CSE 421
Algorithms

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Lecture 21

Network Flow

Network Flow

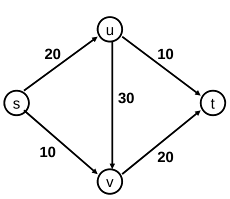
Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

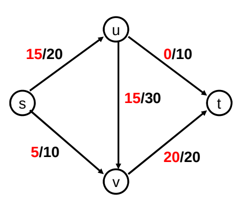
Network Flow Definitions

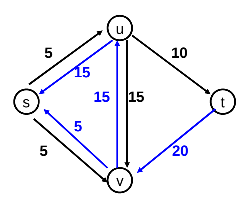
- Capacity
- Source, Sink
- Capacity Condition
- Conservation Condition
- Value of a flow

Flow Example



Flow assignment and the residual graph

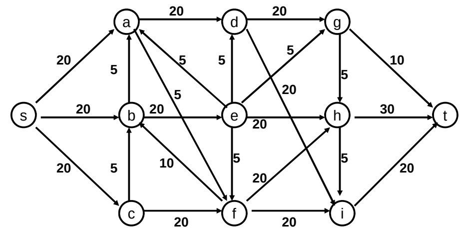




Network Flow Definitions

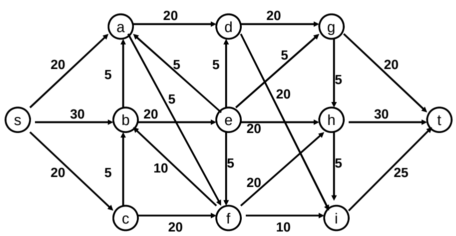
- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

Flow Example



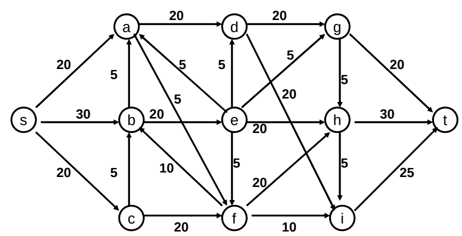
Find a maximum flow

Value of flow:



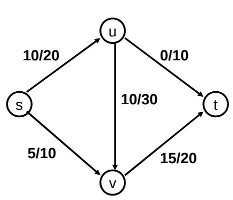
Construct a maximum flow and indicate the flow value

Find a maximum flow

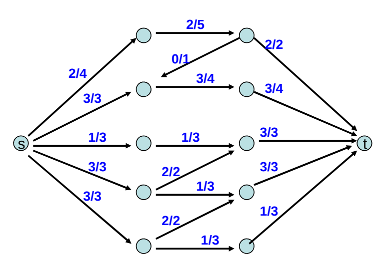


Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



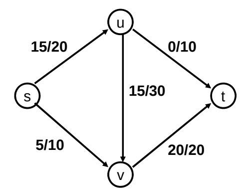
Find two augmenting paths

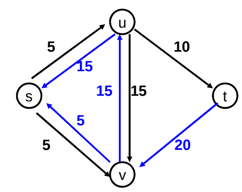


Residual Graph

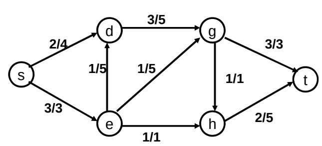
- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

Residual Graph

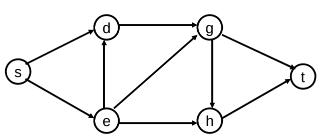




Build the residual graph

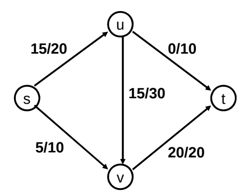


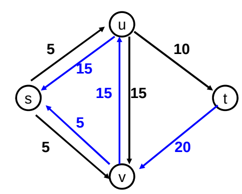
Residual graph:



Augmenting Path Lemma

- Let P = $v_1, v_2, \ldots, v_k$ be a path from s to t with minimum capacity b in the residual graph.
- b units of flow can be added along the path P in the flow graph.





Proof

- Add b units of flow along the path P
- What do we need to verify to show we have a valid flow after we do this?
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Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

Find an s-t path P in G_R with capacity b > 0

Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations