CSE 421: Introduction to Algorithms

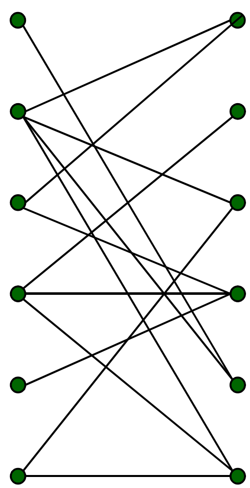
Network Flow

Paul Beame

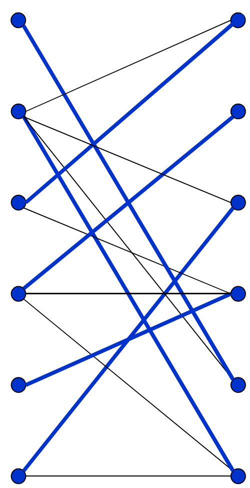
Bipartite Matching

- Given: A bipartite graph G=(V,E)
  - $M \subseteq E$ is a matching in **G** iff no two edges in **M** share a vertex
- Goal: Find a matching **M** in **G** of maximum possible size

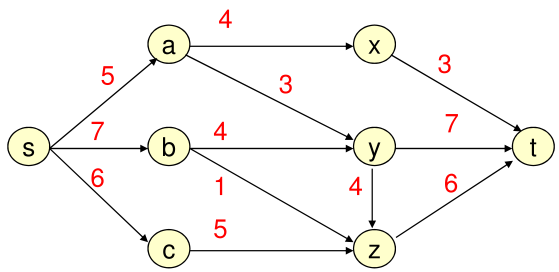
Bipartite Matching



Bipartite Matching

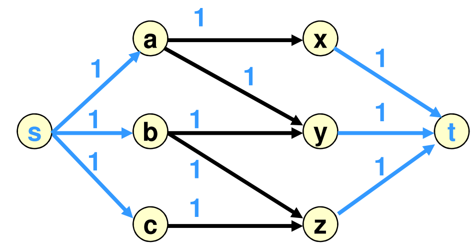


The Network Flow Problem



- How much stuff can flow from s to t?

Bipartite matching as a special case of flow



Net Flow: Formal Definition

Given:

A digraph **G** = (**V**,**E**)

Two vertices **s**,**t** in **V** (source & sink)

A *capacity* **c**(**u**,**v**) ≥ **0** for each (**u**,**v**) ∈ **E** (and **c**(**u**,**v**) = **0** for all non-edges (**u**,**v**))

Find:

A *flow function* **f**: **E** → **R** s.t., for all **u**,**v**:

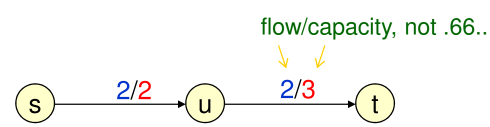
- $0 \le f(u,v) \le c(u,v)$ [Capacity Constraint]
- if $u \ne s,t$, i.e. $f^{out}(u)=f^{in}(u)$ [Flow Conservation]

Maximizing total flow $\nu(f) = f^{out}(s)$

Notation:

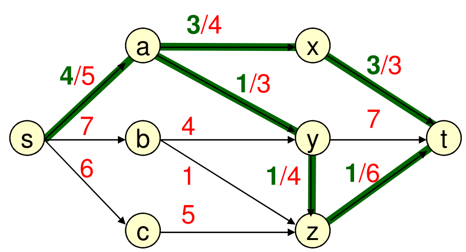
$$f^{in}(v) = \sum_{e=(u,v)\in E} f(u,v) \qquad f^{out}(v) = \sum_{e=(v,w)\in E} f(v,w)$$

Example: A Flow Function



$$f^{in}(u)=f(s,u)=2=f(u,t)=f^{out}(u)$$

Example: A Flow Function

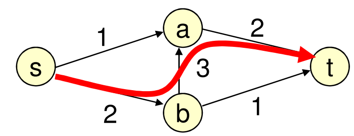


- Not shown: f(u,v) if = 0
- Note: max flow ≥ 4 since
  f is a flow function, with ν(**f**) = 4

Max Flow via a Greedy Alg?

While there is an s → t path in G
  Pick such a path, **p**
  Find **c**, the min capacity of any edge in **p**
  Subtract **c** from all capacities on **p**
  Delete edges of capacity **0**

- This does NOT always find a max flow:



If pick s →b →a →t
first, flow stuck at 2.
But flow 3 possible.

