CSE 421: Intro Algorithms

Dynamic Programming, I
Intro: Fibonacci & Stamps

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Dynamic Programming

Outline:

- General Principles
- Easy Examples – Fibonacci, Licking Stamps
- Meatier examples
  - Weighted interval scheduling
  - String Alignment
  - RNA Structure prediction
  - Maybe others

Some Algorithm Design Techniques, I: Greedy

Greedy algorithms

- Usually builds something a piece at a time
- Repeatedly make the greedy choice - the one that looks the best right away
  - e.g. closest pair in TSP search
- Usually simple, fast if they work (but often don't)

Some Algorithm Design Techniques, II: D & C

Divide & Conquer

- Reduce problem to one or more sub-problems of the same type, i.e., a recursive solution
- Typically, sub-problems are disjoint, and at most a constant fraction of the size of the original
  - e.g. Mergesort, Quicksort, Binary Search, Karatsuba
- Typically, speeds up a polynomial time algorithm

Some Algorithm Design Techniques, III: DP

Dynamic Programming

- Reduce problem to one or more sub-problems of the same type, i.e., a recursive solution
- Useful when the same sub-problems show up repeatedly in the solution
- Sometimes gives exponential speedups

“Dynamic Programming”

Program — A plan or procedure for dealing with some matter

– Webster’s New World Dictionary

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - "it's impossible to use dynamic in a pejorative sense"
  - "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

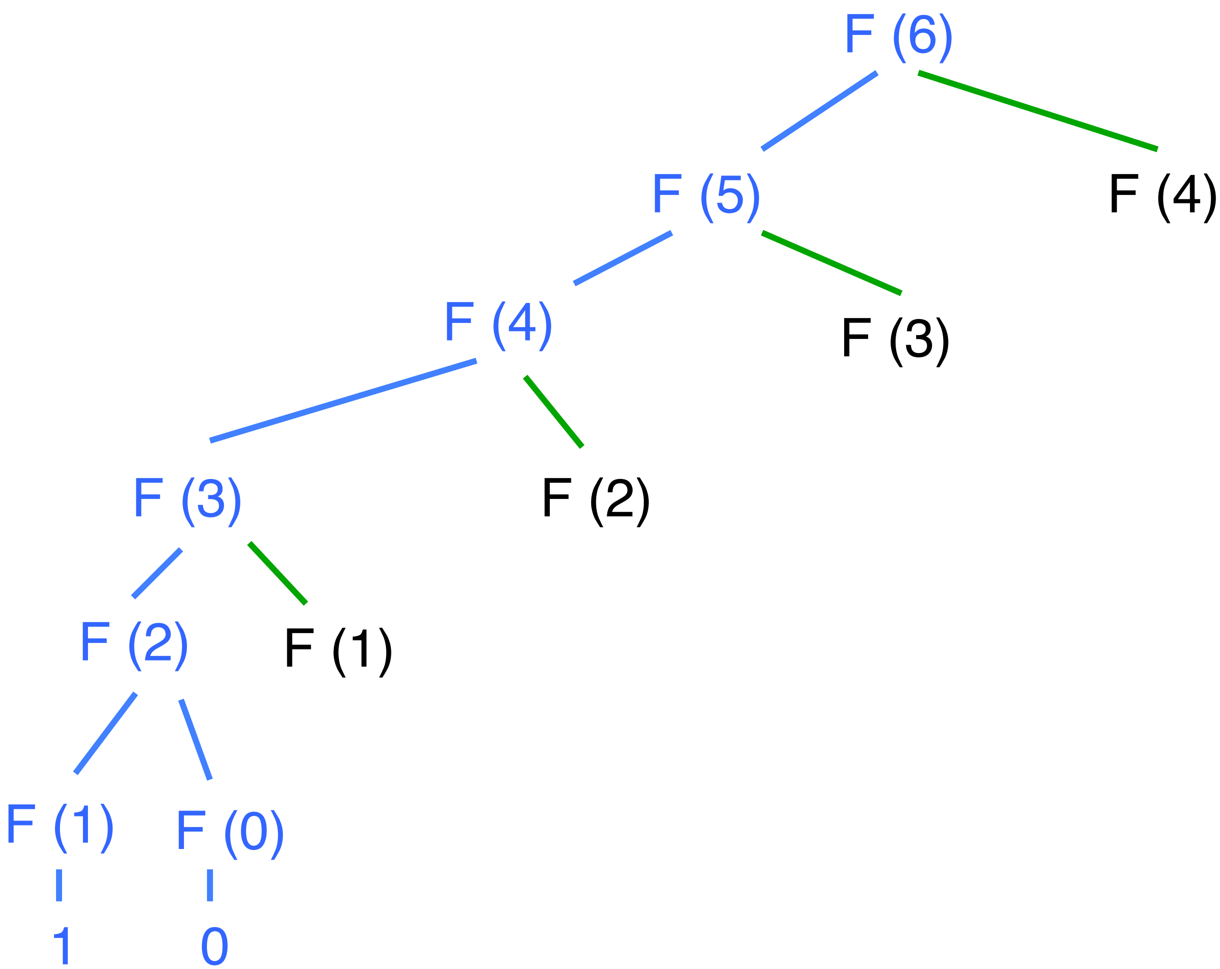
A very simple case: Computing Fibonacci Numbers

