CSE 421: Introduction to Algorithms

Stable Matching

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Matching Residents to Hospitals

- Goal: Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.
- Unstable pair: applicant **x** and hospital **y** are unstable if:
  - **x** prefers **y** to their assigned hospital.
  - **y** prefers **x** to one of its admitted students.
- Stable assignment. Assignment with no unstable pairs.
  - Natural and desirable condition.
  - Individual self-interest will prevent any applicant/hospital deal from being made.

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Stable Matching Problem

- Goal. Given **n** men and **n** women, find a "suitable" matching.
  - Participants rate members of opposite sex.
  - Each man lists women in order of preference from best to worst.
  - Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

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Stable Matching Problem

- Perfect matching: everyone is matched monogamously.
  - Each man gets exactly one woman.
  - Each woman gets exactly one man.
- Stability: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching **M**, an unmatched pair **m-w** is unstable if man **m** and woman **w** prefer each other to current partners.
  - Unstable pair **m-w** could each improve by eloping.
- Stable matching: perfect matching with no unstable pairs.
- Stable matching problem. Given the preference lists of **n** men and **n** women, find a stable matching if one exists.

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Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

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Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?
- A. No. Brenda and Xavier will hook up.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

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Stable Matching Problem

- Q. Is assignment X-A, Y-B, Z-C stable?
- A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

Stable Roommate Problem

- Q. Do stable matchings always exist?
- A. Not obvious a priori.

- Stable roommate problem.
  - **2n** people; each person ranks others from **1** to **2n-1**.
  - Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
David	A	B	C

A-B, C-D ⇒ B-C unstable
A-C, B-D ⇒ A-B unstable
A-D, B-C ⇒ A-C unstable

- Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

- Propose-and-reject algorithm. [Gale-Shapley 1962]
  Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```


Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.

- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

- Claim. Algorithm terminates after at most n^2 iterations of while loop.
- Proof. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Walter	B	C	D	A	E
Xavier	C	D	A	B	E
Yuri	D	A	B	C	E
Zoran	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Brenda	X	Y	Z	V	W
Claire	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

$n(n-1) + 1$ proposals required

Proof of Correctness: Perfection

- Claim. All men and women get matched.
- Proof. (by contradiction)
  - Suppose, for sake of contradiction, that Zoran is not matched upon termination of algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2 (only trading up, never becoming unmatched), Amy was never proposed to.
  - But, Zoran proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

- Claim. No unstable pairs.
- Proof. (by contradiction)
  - Suppose **A-Z** is an unstable pair: each prefers each other to partner in Gale-Shapley matching **S***.

  - Case 1: **Z** never proposed to **A**.
    ⇒ **Z** prefers his GS partner to **A**.
    ⇒ **A-Z** is stable.

    men propose in decreasing order of preference

  - Case 2: **Z** proposed to **A**.
    ⇒ **A** rejected **Z** (right away or later)
    ⇒ **A** prefers her GS partner to **Z**.
    ⇒ **A-Z** is stable.

    women only trade up

  - In either case **A-Z** is stable, a contradiction. ▪

S*
Amy-Yuri
Brenda-Zoran
. . .

Summary

- Stable matching problem. Given **n** men and **n** women, and their preferences, find a stable matching if one exists.
- Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.
- Q. How to implement GS algorithm efficiently?
- Q. If there are multiple stable matchings, which one does GS find?

Implementation for Stable Matching Algorithms

- Problem size
  - **$N=2n^2$** words
    - **2n** people each with a preference list of length **n**
  - **$2n^2$log n** bits
    - specifying an ordering for each preference list takes **nlog n** bits
- Brute force algorithm
  - Try all **n!** possible matchings
  - Do any of them work?
- Gale-Shapley Algorithm
  - **n^2** iterations, each costing constant time as follows:

Efficient Implementation

- Efficient implementation. We describe **$O(n^2)$** time implementation.
- Representing men and women.
  - Assume men are named **1, …, n**.
  - Assume women are named **1', …, n'**.
- Engagements.
  - Maintain a list of free men, e.g., in a queue.
  - Maintain two arrays **wife[m]**, and **husband[w]**.
    - set entry to **0** if unmatched
    - if **m** matched to **w** then **wife[m]=w** and **husband[w]=m**
- Men proposing.
  - For each man, maintain a list of women, ordered by preference.
  - Maintain an array **count[m]** that counts the number of proposals made by man **m**.

