CSE 421: Introduction to Algorithms

Graph Traversal

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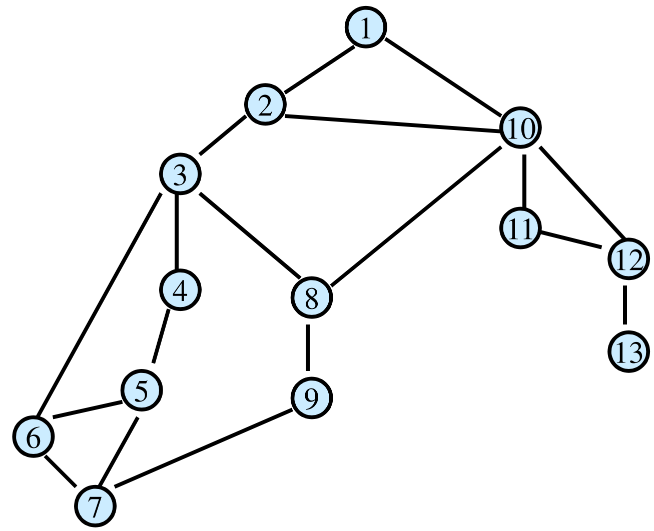
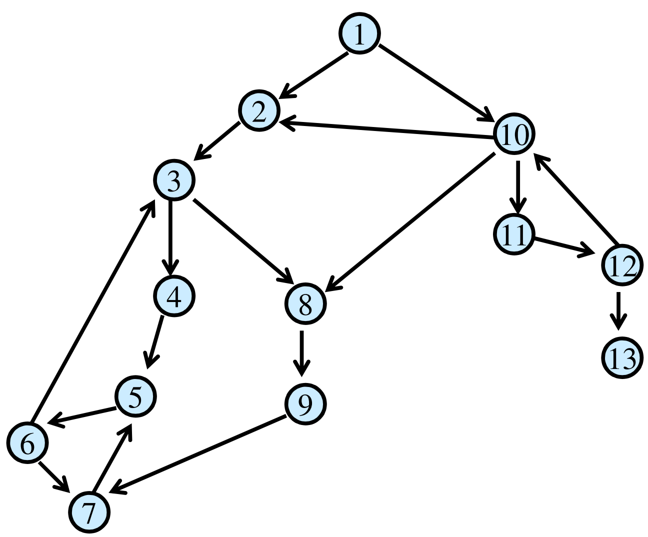
Undirected Graph G = (V,E)



Directed Graph G = (V,E)



Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex s to find all vertices reachable from s
- Three states of vertices
  - **unvisited**
  - **visited/discovered**
  - **fully-explored**

Generic Graph Traversal Algorithm

Find: set **R** of vertices reachable from **s**$\in$**V**

Reachable(**s**):

R$\leftarrow$ {**s**}

While there is a (**u,v**)$\in$**E** where **u**$\in$**R** and **v**$\notin$**R**

Add **v** to **R**

Generic Traversal Always Works

- Claim: At termination **R** is the set of nodes reachable from **s**
- Proof
  - $\subseteq$: For every node **v**$\in$**R** there is a path from s to v
  - $\supseteq$: Suppose there is a node **w**$\notin$**R** reachable from s via a path **P**
    - Take first node **v** on **P** such that **v**$\notin$**R**
    - Predecessor **u** of **v** in **P** satisfies
      - **u** $\in$ **R**
      - **(u,v)**$\in$**E**
    - But this contradicts the fact that the algorithm exited the while loop.