Recall $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$

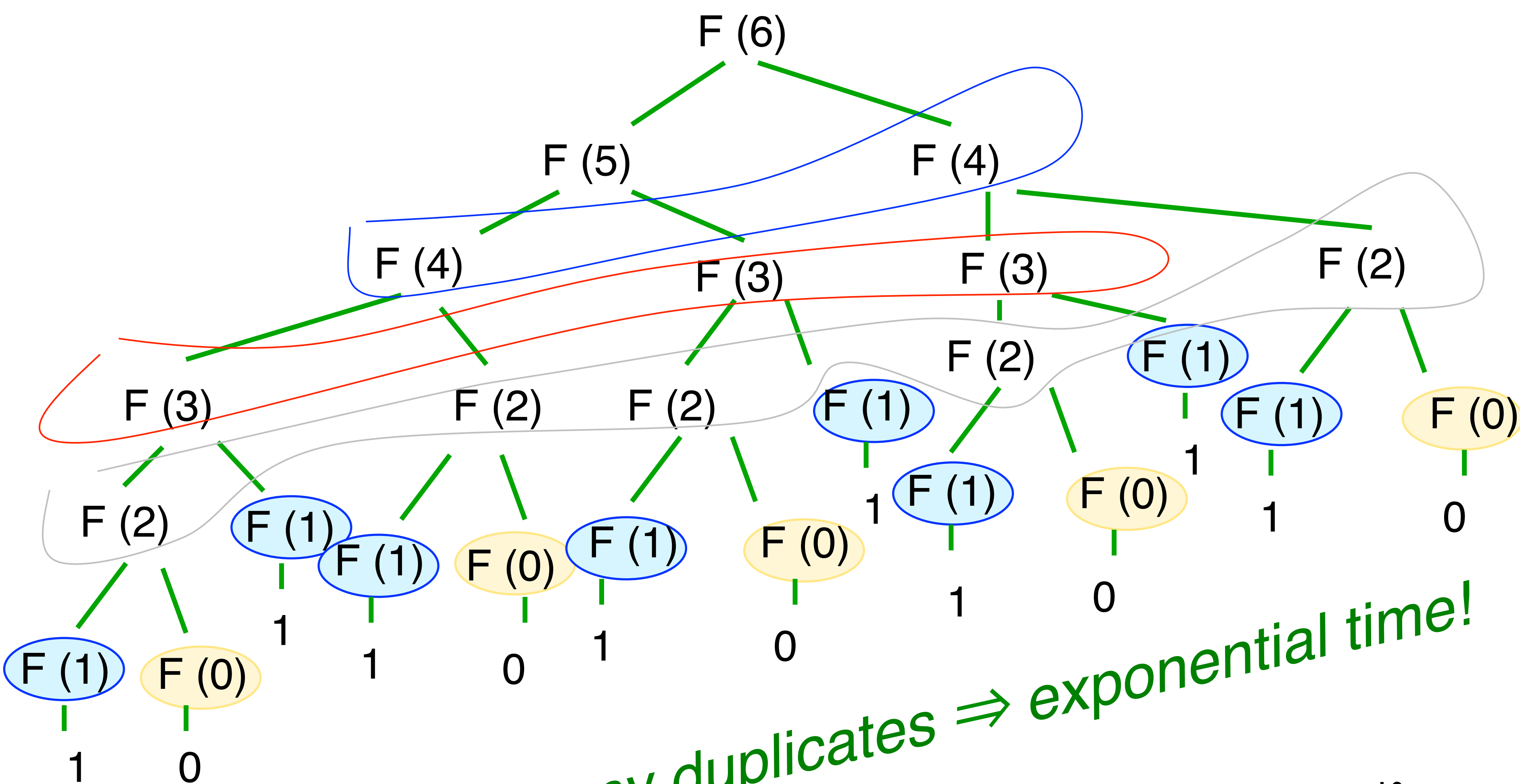
Recursive algorithm:

```
Fibo(n)
    if n=0 then return(0)
    else if n=1 then return(1)
    else return(Fibo(n-1)+Fibo(n-2))
```

Call tree - start



Full call tree



many duplicates ⇒ exponential time!