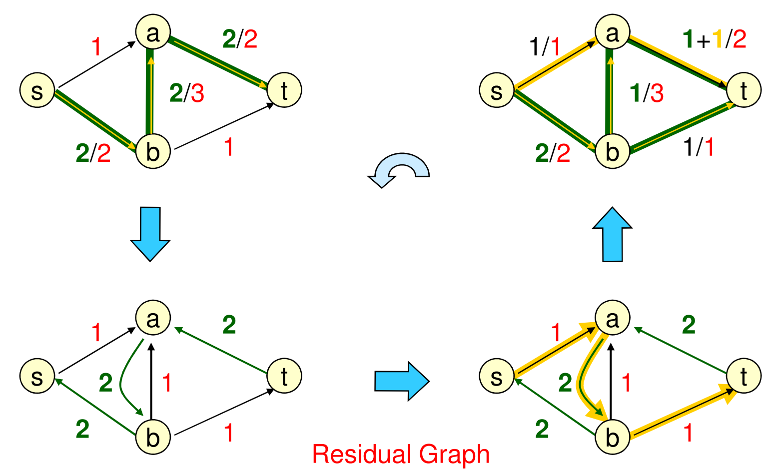
A Brief History of Flow

#	year	discoverer(s)	bound
1	1951	Dantzig	$O(n^2mU)$
2	1955	Ford & Fulkerson	$O(nmU)$
3	1970	Dinitz Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2m)$
5	1972	Edmonds & Karp Dinitz	$O(m^2 \log U)$
6	1973	Dinitz Gabow	$O(nm \log U)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2\sqrt{m})$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg & Tarjan	$O(nm \log(n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log U)$
13	1987	Ahuja et al.	$O(nm \log(n\sqrt{\log U}/(m+2))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\epsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$
19	1994	King et al.	$O(nm \log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log U)$ $O(n^{2/3}m \log(n^2/m) \log U)$
	2012	Orlin + King et al.	$O(nm)$

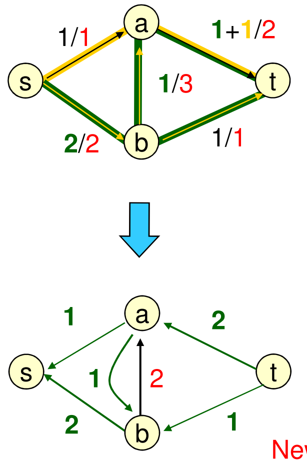
n = # of vertices
m= # of edges
U = Max capacity

Source: Goldberg & Rao, FOCS '97

Greed Revisited: Residual Graph & Augmenting Path



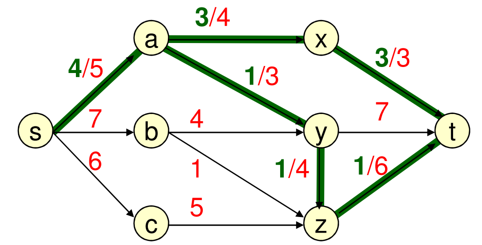
Greed Revisited: An Augmenting Path



New Residual Graph

Residual Capacity

- The *residual capacity* (w.r.t. **f**) of **(u,v)** is $c_f(u,v) = c(u,v) - f(u,v)$ if $f(u,v) \le c(u,v)$ and $c_f(u,v) = f(v,u)$ if $f(v,u) > 0$

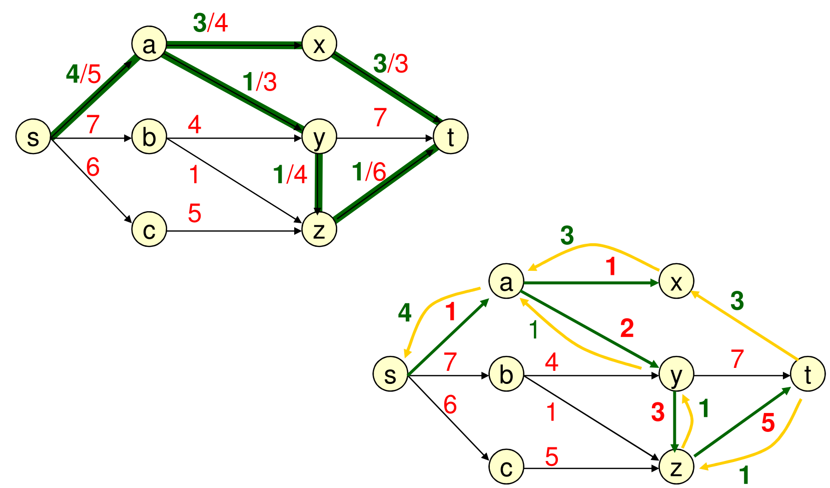


- e.g. $c_f(s,b) = 7$; $c_f(a,x) = 1$; $c_f(x,a) = 3$

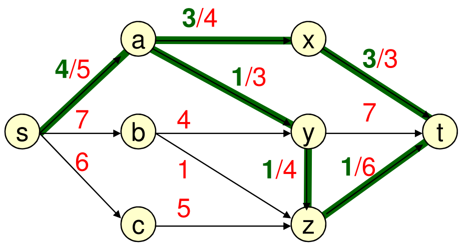
Residual Graph & Augmenting Paths

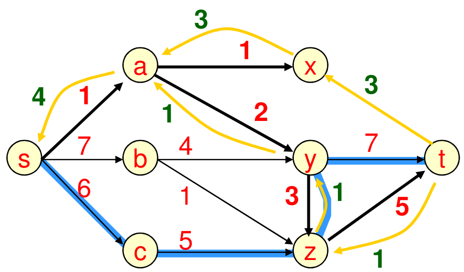
- The *residual graph* (w.r.t. **f**) is the graph $G_f = (V, E_f)$, where
  $E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$
  - Two kinds of edges
    - Forward edges
      - $f(u,v) < c(u,v)$ so $c_f(u,v) = c(u,v) - f(u,v) > 0$
    - Backward edges
      - $f(u,v) > 0$ so $c_f(v,u) \ge -f(v,u) = f(u,v) > 0$
- An *augmenting path* (w.r.t. **f**) is a simple $s \to t$ path in G_f.