Breadth-First Search

- Completely explore the vertices in order of their distance from s
- Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(**s**)

mark **s** "visited"; R$\leftarrow${s}; layer $L_0\leftarrow${s}

while $\mathbf{L_i}$ not empty

$\mathbf{L_{i+1}} \leftarrow \varnothing$

For each **u**$\in$$\mathbf{L_i}$

for each edge {**u**,**v**}

if (**v** is "unvisited")

mark **v** "visited"

Add **v** to set **R** and to layer $\mathbf{L_{i+1}}$

mark **u** "fully-explored"

i $\leftarrow$ **i+1**

Properties of BFS(v)

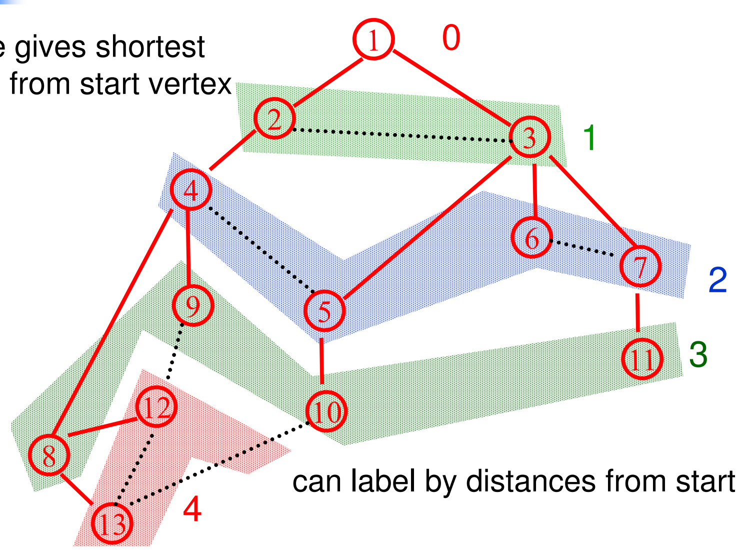
- BFS(s) visits x if and only if there is a path in G from s to x.
- Edges followed to undiscovered vertices define a "breadth first spanning tree" of G
- Layer i in this tree, L_i
  - those vertices u such that the shortest path in G from the root s is of length i.
- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

Properties of BFS

- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers
  - Suppose not
    - Then there would be vertices (x,y) such that $x \in L_i$ and $y \in L_j$ and $j>i+1$
    - Then, when vertices incident to x are considered in BFS y would be added to L_{i+1} and not to L_j

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - Given: vertices u and v in G
  - Is there a path from u to v?
- Idea: create array A such that
  $A[u]$ = smallest numbered vertex that is connected to u
  - question reduces to whether $A[u]=A[v]$?

Q: Why not create an array **Path**[u,v]?

Graph Search Application: Connected Components

- initial state: all **v** unvisited
  for **s**←**1** to **n** do
  if state(**s**) ≠ "fully-explored" then
  BFS(**s**): setting **A**[**u**] ←**s** for each **u** found
  (and marking **u** visited/fully-explored)
  endif
  endfor
- Total cost: **O(n+m)**
  - each vertex is touched once in this outer procedure and the edges examined in the different BFS runs are disjoint
  - works also with Depth First Search

DFS(u) – Recursive version

Global Initialization: mark all vertices "unvisited"

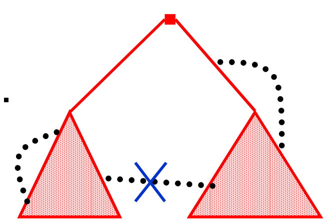
DFS(**u**)
  mark **u** "visited" and add **u** to **R**
  for each edge {**u**,**v**}
    if (**v** is "unvisited")
      DFS(**v**)
  end for
  mark **u** "fully-explored"

Properties of DFS(s)

- Like BFS(**s**):
  - DFS(**s**) visits **x** if and only if there is a path in **G** from **s** to **x**
  - Edges into undiscovered vertices define a "depth first spanning tree" of **G**
- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- No cross edges.