Two Alternative Fixes

- Memoization ("Caching")
  - Compute on demand, but don't re-compute:
    - Save answers from all recursive calls
    - Before a call, test whether answer saved
- Dynamic Programming (not memoized)
  - Pre-compute, don't re-compute:
    - Recursion become iteration (top-down → bottom-up)
    - Anticipate and pre-compute needed values
- DP usually cleaner, faster, simpler data structs

Fibonacci - Memoized Version

initialize: F[i] ← undefined for all i > 1

F[0] ← 0

F[1] ← 1

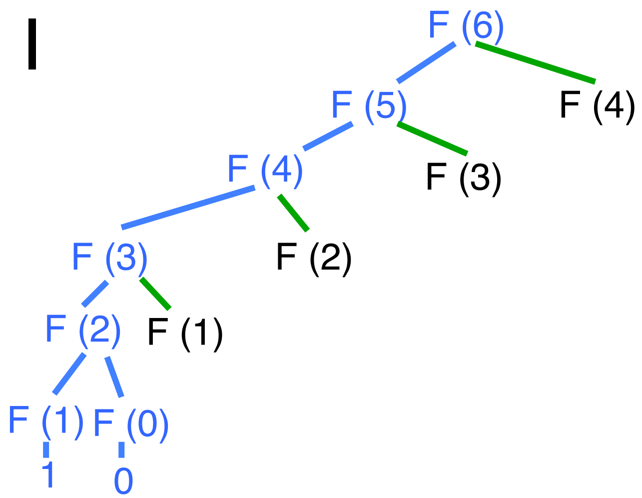
FiboMemo(n):

    if(F[n] undefined) {

        F[n] ← FiboMemo(n-2)+FiboMemo(n-1)

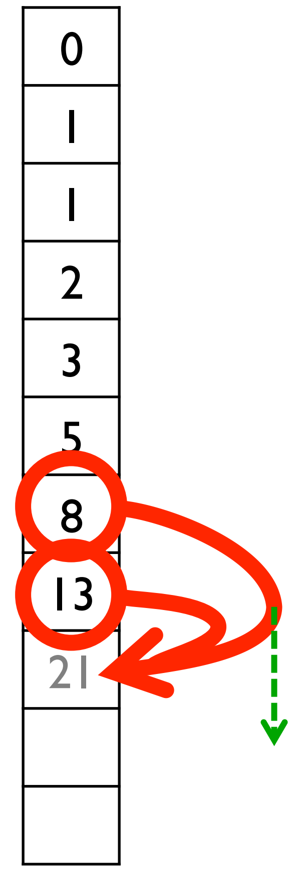
    }

    return(F[n])



Fibonacci - Dynamic Programming Version

```
FiboDP(n):
    F[0] ← 0
    F[1] ← 1
    for 1 = 2 to n do
        F[i]  ← F[i-1]+F[i-2]
    end
    return(F[n])
```



For this problem, keeping only last 2 entries instead of full array suffices, but about the same speed

Dynamic Programming

Useful when

- Same recursive sub-problems occur *repeatedly*
- Parameters of these recursive calls anticipated
- The solution to whole problem can be solved without knowing the *internal* details of how the sub-problems are solved
  - "principle of optimality" – more below

Making change

Given:

- Large supply of 1¢, 5¢, 10¢, 25¢, 50¢ coins
- An amount N

Problem: choose fewest coins totaling N

Cashier's (greedy) algorithm works:

- Give as many as possible of the next biggest denomination

Licking Stamps

Given:

- Large supply of 5¢, 4¢, and 1¢ stamps
- An amount N

Problem: choose fewest stamps totaling N

How to Lick 27¢

# of 5¢ stamps	# of 4 ¢ stamps	# of 1¢ stamps	total number
5	0	2	7
4	1	3	8
3	3	0	6

Morals: Greed doesn't pay; success of "cashier's alg" depends on coin denominations

A Simple Algorithm

At most N stamps needed, etc.

```
for a = 0, …, N {
   for b = 0, …, N {
      for c = 0, …, N {
         if (5a+4b+c == N && a+b+c is new min)
            {retain (a,b,c);}}}
output retained triple;
```

Time: $O(N^3)$

(Not too hard to see some optimizations, but we're after bigger fish…)

Better Idea

Theorem: If last stamp in an opt sol has value v, then previous stamps are *opt sol for N-v.*

Proof: if not, we could improve the solution for N by using opt for N-v.