Efficient Implementation

- Women rejecting/accepting.
  - Does woman **w** prefer man **m** to man **m'**?
  - For each woman, create inverse of preference list of men.
  - Constant time access for each query after **O(n)** preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man **3** to **6** since `inverse[3]=2 < 7=inverse[6]`

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

	1st	2nd	3rd
Xavier	A	B	C
Yuri	B	A	C
Zoran	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Brenda	X	Y	Z
Claire	X	Y	Z

- An instance with two stable matchings.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?
- Def. Man **m** is a valid partner of woman **w** if there exists some stable matching in which they are matched.
- Man-optimal assignment. Each man receives **best** valid partner (according to his preferences).
- Claim. All executions of GS yield a **man-optimal** assignment, which is a stable matching!
  - No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
  - Simultaneously best for each and every man.

Man Optimality

S* S
Amy-Yuri
Brenda-Zoran
. . .

- Claim. GS matching **S*** is man-optimal.
- Proof. (by contradiction)
  - Suppose some man is paired with someone other than his best partner. Men propose in decreasing order of preference ⇒ some man is rejected by a valid partner.
  - Let **Y** be first such man, and let **A** be the first valid woman that rejects him.
  - Let **S** be a stable matching where **A** and **Y** are matched.
  - In building **S***, when **Y** is rejected, **A** forms (or reaffirms) engagement with a man, say **Z**, whom she prefers to **Y**.
  - Let **B** be **Z**'s partner in **S**.
  - In building **S***, **Z** is not rejected by any valid partner at the point when **Y** is rejected by **A**.
  - Thus, **Z** prefers **A** to **B**.
  - But **A** prefers **Z** to **Y**.
  - Thus **A-Z** is unstable in **S**. ▪

since this is the first rejection by a valid partner

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Stable Matching Summary

- Stable matching problem. Given preference profiles of **n** men and **n** women, find a stable matching.

no man and woman prefer to be with each other than with their assigned partner

- Gale-Shapley algorithm. Finds a stable matching in **O(n²)** time.
- Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of **m** if there exist some stable matching where **m** and **w** are paired

- Q. Does man-optimality come at the expense of the women?

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Woman Pessimality

- Woman-pessimal assignment. Each woman receives worst valid partner.
- Claim. GS finds woman-pessimal stable matching **S***.
- Proof.
  - Suppose **A-Z** matched in **S***, but **Z** is not worst valid partner for **A**.
  - There exists stable matching **S** in which **A** is paired with a man, say **Y**, whom she likes less than **Z**.
  - Let **B** be **Z**'s partner in **S**.
  - **Z** prefers **A** to **B**. ← man-optimality of **S***
  - Thus, **A-Z** is an unstable in **S**. ▪

S* S
Amy-Yuri
Brenda-Zoran
. . .

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Extensions: Matching Residents to Hospitals

- Ex: Men ≈ hospitals, Women ≈ med school residents.
- Variant 1. Some participants declare others as unacceptable.
- Variant 2. Unequal number of men and women.

e.g. resident **A** unwilling to work in Cleveland

- Variant 3. Limited polygamy.

e.g. hospital **X** wants to hire **3** residents

- Def. Matching **S** is unstable if there is a hospital **h** and resident **r** such that:
  - **h** and **r** are acceptable to each other; and
  - either **r** is unmatched, or **r** prefers **h** to her assigned hospital; and
  - either **h** does not have all its places filled, or **h** prefers **r** to at least one of its assigned residents.

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Application: Matching Residents to Hospitals

- NRMP. (National Resident Matching Program)
  - Original use just after WWII. ← predates computer usage
  - Ides of March, 23,000+ residents.
- Rural hospital dilemma.
  - Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
  - Rural hospitals were under-subscribed in NRMP matching.
  - How can we find stable matching that benefits "rural hospitals"?
- Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

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Lessons Learned

- Powerful ideas learned in course.
  - Isolate underlying structure of problem.
  - Create useful and efficient algorithms.
- Potentially deep social ramifications. [legal disclaimer]

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