No cross edges in DFS on undirected graphs

- Claim: During **DFS(x)** every vertex marked visited is a descendant of **x** in the DFS tree **T**
- Claim: For every **x,y** in the DFS tree **T**, if **(x,y)** is an edge not in **T** then one of **x** or **y** is an ancestor of the other in **T**
- Proof:
  - One of **x** or **y** is visited first, suppose WLOG that **x** is visited first and therefore **DFS(x)** was called before **DFS(y)**
    - During **DFS(x)**, the edge **(x,y)** is examined
  - Since **(x,y)** is a not an edge of **T**, **y** was visited when the edge **(x,y)** was examined during **DFS(x)**
  - Therefore **y** was visited during the call to **DFS(x)** so **y** is a descendant of **x**.

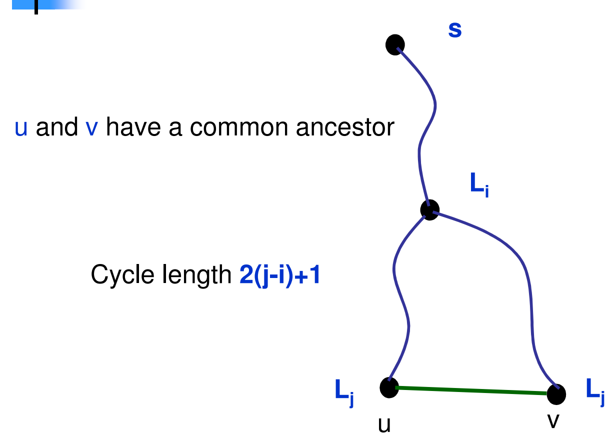
Applications of Graph Traversal: Bipartiteness Testing

- Easy: A graph G is not bipartite if it contains an odd length cycle
- WLOG: G is connected
  - Otherwise run on each component
- Simple idea: start coloring nodes starting at a given node s
  - Color s red
  - Color all neighbors of s blue
  - Color all their neighbors red
  - If you ever hit a node that was already colored
    - the same color as you want to color it, ignore it
    - the opposite color, output error

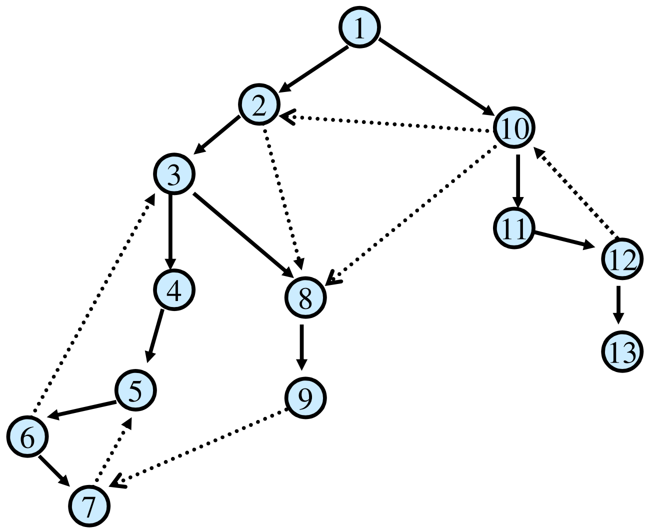
BFS gives Bipartiteness

- Run BFS assigning all vertices from layer $\mathbf{L_i}$ the color **i** mod **2**
  - i.e. red if they are in an even layer, blue if in an odd layer
- If there is an edge joining two vertices from the same layer then output “Not Bipartite”

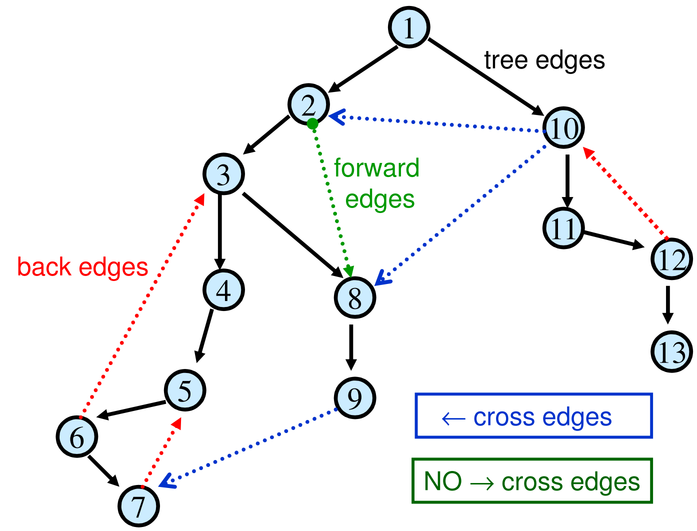
Why does it work?



DFS(v) for a directed graph



DFS(v)



Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree

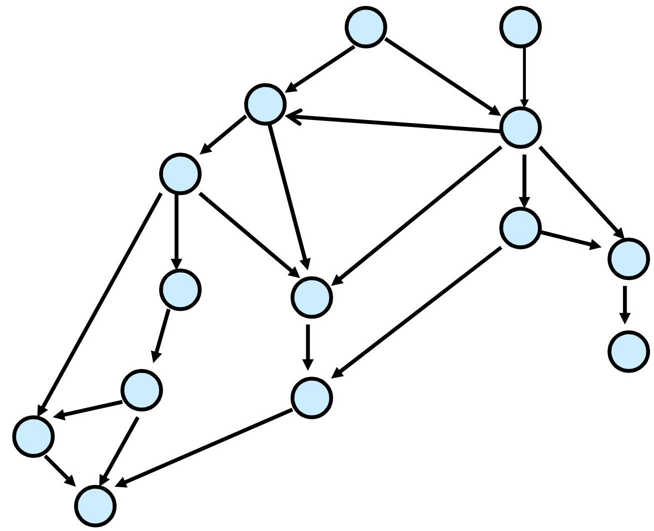
Directed Acyclic Graphs

- A directed graph G=(V,E) is acyclic if it has no directed cycles
- Terminology: A directed acyclic graph is also called a DAG

Topological Sort

- **Given:** a directed acyclic graph (DAG) **G=(V,E)**
- **Output:** numbering of the vertices of **G** with distinct numbers from **1** to **n** so edges only go from lower number to higher numbered vertices
- Applications
  - nodes represent tasks
  - edges represent precedence between tasks
  - topological sort gives a sequential schedule for solving them

Directed Acyclic Graph



In-degree 0 vertices

- Every DAG has a vertex of in-degree 0
- Proof: By contradiction
  - Suppose every vertex has some incoming edge
  - Consider following procedure:

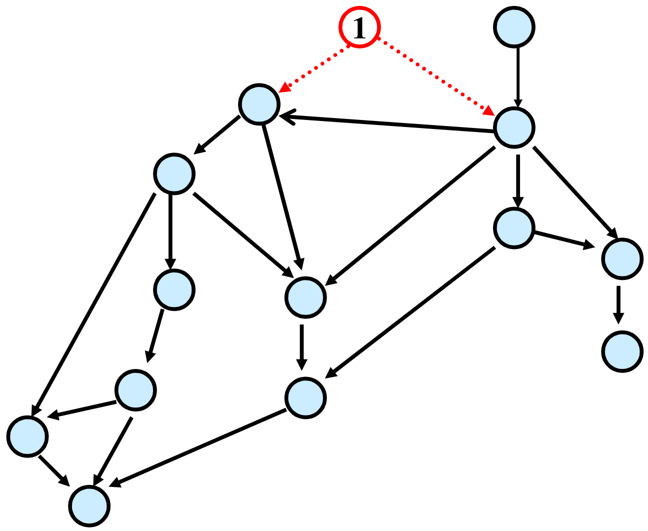
    while (true) do
      v←some predecessor of **v**

  - After **n+1** steps where **n**=|**V**| there will be a repeated vertex
    - This yields a cycle, contradicting that it is a DAG