A Residual Network



An Augmenting Path





Augmenting A Flow

augment(**f**,**P**)

$\mathbf{c_P} \leftarrow \min_{(u,v)\in P} \mathbf{c_f}(\mathbf{u},\mathbf{v})$ "bottleneck(P)"

for each $\mathbf{e} \in \mathbf{P}$

if **e** is a forward edge then

increase **f**(**e**) by $\mathbf{c_P}$

else (**e** is a backward edge)

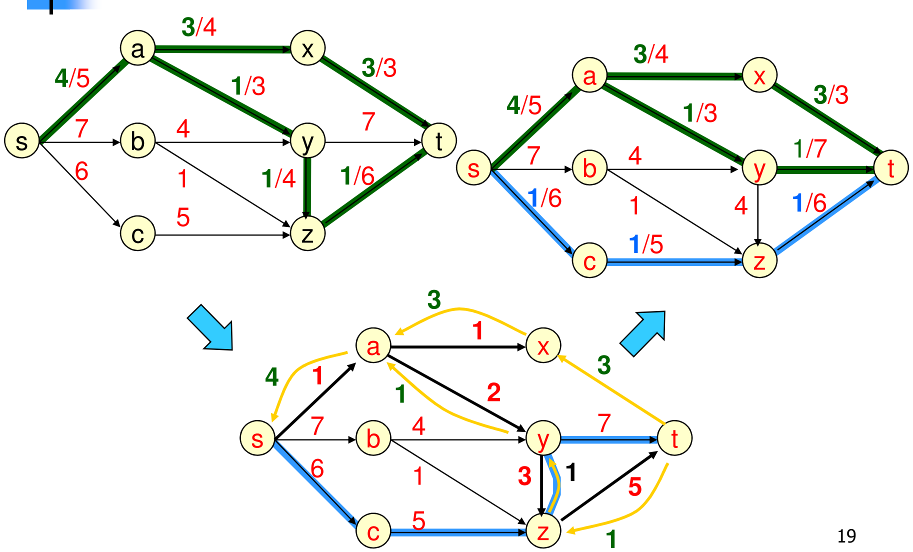
decrease **f**(**e**) by $\mathbf{c_P}$

endif

endfor

return(**f**)

Augmenting A Flow



Claim 7.1

If $\mathbf{G_f}$ has an augmenting path **P**, then the function **f'=**augment(**f**,**P**) is a legal flow.

Proof:

- **f'** and **f** differ only on the edges of **P** so only need to consider such edges (**u**,**v**)

Proof of Claim 7.1

- If $(\mathbf{u},\mathbf{v})$ is a forward edge then
  $\mathbf{f'}(\mathbf{u},\mathbf{v})=\mathbf{f}(\mathbf{u},\mathbf{v})+\mathbf{c_P} \leq \mathbf{f}(\mathbf{u},\mathbf{v})+\mathbf{c_f}(\mathbf{u},\mathbf{v})$
  $= \mathbf{f}(\mathbf{u},\mathbf{v})+\mathbf{c}(\mathbf{u},\mathbf{v})-\mathbf{f}(\mathbf{u},\mathbf{v})$
  $=\mathbf{c}(\mathbf{u},\mathbf{v})$
- If $(\mathbf{u},\mathbf{v})$ is a backward edge then **f** and **f'** differ on flow along $(\mathbf{v},\mathbf{u})$ instead of $(\mathbf{u},\mathbf{v})$
  $\mathbf{f'}(\mathbf{v},\mathbf{u})=\mathbf{f}(\mathbf{v},\mathbf{u})-\mathbf{c_P} \geq \mathbf{f}(\mathbf{v},\mathbf{u})-\mathbf{c_f}(\mathbf{u},\mathbf{v})$
  $= \mathbf{f}(\mathbf{v},\mathbf{u})-\mathbf{f}(\mathbf{v},\mathbf{u})=\mathbf{0}$
- Other conditions like flow conservation still met

Ford-Fulkerson Method

Start with **f=0** for every edge

While $\mathbf{G_f}$ has an augmenting path,
augment

- Questions:
  - Does it halt?
  - Does it find a maximum flow?
  - How fast?