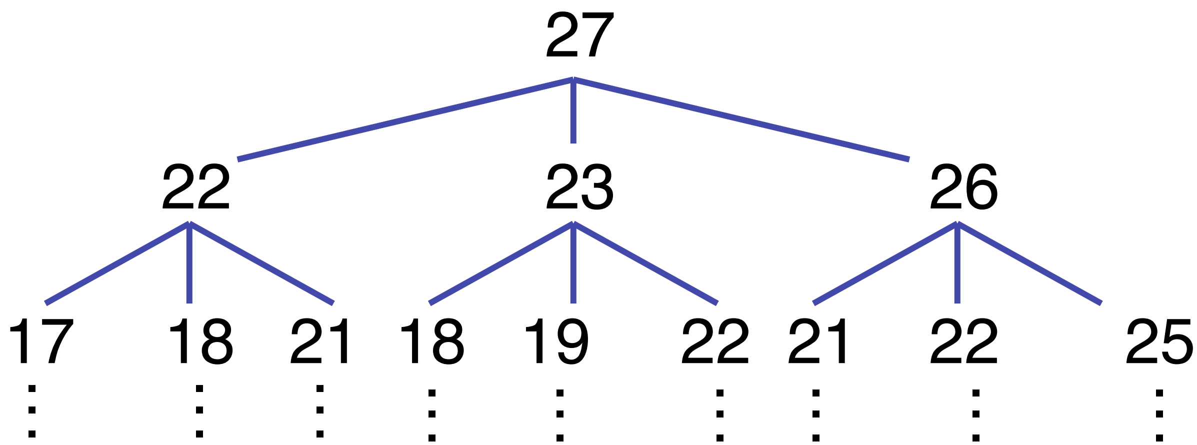
Alg: for i = 1 to n:

$$M(i) = \min \begin{Bmatrix} 0 & i=0 \\ 1+M(i-5) & i \geq 5 \\ 1+M(i-4) & i \geq 4 \\ 1+M(i-1) & i \geq 1 \end{Bmatrix}$$

where *M(i)* = min number of stamps totaling *i*¢

New Idea: Recursion

$$M(i) = \min \begin{cases} 0 & i = 0 \\ 1 + M(i-5) & i \geq 5 \\ 1 + M(i-4) & i \geq 4 \\ 1 + M(i-1) & i \geq 1 \end{cases}$$



Time: > $3^{N/5}$

Another New Idea: Avoid Recomputation

Tabulate values of solved subproblems

- Top-down: "memoization"
- Bottom up (better):

for i = 0, …, N do $M[i] = \min \begin{Bmatrix} 0 & i=0 \\ 1+M[i-5] & i \geq 5 \\ 1+M[i-4] & i \geq 4 \\ 1+M[i-1] & i \geq 1 \end{Bmatrix}$

Time: O(N)

Finding *How Many* Stamps

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

1+Min(3,1,3) = 2

Finding *Which* Stamps: Trace-Back

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

4¢

1+Min(3,**1**,3) = **2**

Trace-Back

Way 1: tabulate all

add data structure storing back-pointers indicating which predecessor gave the min. (more space, maybe less time)

Way 2: re-compute just what's needed

```
TraceBack(i):
  if i == 0 then return;
  for d in {1, 4, 5} do
    if M[i] == 1 + M[i - d]
      then break;
  print d;
  TraceBack(i - d);
```

$$M[i] = \min \left\{ \begin{array}{ll} 0 & i=0 \\ 1+M[i-5] & i\geq 5 \\ 1+M[i-4] & i\geq 4 \\ 1+M[i-1] & i\geq 1 \end{array} \right\}$$

Complexity Note

O(N) is better than $O(N^3)$ or $O(3^{N/5})$

But still *exponential* in input size (log N bits)

(E.g., miserable if N is 64 bits – $c \cdot 2^{64}$ steps & 2^{64} memory.)

Note: can do in O(1) for fixed denominations, e.g., 5¢, 4¢, and 1¢ (how?) but not in general. See "NP-Completeness" later.

Elements of Dynamic Programming

What feature did we use?

What should we look for to use again?

“Optimal Substructure”

- Optimal solution contains optimal subproblems
- A non-example: min (number of stamps mod 2)

“Repeated Subproblems”

- The same subproblems arise in various ways