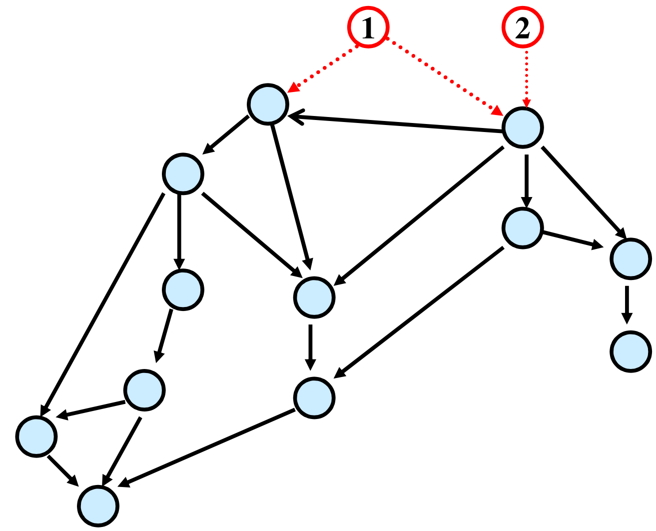
Topological Sort

- Can do using DFS
- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.

Topological Sort



Topological Sort



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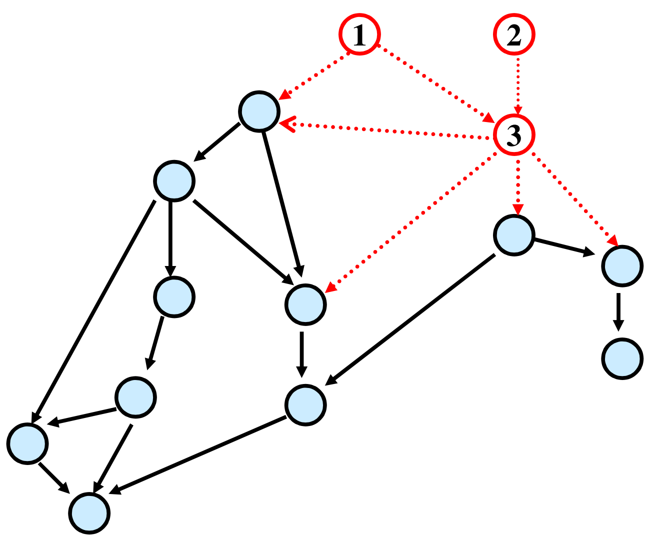

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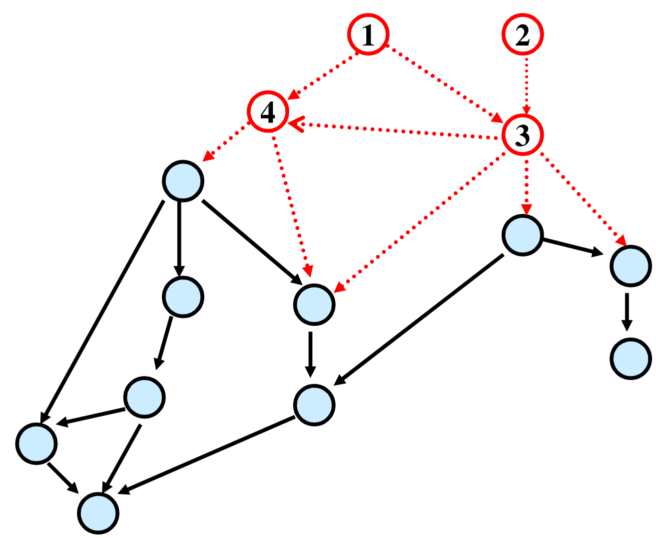