Observations about Ford-Fulkerson Algorithm

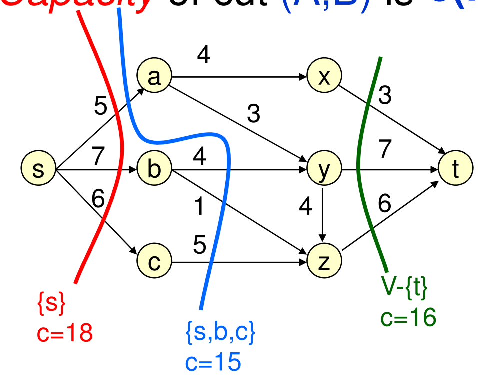
- At every stage the capacities and flow values are always integers (if they start that way)
- The flow value $\nu(\mathbf{f'})=\nu(\mathbf{f})+\mathbf{c_P}>\nu(\mathbf{f})$ for $\mathbf{f'}=\text{augment}(\mathbf{f},\mathbf{P})$
  - Since edges of residual capacity **0** do not appear in the residual graph
- Let $\mathbf{C}=\sum_{(\mathbf{s},\mathbf{u})\in \mathbf{E}} \mathbf{c}(\mathbf{s},\mathbf{u})$
  - $\nu(\mathbf{f})\leq\mathbf{C}$
  - **F-F** does at most **C** rounds of augmentation since flows are integers and increase by at least **1** per step

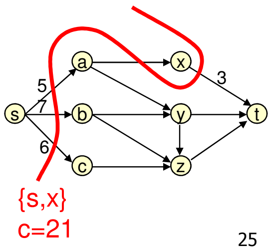
Running Time of Ford-Fulkerson

- For **f=0**, $\mathbf{G_f}=\mathbf{G}$
- Finding an augmenting path in G_f is graph search $O(\mathbf{n}+\mathbf{m})=O(\mathbf{m})$ time
- Augmenting and updating $\mathbf{G_f}$ is $O(\mathbf{n})$ time
- Total $O(\mathbf{mC})$ time
- Does is find a maximum flow?
  - Need to show that for every flow **f** that isn't maximum $\mathbf{G_f}$ contains an **s-t**-path

Cuts

- A partition (**A**,**B**) of **V** is an *s-t-cut* if
  - $s \in A$, $t \in B$
- *Capacity* of cut (A,B) is $c(A,B) = \sum_{u \in A, v \in B} c(u,v)$



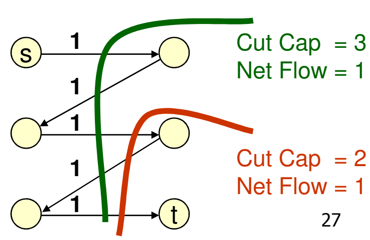


Convenient Definition

- $f^{out}(A) = \sum_{v \in A,\, w \notin A} f(v,w)$
- $f^{in}(A) = \sum_{v \in A,\, u \notin A} f(u,v)$

Claims 7.6 and 7.8

- For any flow **f** and any cut (**A**,**B**),
  - the net flow across the cut equals the total flow, i.e., $\nu(f) = f^{out}(A) - f^{in}(A)$, and
  - the net flow across the cut cannot exceed the capacity of the cut, i.e. $f^{out}(A) - f^{in}(A) \le c(A,B)$
- Corollary :
  Max flow ≤ Min cut



Proof of Claim 7.6

- Consider a set **A** with $s \in A$, $t \notin A$
- $f^{out}(A) - f^{in}(A) = \sum_{v \in A,\, w \notin A} f(v,w) - \sum_{v \in A,\, u \notin A} f(u,v)$
- We can add flow values for edges with both endpoints in **A** to **both** sums and they would cancel out so
- $f^{out}(A) - f^{in}(A) = \sum_{v \in A,\, w \in V} f(v,w) - \sum_{v \in A,\, u \in V} f(u,v)$
  $= \sum_{v \in A} \left(\sum_{w \in V} f(v,w) - \sum_{u \in V} f(u,v)\right)$
  $= \sum_{v \in A} f^{out}(v) - f^{in}(v)$
  $= f^{out}(s) - f^{in}(s)$

  since all other vertices have $f^{out}(v) = f^{in}(v)$
- $\nu(f) = f^{out}(s)$ and $f^{in}(s) = 0$

Proof of Claim 7.8

- $\nu(f)=f^{out}(A)-f^{in}(A)$
  $\leq f^{out}(A)$
  $= \sum_{v\in A,\ w\notin A} f(v,w)$
  $\leq \sum_{v\in A,\ w\notin A} c(v,w)$
  $\leq \sum_{v\in A,\ w\in B} c(v,w)$
  $=c(A,B)$

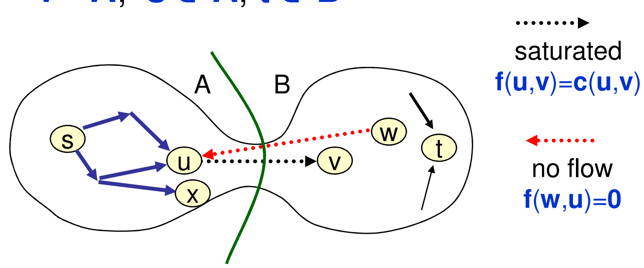
Max Flow / Min Cut Theorem

Claim 7.9 For any flow **f**, if G_f has no augmenting path then there is some **s**-**t**-cut (**A**,**B**) such that $\nu(f)=c(A,B)$ (proof on next slide)

