



CSE 421 Algorithms

Richard Anderson
Lecture 22
Network Flow



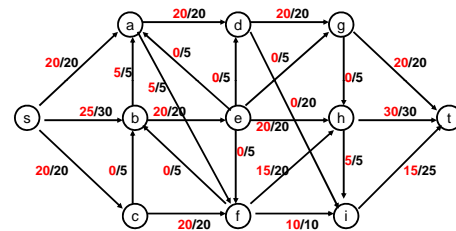
Review

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, $c(e) \geq 0$
- Problem, assign flows $f(e)$ to the edges such that:
 - $0 \leq f(e) \leq c(e)$
 - Flow is conserved at vertices other than s and t
 - Flow conservation: flow going into a vertex equals the flow going out
 - The flow leaving the source is as large as possible

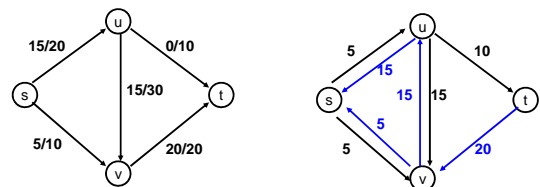
Find a maximum flow



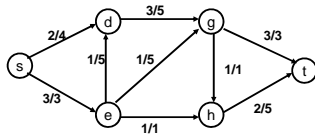
Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G , Residual Graph G_R
 - G : edge e from u to v with capacity c and flow f
 - G_R : edge e' from u to v with capacity $c - f$
 - G_R : edge e'' from v to u with capacity f

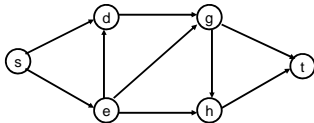
Residual Graph



Build the residual graph

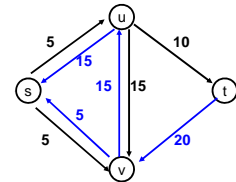
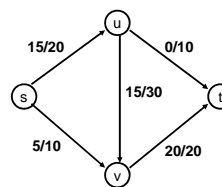


Residual graph:



Augmenting Path Lemma

- Let $P = v_1, v_2, \dots, v_k$ be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.



Proof

- Add b units of flow along the path P
- What do we need to verify to show we have a valid flow after we do this?
-
-

Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

Find an s - t path P in G_R with capacity $b > 0$

Add b units along in G

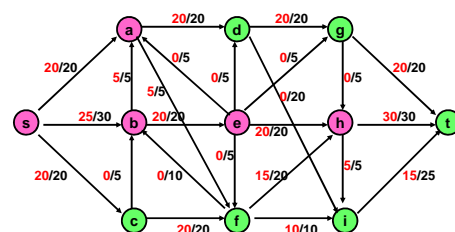
If the sum of the capacities of edges leaving S is at most C , then the algorithm takes at most C iterations

Cuts in a graph

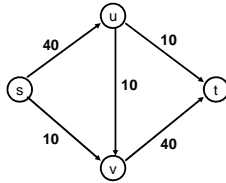
- Cut: Partition of V into disjoint sets S, T with s in S and t in T .
- $\text{Cap}(S, T)$: sum of the capacities of edges from S to T
- $\text{Flow}(S, T)$: net flow out of S
 - Sum of flows out of S minus sum of flows into S
- $\text{Flow}(S, T) \leq \text{Cap}(S, T)$

What is $\text{Cap}(S, T)$ and $\text{Flow}(S, T)$

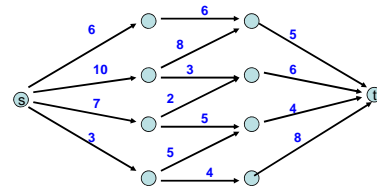
$S = \{s, a, b, e, h\}$, $T = \{c, f, i, d, g, t\}$



Minimum value cut

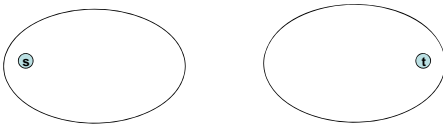


Find a minimum value cut

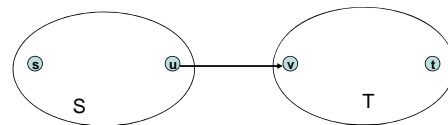


MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in G_R reachable from s with paths of positive capacity



Let S be the set of vertices in G_R reachable from s with paths of positive capacity



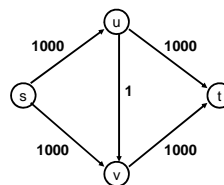
What can we say about the flows and capacity between u and v?

Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.
- If we want to find a minimum cut, we begin by looking for a maximum flow.

Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible



Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
 - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
 - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
 - $O(mn \log n)$ time algorithm for network flow