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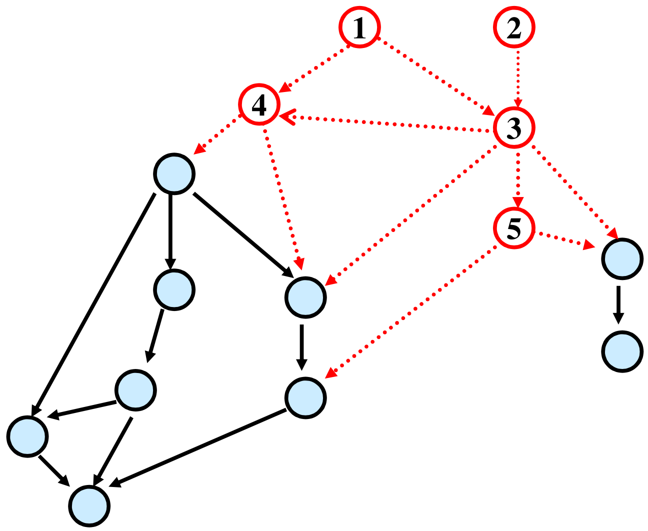

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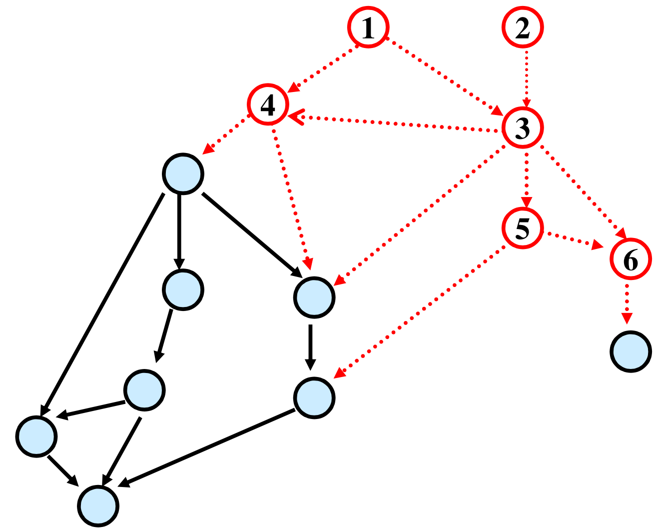

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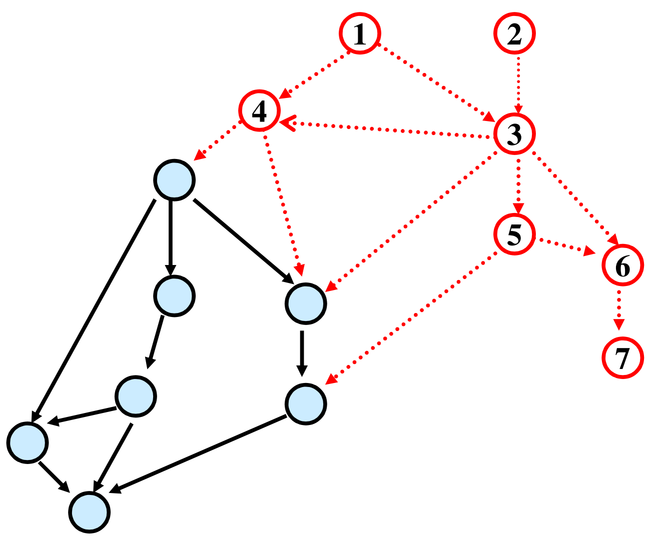

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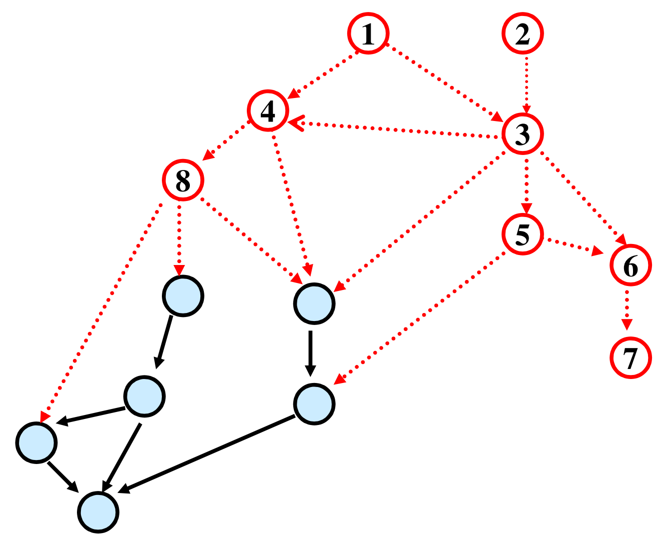

