CSE 421
Algorithms

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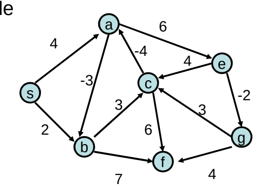
Lecture 10-11

Minimum Spanning Trees

Shortest Paths

- Negative Cost Edges
  - Dijkstra's algorithm assumes positive cost edges
  - For some applications, negative cost edges make sense
  - Shortest path not well defined if a graph has a negative cost cycle

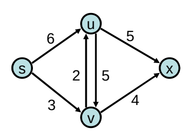


Negative Cost Edge Preview

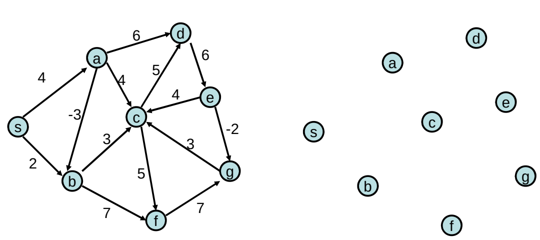
- Topological Sort can be used for solving the shortest path problem in directed acyclic graphs
- Bellman-Ford algorithm finds shortest paths in a graph with negative cost edges (or reports the existence of a negative cost cycle).

Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path

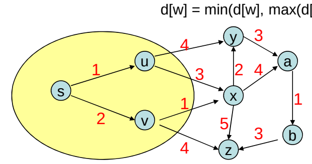


Compute the bottleneck shortest paths



Dijkstra's Algorithm for Bottleneck Shortest Paths

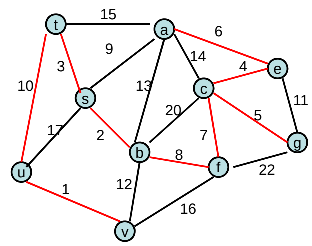
```
S = {};    d[s] = negative infinity;    d[v] = infinity for v != s
While S != V
        Choose v in V-S with minimum d[v]
        Add v to S
        For each  w in the neighborhood of v
                d[w] = min(d[w], max(d[v], c(v, w)))
```



Minimum Spanning Tree

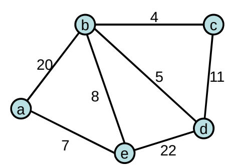
- Introduce Problem
- Demonstrate three different greedy algorithms
- Provide proofs that the algorithms work

Minimum Spanning Tree



Greedy Algorithms for Minimum Spanning Tree

- Extend a tree by including the cheapest out going edge
- Add the cheapest edge that joins disjoint components
- Delete the most expensive edge that does not disconnect the graph

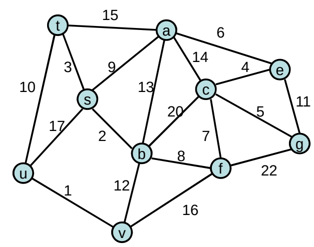


Greedy Algorithm 1
Prim's Algorithm

- Extend a tree by including the cheapest out going edge

Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion

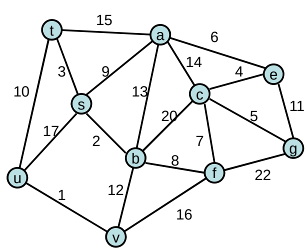


Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components

Construct the MST with Kruskal's algorithm

Label the edges in order of insertion

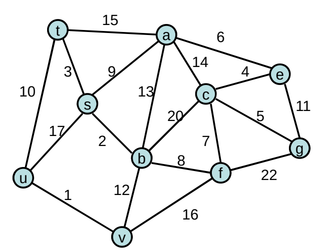


Greedy Algorithm 3
Reverse-Delete Algorithm

- Delete the most expensive edge that does not disconnect the graph

Construct the MST with the reverse-delete algorithm

Label the edges in order of removal



Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S
- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim's Algorithm computes a MST
- Kruskal's Algorithm computes a MST

Reverse-Delete Algorithm

- Lemma: The most expensive edge on a cycle is never in a minimum spanning tree

Dealing with the assumption of no equal weight edges

- Force the edge weights to be distinct
  - Add small quantities to the weights
  - Give a tie breaking rule for equal weight edges

Dijkstra's Algorithm for Minimum Spanning Trees

S = {}; d[s] = 0; d[v] = infinity for v != s

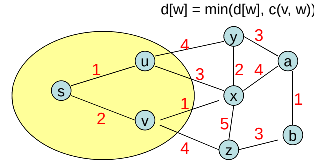
While S != V

Choose v in V-S with minimum d[v]