- We know by Claims 7.6 & 7.8 that any flow **f'** satisfies $\nu(f') \leq c(A,B)$ and we know that F-F runs for finite time until it finds a flow **f** satisfying conditions of Claim 7.9
  - Therefore by 7.9 for any flow **f'**, $\nu(f') \leq \nu(f)$
- Corollary (1) F-F computes a maximum flow in **G**
  (2) For any graph **G**, the value $\nu(f)$ of a maximum flow = minimum capacity $c(A,B)$ of any **s**-**t**-cut in **G**

Claim 7.9

Let **A** = { **u** | ∃ an path in G_f from **s** to **u** }
B = **V** - **A**; $s \in A$, $t \in B$

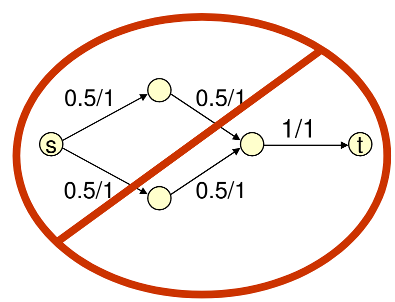


This is true for **every** edge crossing the cut, i.e.

$f^{out}(A) = \sum_{u\in A, v\in B} f(u,v) = \sum_{u\in A, v\in B} c(u,v) = c(A,B)$ and $f^{in}(A)=0$ so $\nu(f)=f^{out}(A)-f^{in}(A)=c(A,B)$

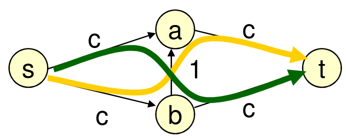
Flow Integrality Theorem

If all capacities are integers
- The max flow has an integer value
- Ford-Fulkerson method finds a max flow in which $f(u,v)$ is an integer for all edges (u,v)



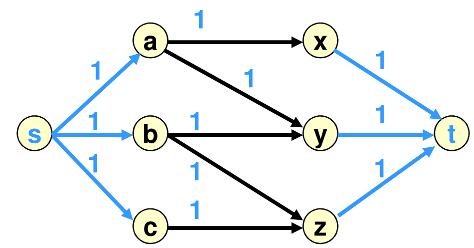
Corollaries & Facts

- If Ford-Fulkerson terminates, then it's found a max flow.
- It will terminate if **c(e)** integer or rational (but may not if they're irrational).
- However, may take exponential time, even with integer capacities:



c = **10^9**, say

Bipartite matching as a special case of flow



Integer flows implies each flow is just a subset of the edges

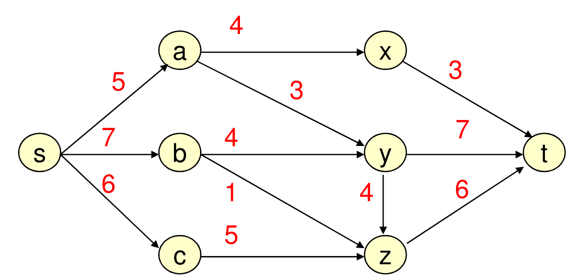
Therefore flow corresponds to a matching

O(**mC**)=O(**nm**) running time

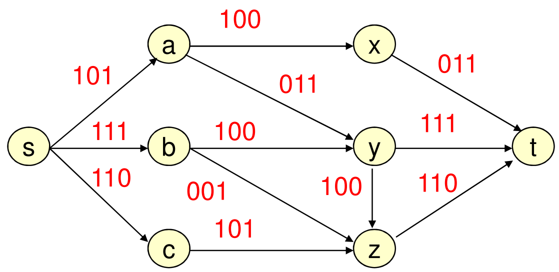
Capacity-scaling algorithm

- General idea:
  - Choose augmenting paths **P** with 'large' capacity $\mathbf{c_P}$
  - Can augment flows along a path **P** by any amount $\Delta \leq \mathbf{c_P}$
    - Ford-Fulkerson still works
  - Get a flow that is maximum for the high-order bits first and then add more bits later

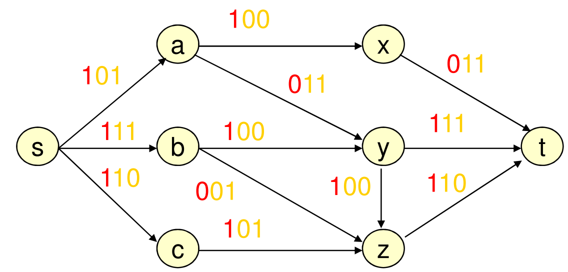
Capacity Scaling



Capacity Scaling

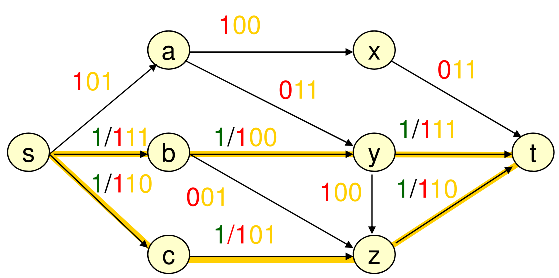


Capacity Scaling Bit 1



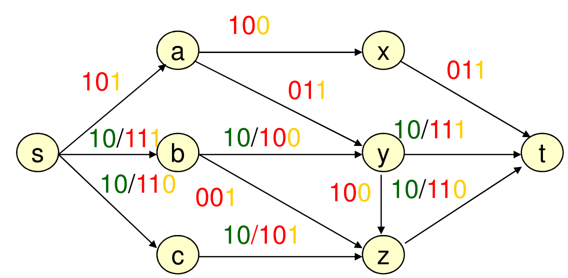
Capacity on each edge is at most **1**
(either **0** or **1** times **Δ=4**)

