:

$\{u,v\}$ is critical on augmenting path p if it's closest to s having min residual capacity.

Won't be critical again until farther from s.

So each edge critical at most n times.

Augmentation vs BFS Level



Corollary

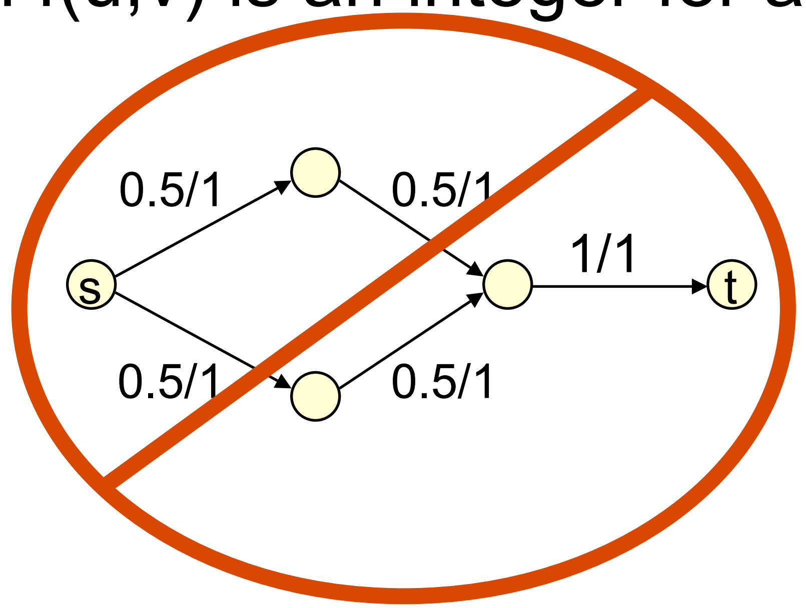
Edmonds-Karp runs in $O(nm^2)$

Flow Integrality Theorem

If all capacities are integers

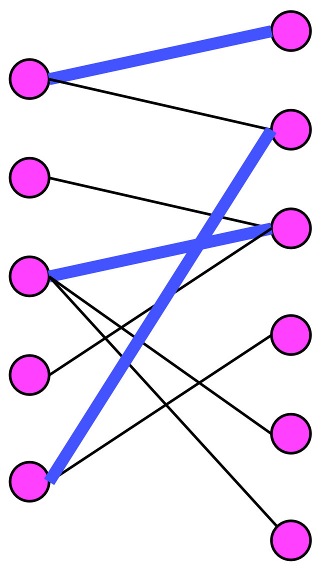
» Some max flow has an integer value

» Ford-Fulkerson method finds a max flow in which f(u,v) is an integer for all edges (u,v)



A valid flow,
but unnecessary

Bipartite Maximum Matching



Bipartite Graphs:

- $G = (V,E)$
- $V = L \cup R \quad (L \cap R = \varnothing)$
- $E \subseteq L \times R$

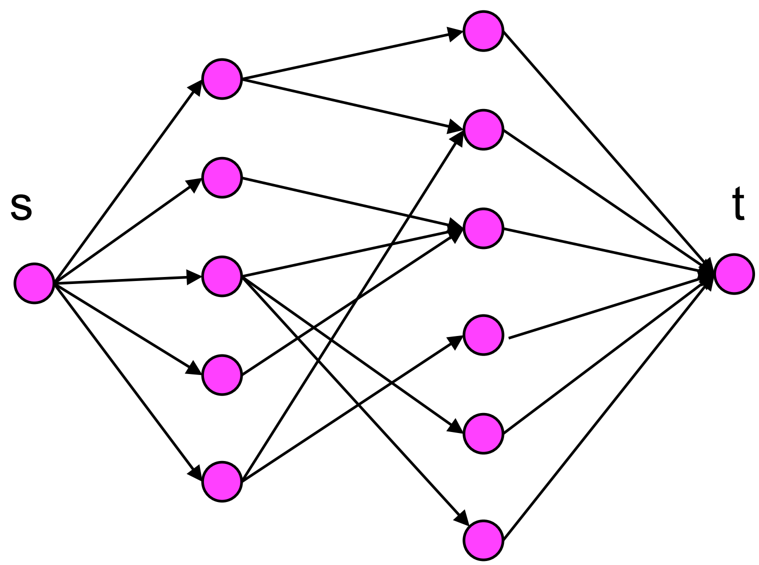
Matching:

