Name: ______________________________

# CSE 421
# Final Exam
March 13, 2007

**Instructions:**

- You have 1 hour and 50 minutes to complete the exam.
- Feel free to ask for clarification if something is unclear.
- Please do not turn the page until you are instructed to do so.
- The exam is open book, open notes.
- Good luck!

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|---|---|
| 1 | /14 |
| 2 | / 6 |
| 3 | /12 |
| 4 | /15 |
| 5 | /15 |
| 6 | /13 |
| total | /75 |

1. (14 points, 2 points for each correct answer, -2 points for each incorrect answer) Indicate for each of the following if it is **true or false** by circling the appropriate answer.

   - *True or False:* You are given a vector $(a_1, a_2, \ldots, a_n)$. You are also given the recurrence $f(i) = \min_{1 \le j < i}(j + a_j + f(j))$, and you are told $f(1) = 0$. Then dynamic programming allows you to calculate $f(n)$ in time $O(n)$.
   - *True or False:* Consider the problem of shortest paths in a graph $G$ where edges can have negative weights. Recall that we defined $Opt(i, v)$ to be the length of the shortest path from $v$ to $t$ that uses at most $i$ edges. Suppose that there is a some vertex $w$ such that $Opt(n, w) \neq Opt(n-1, w)$ (where $n$ is the number of nodes in the graph). Then $G$ has a negative cycle.
   - *True or False:* Same setup as previous question. Suppose that $Opt(1, v) = Opt(2, v)$ for all $v$ (in a graph where there are $n > 2$ vertices). Then $G$ has no negative cycle from which $t$ can be reached.
   - *True or False:* We know of a problem in $\mathcal{NP}$ that is also in $\mathcal{P}$.
   - *True or False* Suppose that $X$ and $Y$ are both in $\mathcal{P}$. Then there is a polytime reduction from $X$ to $Y$.
   - *True or False:* Suppose that $X \le_{\mathcal{P}} Y$, $X$ is NP-complete and $Y \in \mathcal{NP}$. Then $Y \le_{\mathcal{P}} X$.
   - *True or False:* Suppose that $X$ and $Y$ are both in $\mathcal{NP}$, and that $SAT \le_{\mathcal{P}} X$ and $SAT \le_{\mathcal{P}} Y$. Then $X \le_{\mathcal{P}} Y$.

2. (6 points, 2 points for each correct answer, -2 points for each incorrect answer) Indicate for each of the following if it is **true or false** by circling the appropriate answer. In all of the following, you are given an s-t flow network $G$, where $c(u, v)$ is the capacity of edge $(u, v)$.

   - *True or False:* Let $f$ be a maximum flow in $G$, where $f(u, v)$ is the flow on edge $(u, v)$. Let $(A_1, B_1)$ and $(A_2, B_2)$ be two different minimum s-t cuts. Then $\sum_{u \in A_1, v \in B_1} f(u, v) = \sum_{u \in A_2, v \in B_2} c(u, v)$. (Pay close attention to the subscripts.)
   - *True or False:* Let $f$ be a maximum flow in $G$. Let $(A_1, B_1)$ and $(A_2, B_2)$ be two different minimum s-t cuts. Then $\sum_{u \in B_1, v \in A_1,} f(u, v) = \sum_{u \in B_2, v \in A_2} f(u, v)$. (Pay close attention to the subscripts.)
   - *True or False:* Let $f$ be a maximum flow in $G$. Let $(A_1, B_1)$ and $(A_2, B_2)$ be two different minimum s-t cuts. Then $\sum_{u \in A_1, v \in B_1,} f(u, v) - \sum_{u \in A_1, v \in B_1,} c(u, v) = \sum_{u \in B_2, v \in A_2} f(u, v)$. (Pay close attention to the subscripts.)

3. (12 points, 4 points each) Consider a recursive divide and conquer algorithm that satisfies the following recurrence on its running time:

$$T(n) = 4T(n/3) + n,$$

with $T(1) = 1$. In the following you may assume that $n$ is a power of 3.

- How many subproblems are there at depth $k$ in the recursion tree? (The number of subproblems at depth 0, namely at the root of the tree, is 1.)
- What is the size of each subproblem at depth $k$ of the recursion tree? (The size of the subproblem at depth 0, namely at the root of the tree, is $n$).
- What is the running time $T(n)$ of this algorithm. Express it in big Oh notation as $O(n^a)$ for an appropriate choice of the value $a$.

4. (15 points, 5 points each) Recall the knapsack problem: We are given $n$ items and a "knapsack". The knapsack can carry a weight up to $W$. (Assume $W$ is an integer.) Each item $i$ has an integer value $v_i$ and an integer weight $0 < w_i < W$. The goal is to choose a subset $S$ of the items to fill the knapsack with so that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximized. Let $V^*$ be the optimum value, i.e. $V^* = \sum_{i \in S^*} v_i$ for the optimal subset $S^*$.

   In Section 6.4, one dynamic programming approach to this problem is given. Here we explore another.

   Define $OPT(i, v)$ to be the weight of the minimum weight subset of items $1 \ldots i$ that yields a total value of exactly $v$. We get a subproblem for each $0 \leq i \leq n$ and each integer $v$ such that $0 \leq v \leq V = \sum_{1 \leq i \leq n} v_i$. (You can define $OPT(i, v) = \infty$ if there is no subset of $1..i$ that yields value exactly $v$.)

   - Write a recurrence for $OPT(i, v)$. Be sure to also specify the base cases ($OPT(0, v)$ and $OPT(i, 0)$).
   - Given the values of $OPT(i, v)$ for $0 \leq i \leq n$ and $0 \leq v \leq V = \sum_{1 \leq i \leq n} v_i$, how would you compute $V^*$?
   - What is the running time of this dynamic programming procedure for computing $V^*$? Include the time to compute the values of $OPT(i, v)$ for all $0 \leq i \leq n$ and $0 \leq v \leq V$. Your answer should be expressed in terms of $n$ and $V = \sum_{1 \leq i \leq n} v_i$. No need to give the algorithm, just the running time.

5. (15 points, 5 points each) Consider the following network, with edge capacities as shown.

(a) What is the value of the maximum flow in the network? I'm just interested in the single number $\nu(f^*)$.

(b) Give a minimum cut in this network. (Specify which nodes are on the $s$ side of the cut.)

(c) Is there more than one minimum cut in this network? If so, specify which nodes are on the $s$ side of a minimum cut different from the one you gave in part (b).

6. (13 points) Prove that the following problem, called `MAXSAT`, is NP-complete. Given a 3-CNF formula $\Phi$ and an integer $g$, is there a truth assignment that satisfies at least $g$ clauses?