Capacity Scaling Bit 1



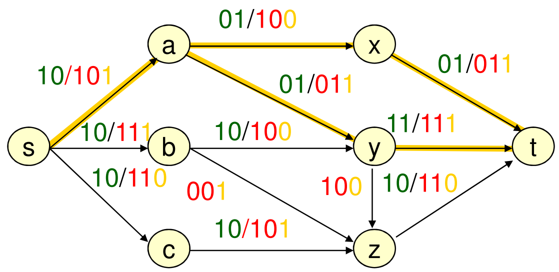
O(nm) time

Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**
(either **0** or **1** times **Δ=2**)

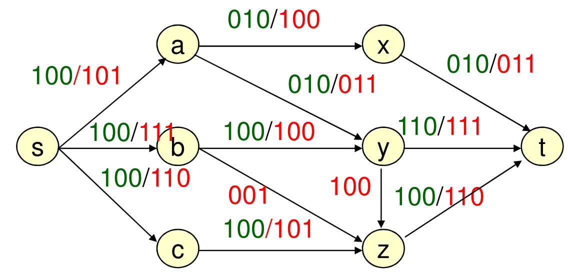
Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**

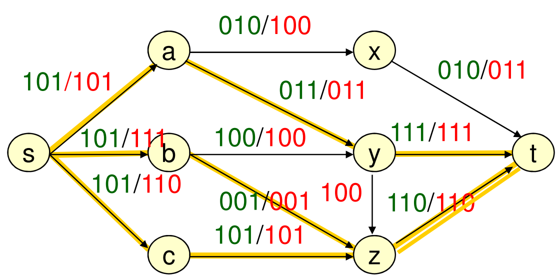
⇒ ≤ **m** augmentations

Capacity Scaling Bit 3



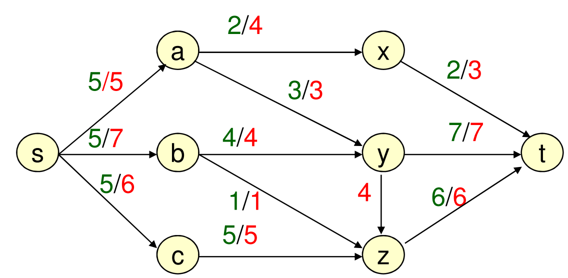
Residual capacity across min cut is at most **m**
(either **0** or **1** times **Δ=1**)

Capacity Scaling Bit 3

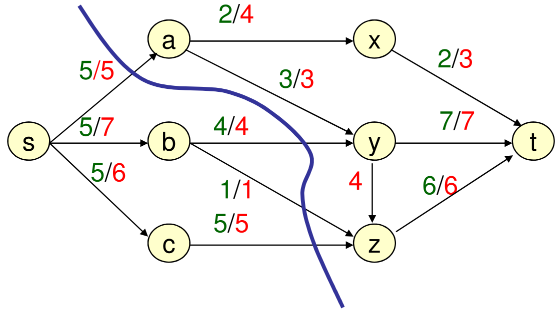


After ≤ **m** augmentations

Capacity Scaling Final



Capacity Scaling Min Cut



Total time for capacity scaling

- $\log_2$ **U** rounds where **U** is largest capacity
- At most **m** augmentations per round
  - Let c_i be the capacities used in the i^{th} round and f_i be the maxflow found in the i^{th} round
    - For any edge (u,v), $c_{i+1}(u,v) \leq 2c_i(u,v)+1$
  - $i+1^{st}$ round starts with flow $f = 2 f_i$
  - Let (A,B) be a min cut from the i^{th} round
    - $\nu(f_i)=c_i(A,B)$ so $\nu(f)=2c_i(A,B)$
  - $\nu(f_{i+1}) \leq c_{i+1}(A,B) \leq 2c_i(A,B)+m = \nu(f)+m$
- $O(m)$ time per augmentation
- Total time $O(m^2 \log U)$

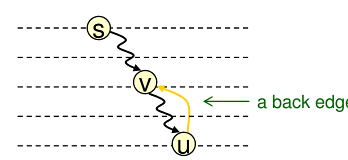
Edmonds-Karp Algorithm

- Use a shortest augmenting path
  (via Breadth First Search in residual graph)
- Time: $O(n\, m^2)$

BFS/Shortest Path Lemmas

Distance from **s** in G_f is never reduced by:

- Deleting an edge
  Proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  Proof: BFS is unchanged, since **v** visited before **(u,v)** examined