- A set of edges $M \subseteq E$ such that no two edges touch a common vertex

Problem:

- Find a matching M of maximum size

Reducing Matching to Flow



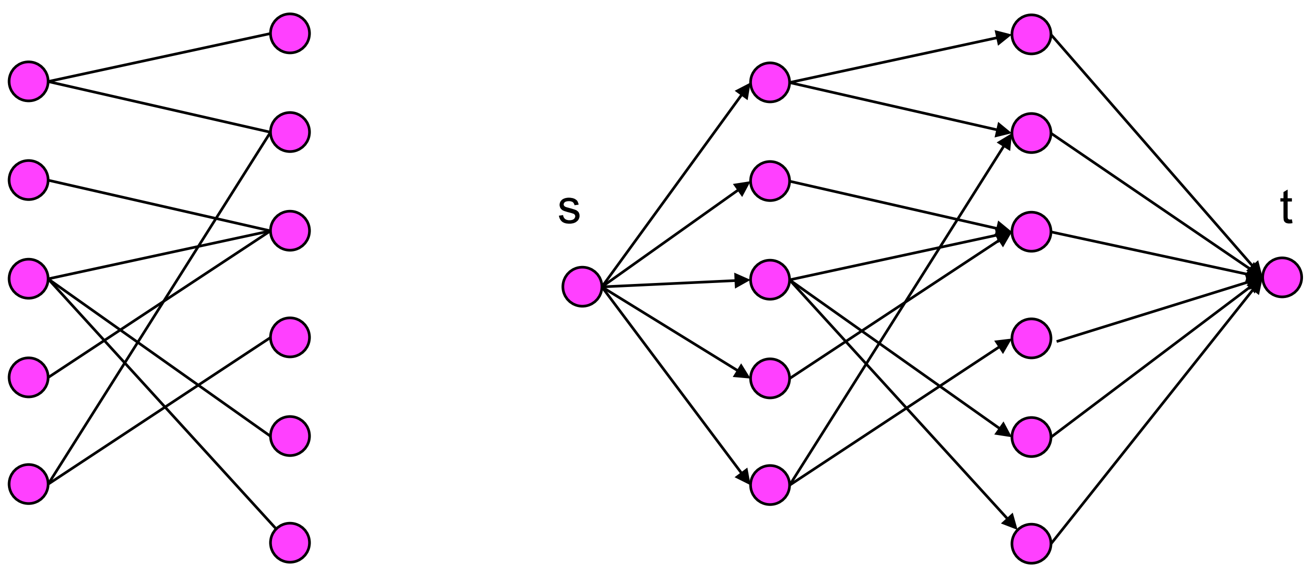
Given bipartite G, build flow network N as follows:

- Add source s, sink t
- Add edges s → L
- Add edges R→ t
- All edge capacities 1

Theorem:
Max flow iff max matching

Reducing Matching to Flow

Theorem: Max matching size = max flow value



M → f? Easy – send flow only through M

f → M? Flow integrality Thm, + cap constraints

Notes on Matching

- Max Flow Algorithm is probably overly general here
- But most direct matching algorithms use "augmenting path" type ideas similar to that in max flow – See text & homework
- Time $mn^{1/2}$ possible via Edmonds-Karp

7.12 Baseball Elimination

Some slides by Kevin Wayne

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- » Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.
- » $w_i + g_i < w_j \Rightarrow$ team i eliminated.
- » Only reason sports writers appear to be aware of.
- » Sufficient, but not necessary!

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play g_i	Against = g_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- Philly can win 83, but still eliminated . . .
- If Atlanta loses a game, then some other team wins one.

Remark. Depends on *both* how many games already won and left to play, *and* on whom they're against.

Baseball Elimination

Baseball elimination problem.