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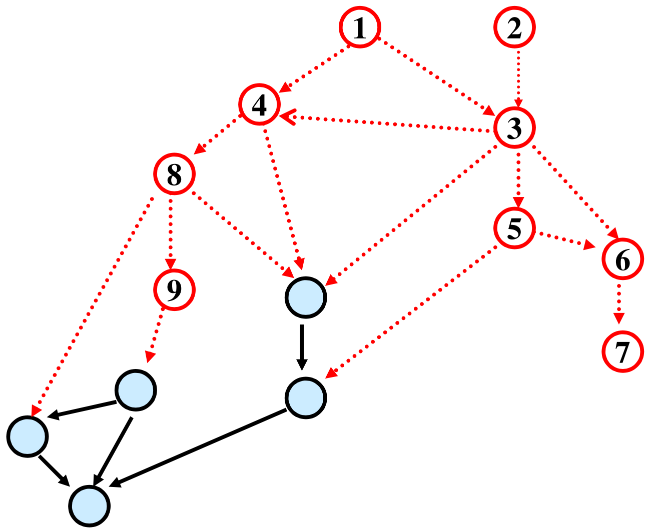

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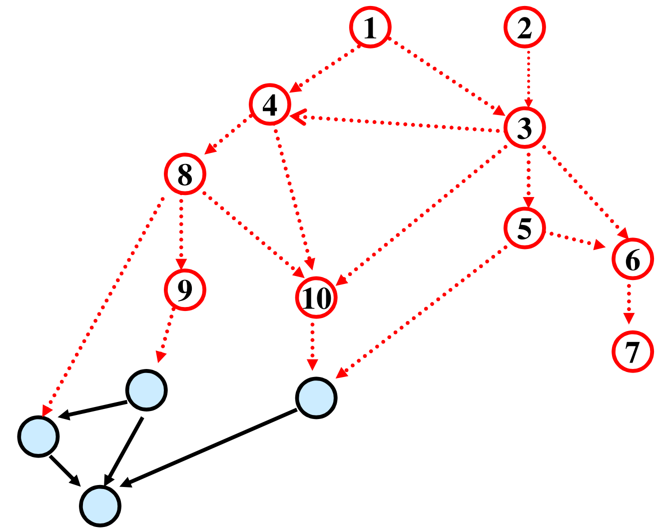

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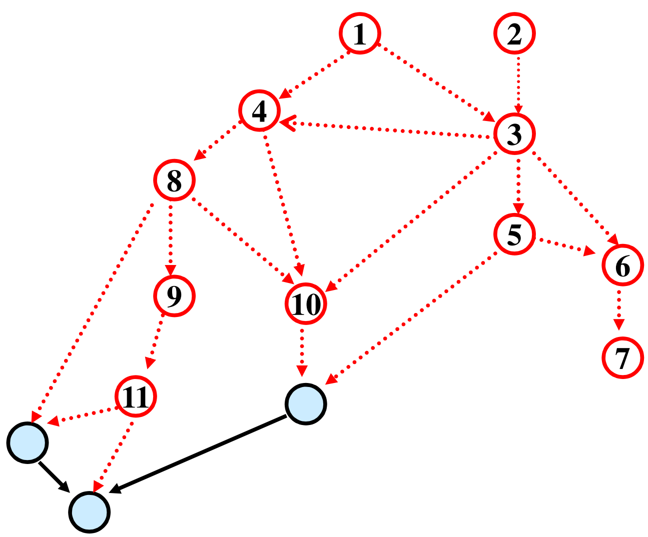