Add v to S

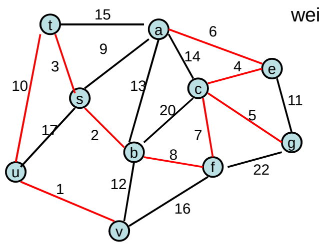
For each w in the neighborhood of v

d[w] = min(d[w], c(v, w))



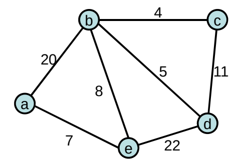
Minimum Spanning Tree

Undirected Graph G=(V,E) with edge weights



Greedy Algorithms for Minimum Spanning Tree

- [Prim] Extend a tree by including the cheapest out going edge
- [Kruskal] Add the cheapest edge that joins disjoint components
- [ReverseDelete] Delete the most expensive edge that does not disconnect the graph

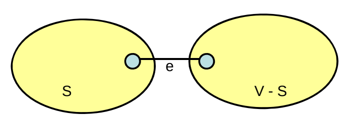


Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct

Edge inclusion lemma

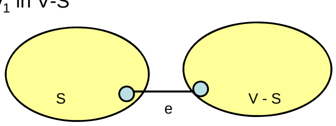
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



e is the minimum cost edge between S and V-S

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim's Algorithm computes a MST
- Kruskal's Algorithm computes a MST

- Show that when an edge is added to the MST by Prim or Kruskal, the edge is the minimum cost edge between S and V-S for some set S.

Prim's Algorithm

S = { }; T = { };

while S != V

choose the minimum cost edge
e = (u,v), with u in S, and v in V-S

add e to T

add v to S

Prove Prim's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Kruskal's Algorithm

Let C = {{v_1}, {v_2}, . . ., {v_n}}; T = { }

while |C| > 1

Let e = (u, v) with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by C_i U C_j

Add e to T

Prove Kruskal's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Reverse-Delete Algorithm

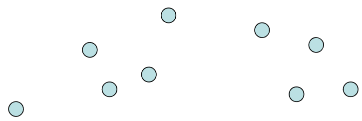
- Lemma: The most expensive edge on a cycle is never in a minimum spanning tree

Dealing with the assumption of no equal weight edges

- Force the edge weights to be distinct
  - Add small quantities to the weights
  - Give a tie breaking rule for equal weight edges

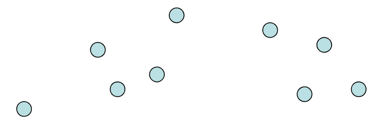
Application: Clustering

- Given a collection of points in an r-dimensional space, and an integer K, divide the points into K sets that are closest together

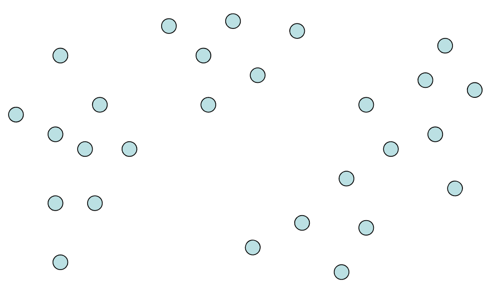


Distance clustering

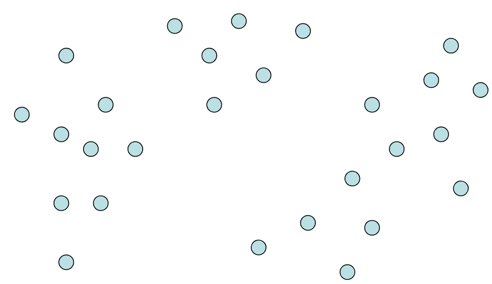
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - dist (S_1, S_2) = min {dist(x, y) | x in S_1, y in S_2}



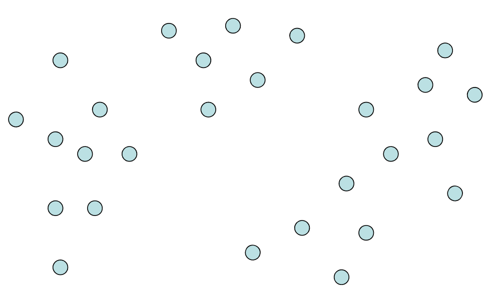
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

Let C = $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$; T = { }

while |C| > K

    Let e = (u, v) with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

    Replace C_i and C_j by $C_i \cup C_j$

K-clustering