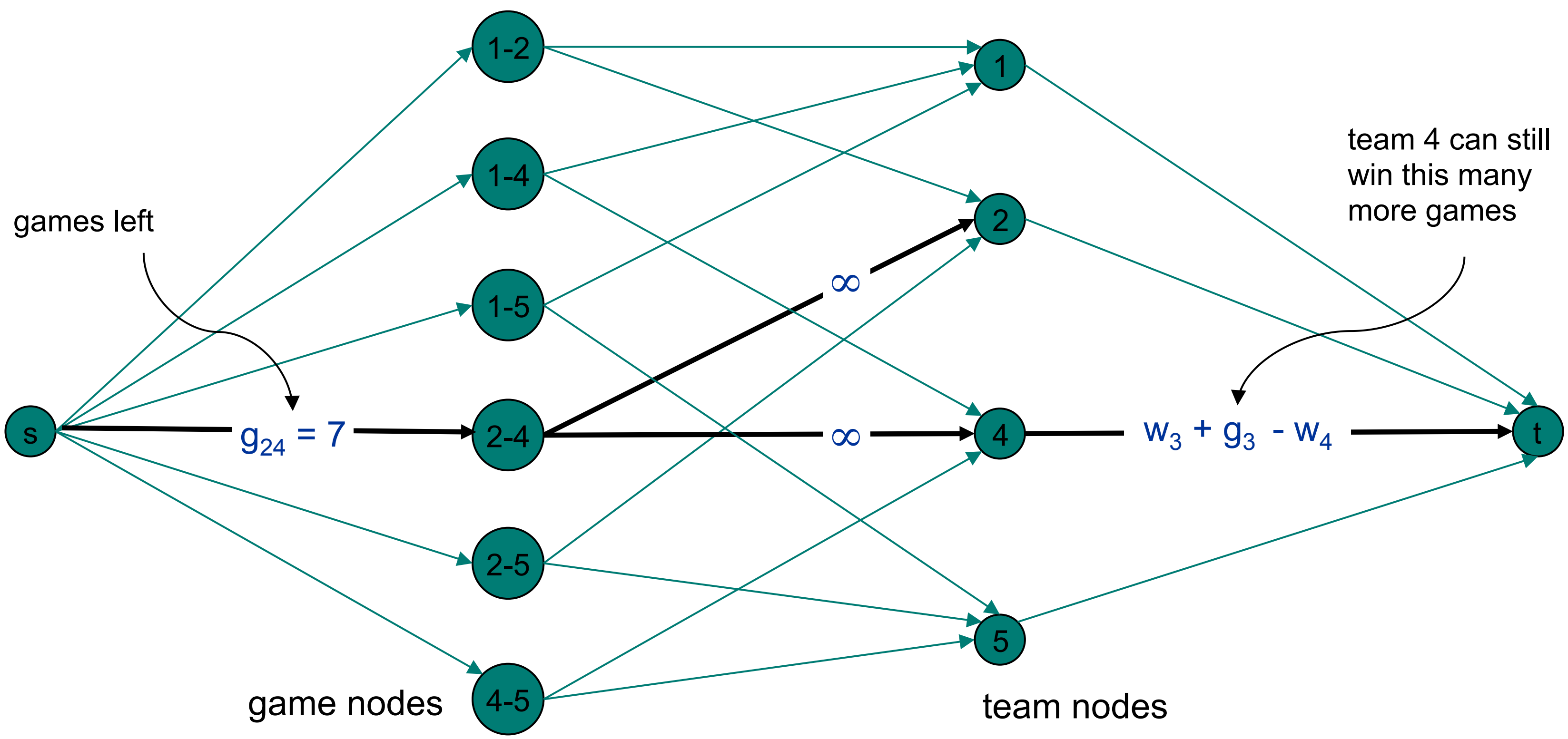
- Set of teams S.
- Distinguished team $s \in S$.
- Team x has won w_x games already.
- Teams x and y play each other g_{xy} additional times.
- Is there any outcome of the remaining games in which team s finishes with the most (or tied for the most) wins?

Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

Assume team 3 wins all remaining games $\Rightarrow$ $w_3 + g_3$ wins.