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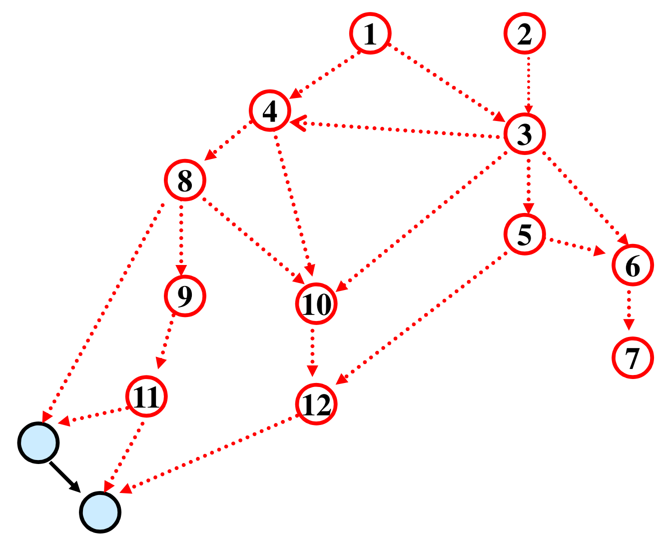

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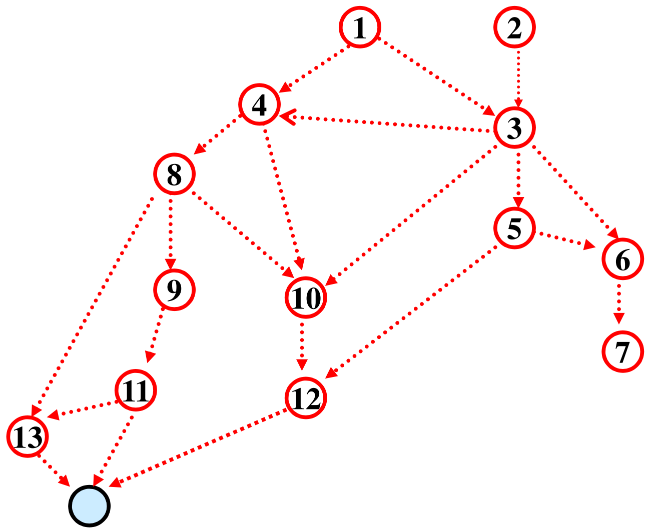

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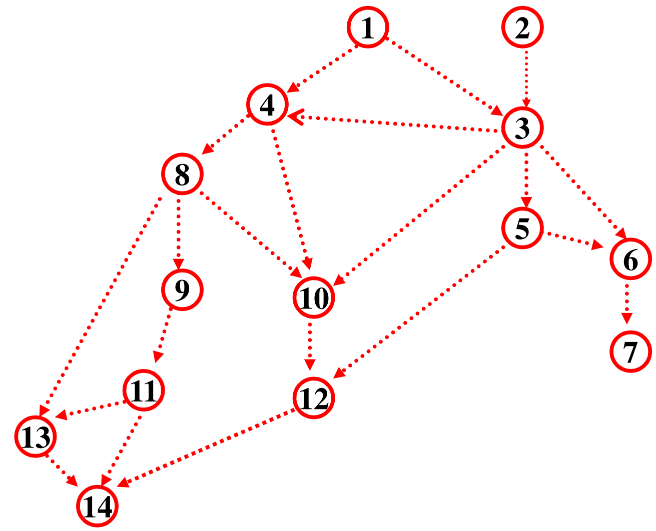

Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex **O**(**m**+**n**)
- Maintain a queue (or stack) of vertices of in-degree **0**
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by **1** and add them to the queue if their degree drops to **0**

Total cost **O**(**m**+**n**)