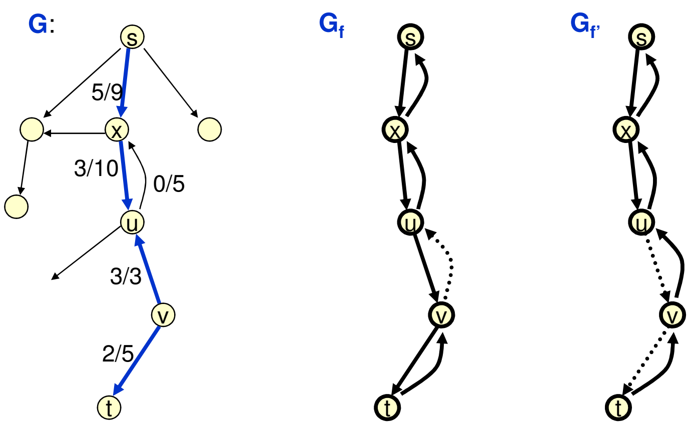


Key Lemma

Let f be a flow, G_f the residual graph, and P a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

Proof: Augmentation along **P** only deletes forward edges, or adds back edges that go to previous vertices along **P**

Augmentation vs BFS



Theorem

The Edmonds-Karp Algorithm performs O(**mn**) flow augmentations

Proof:

Call (**u**,**v**) critical for augmenting path **P** if it's closest to **s** having min residual capacity

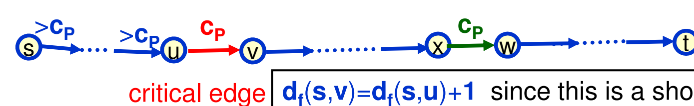
It will disappear from G_f after augmenting along **P**

In order for **(u,v)** to be critical again the **(u,v)** edge must re-appear in G_f but that will only happen when the distance to **u** has increased by **2** (next slide)

It won't be critical again until farther from **s** so each edge critical at most **n/2** times

Critical Edges in G_f

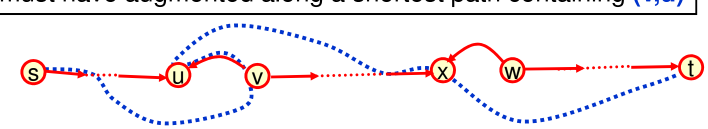
Shortest s-t path **P** in G_f



$d_f(s,v)=d_f(s,u)+1$ since this is a shortest path

After augmenting along **P**

For (**u**,**v**) to be critical later for some flow **f'** it must be in $G_{f'}$ so must have augmented along a shortest path containing **(v,u)**



Then we must have $d_{f'}(s,u)=d_{f'}(s,v)+1 \geq d_f(s,v)+1=d_f(s,u)+2$

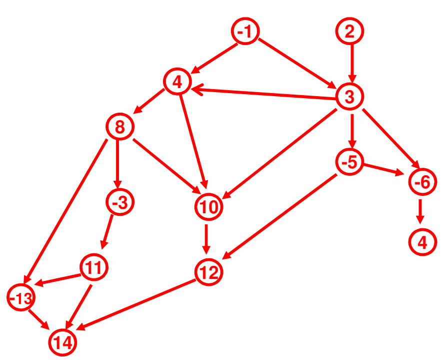
Corollary

- Edmonds-Karp runs in $O(nm^2)$ time

Project Selection a.k.a. The Strip Mining Problem

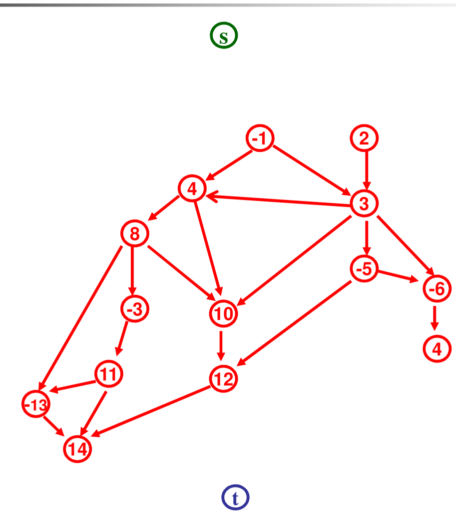
- Given
  - a directed acyclic graph $G=(V,E)$ representing precedence constraints on tasks (a task points to its **predecessors**)
  - a profit value $p(v)$ associated with each task $v\in V$ (may be positive or negative)
- Find
  - a set $A\subseteq V$ of tasks that is closed under predecessors, i.e. if $(u,v)\in E$ and $u\in A$ then $v\in A$, that maximizes $\text{Profit}(A)=\sum_{v\in A} p(v)$

Project Selection Graph



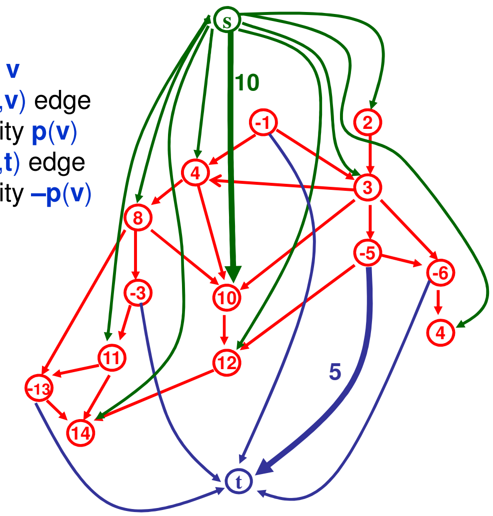
Each task points to its predecessor tasks