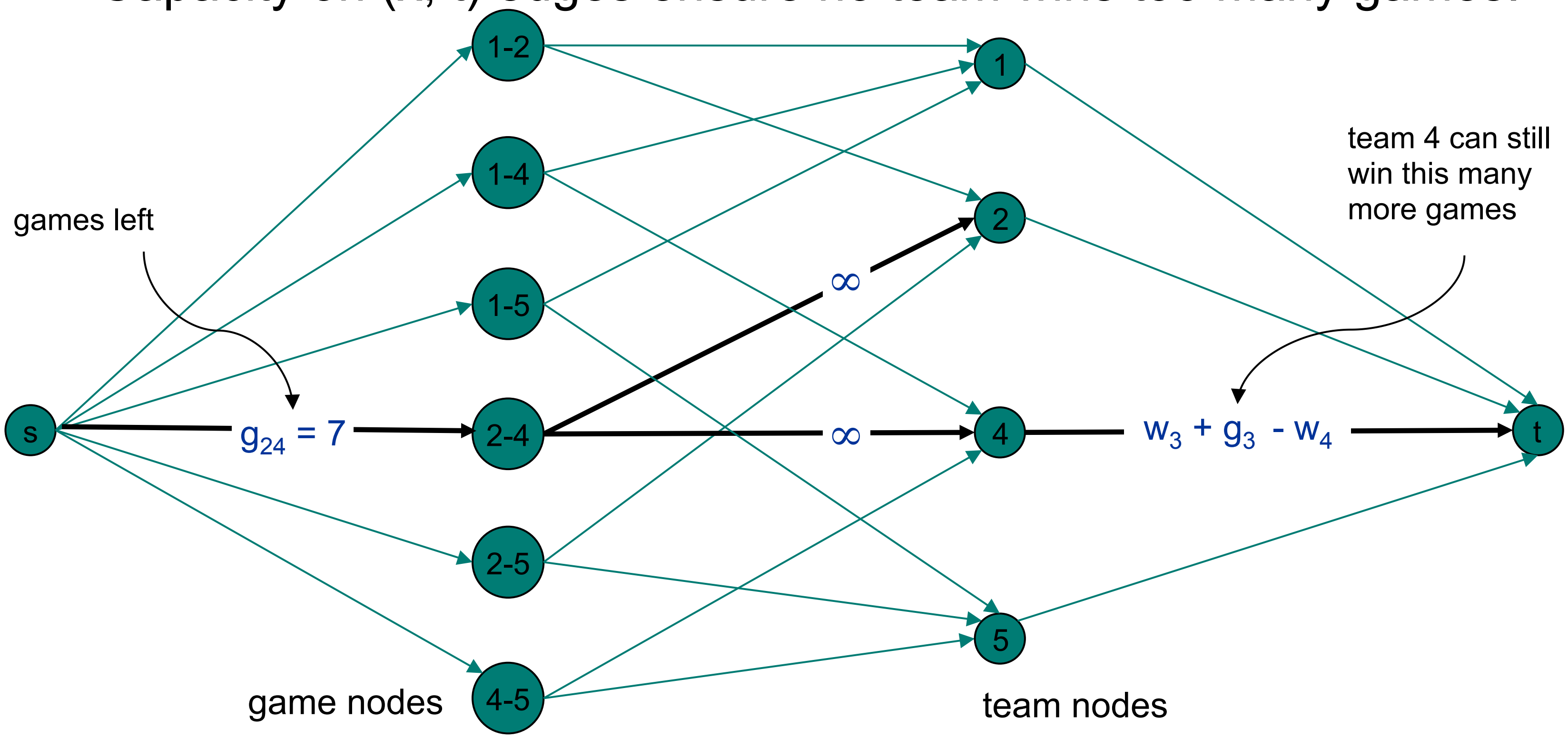
Divvy remaining games so that all teams have $\leq$ $w_3 + g_3$ wins.



Baseball Elimination: Max Flow Formulation

Theorem. Team 3 is not eliminated iff max flow saturates all edges leaving source.

- Integrality $\Rightarrow$ each remaining x-y game added to # wins for x or y.
- Capacity on (x, t) edges ensure no team wins too many games.



Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against $= g_{ij}$				
				NY	Bal	Bos	Tor	Det
NY	75	59	28	-	3	8	7	3
Baltimore	71	63	28	3	-	2	7	4
Boston	69	66	27	8	2	-	0	0
Toronto	63	72	27	7	7	0	-	-
Detroit	49	86	27	3	4	0	0	-

AL East: August 30, 1996

Which teams have a chance of finishing the season with most wins?

Detroit could finish season with 49 + 27 = 76 wins.

Baseball Elimination: Explanation for Sports Writers

Team i	Wins w_i	Losses l_i	To play g_i	Against $=g_{ij}$				
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AL East: August 30, 1996

Which teams could finish the season with most wins?

- Detroit could finish season with 49 + 27 = 76 wins.

Certificate of elimination. R = {NY, Bal, Bos, Tor}

- Have already won w(R) = 278 games.
- Must win at least r(R) = 27 more.
- Average team in R wins at least 305/4 > 76 games.

Baseball Elimination: Explanation for Sports Writers

Certificate of elimination

$$T \subseteq S, \quad w(T) := \overbrace{\sum_{i \in T} w_i}^{\#\text{ wins}}, \quad g(T) := \overbrace{\sum_{\{x,y\} \subseteq T} g_{xy}}^{\#\text{ remaining games}},$$

If $\overbrace{\dfrac{w(T)+g(T)}{|T|}}^{\text{LB on avg \# games won}} > w_z + g_z$ then z eliminated (by subset T).

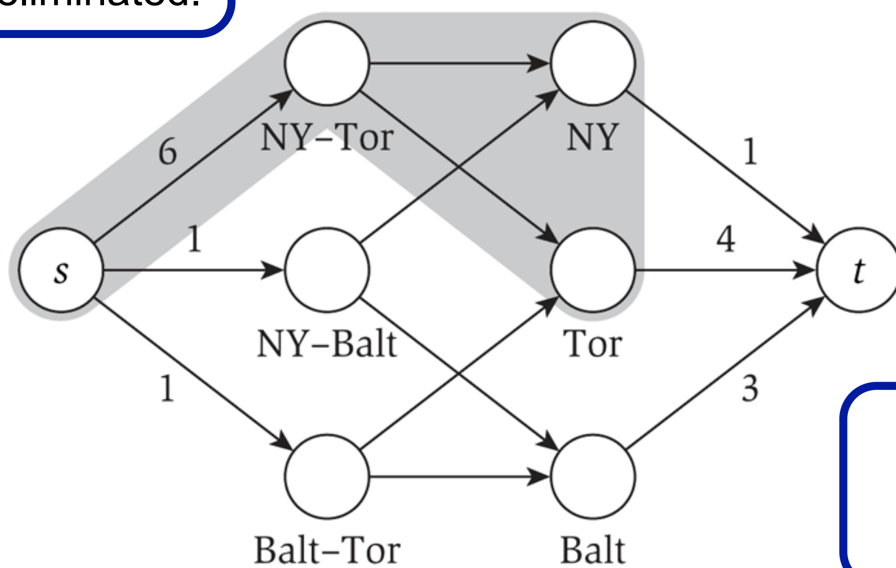
Theorem. [Hoffman-Rivlin 1967] Team z is eliminated iff there exists a subset T* that eliminates z.

Proof idea. Let T* = teams on source side of min cut.

	w	l	g	NY	Balt	Tor	Bos
NY	90		11	-	1	6	4
Baltimore	88		6	1	-	1	4
Toronto	87		10	6	1	-	4
Boston	79		12	4	4	4	-

g* = 1+6+1 = 8

(90 + 87 + 6) / 2 > 91,
so the set T = {NY, Tor}
proves Boston is eliminated.



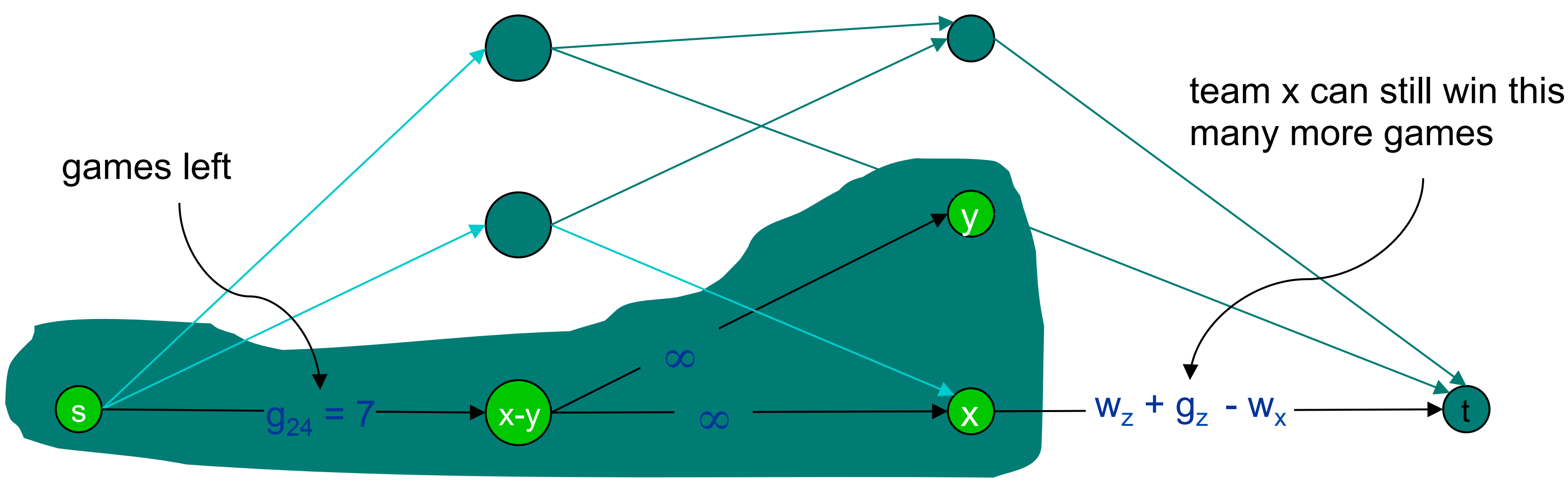
Note: T={NY,Tor, Balt} is
NOT a certificate, since
(90+88+87+8)/3 = 91

Fig 7.21 Min cut ⇒ no flow of value g*, so Boston eliminated.

Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

- Use max flow formulation, and consider min cut (A, B).
- Define T* = team nodes on source side of min cut.
- Observe x-y $\in$ A iff both x $\in$ T* and y $\in$ T*.
  - infinite capacity edges ensure if x-y $\in$ A then x $\in$ A and y $\in$ A
  - if x $\in$ A and y $\in$ A but x-y $\notin$ T*, then adding x-y to A decreases capacity of cut



Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

- Use max flow formulation, and consider min cut (A, B).
- Define T* = team nodes on source side of min cut.
- Observe x-y ∈ A iff both x ∈ T* and y ∈ T*.

$$
\begin{aligned}
g(S-\{z\}) \quad &> \quad cap(A, B) \\
&= \overbrace{g(S-\{z\}) - g(T^*)}^{\text{capacity of game edges leaving A}} + \overbrace{\sum_{x \in T^*} (w_z + g_z - w_x)}^{\text{capacity of team edges leaving A}} \\
&= g(S-\{z\}) - g(T^*) - w(T^*) + |T^*|(w_z + g_z)
\end{aligned}
$$

Rearranging:

$$
w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}
$$

Matching & Baseball: Key Points

Can (sometimes) take problems that seemingly have *nothing* to do with flow & reduce them to a flow problem

How? Build a clever network; map allocation of stuff in original problem (match edges; wins) to allocation of flow in network. Clever edge capacities constrain solution to mimic original problem in some way. Integrality useful.

Matching & Baseball: Key Points

Furthermore, in the baseball example, min cut can be translated into a succinct *certificate* or *proof* of some property that is much more transparent than “see, I ran max-flow and it says flow must be less than g*”.

These examples suggest why max flow is so important – *it’s a very general tool used in many other algorithms*.