Extended Graph



Extended Graph G'

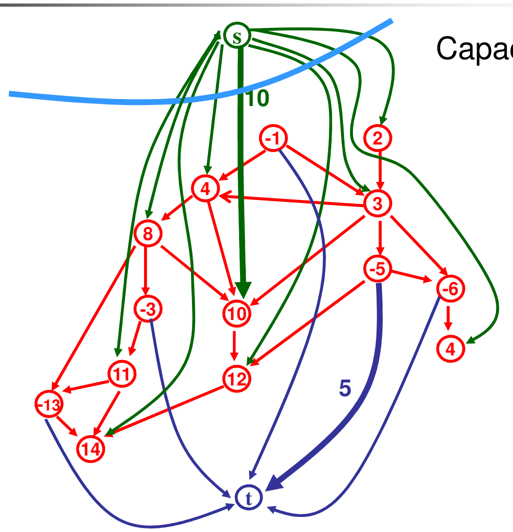
For each vertex **v**
If **p**(**v**)≥**0** add (**s**,**v**) edge
with capacity **p**(**v**)
If **p**(**v**)<**0** add (**v**,**t**) edge
with capacity **–p**(**v**)



Extended Graph G'

- Want to arrange capacities on edges of **G** so that for minimum **s-t**-cut (**S**,**T**) in **G'**, the set **A**=**S**-{**s**}
  - satisfies precedence constraints
  - has maximum possible profit in **G**
- Cut capacity with **S**={**s**} is just $C=\sum_{v:\ p(v)\geq 0} p(v)$
  - Profit(**A**) ≤ **C** for any set **A**
- To satisfy precedence constraints don't want any original edges of **G** going forward across the minimum cut
  - That would correspond to a task in **A**=**S**-{**s**} that had a predecessor not in **A**=**S**-{**s**}
- Set capacity of each of the edges of **G** to **C+1**
  - The minimum cut has size at most **C**

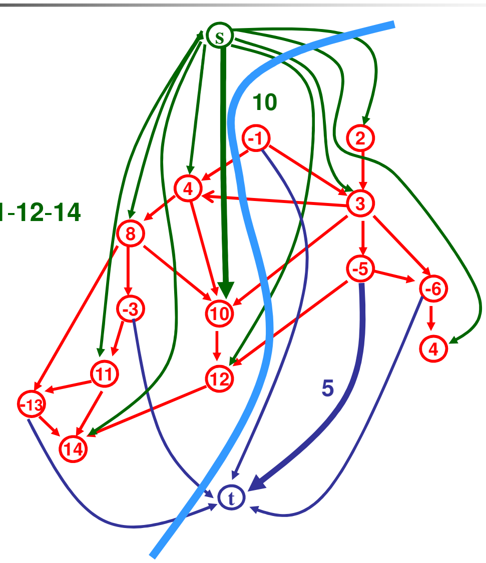
Extended Graph G'



Capacity **C**

Extended Graph G'

Cut value
=**13**+**3**+**2**+**3**+**4**
=**13**+**3**
+**C-4-8-10-11-12-14**



Project Selection

- **Claim** Any **s**-**t**-cut (**S**,**T**) in **G'** such that **A**=**S**-{**s**} satisfies precedence constraints has capacity

  $c(\mathbf{S},\mathbf{T})=\mathbf{C} - \sum_{v\in A} \mathbf{p}(\mathbf{v}) = \mathbf{C} - \text{Profit}(\mathbf{A})$

- **Corollary** A minimum cut (**S**,**T**) in **G'** yields an optimal solution **A**=**S**-{**s**} to the profit selection problem

- **Algorithm** Compute maximum flow **f** in **G'**, find the set **S** of nodes reachable from **s** in $\mathbf{G'_f}$ and return **S**-{**s**}

Proof of Claim

- **A=S-**{**s**} satisfies precedence constraints
  - No edge of **G** crosses forward out of **A** since those edges have capacity **C+1**
  - Only forward edges cut are of the form (**v**,**t**) for **v**∈**A** or (**s**,**v**) for **v**∉**A**
  - The (**v**,**t**) edges for **v**∈**A** contribute

    $\sum_{v\in A:p(v)<0} -\mathbf{p}(\mathbf{v}) = -\sum_{v\in A:p(v)<0} \mathbf{p}(\mathbf{v})$
  - The (**s**,**v**) edges for **v**∉**A** contribute

    $\sum_{v\notin A:\, p(v)\geq 0} \mathbf{p}(\mathbf{v})=\mathbf{C}-\sum_{v\in A:\, p(v)\geq 0} \mathbf{p}(\mathbf{v})$
  - Therefore the total capacity of the cut is

    $\mathbf{c}(\mathbf{S},\mathbf{T}) = \mathbf{C} - \sum_{v\in A} \mathbf{p}(\mathbf{v}) = \mathbf{C}-\text{Profit}(\mathbf{A})$