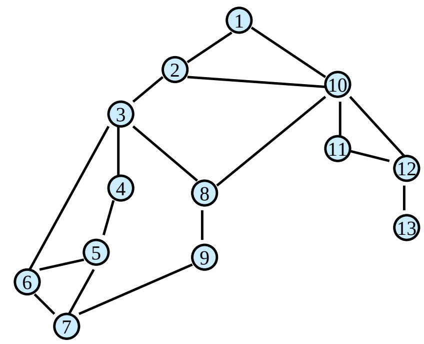
CSE 421: Introduction to Algorithms

Graph Traversal

Paul Beame

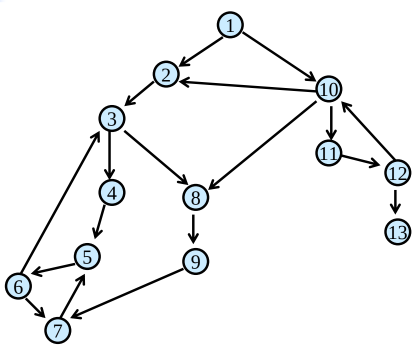
1

Undirected Graph G = (V,E)



2

Directed Graph G = (V,E)



3

Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex s to find all vertices reachable from s
- Three states of vertices
  - **unvisited**
  - **visited/discovered**
  - **fully-explored**

4

Generic Graph Traversal Algorithm

Find: set **R** of vertices reachable from **s∈V**

Reachable(**s**):
- **R**← {**s**}
- While there is a (**u**,**v**)∈**E** where **u**∈**R** and **v**∉**R**
  - Add **v** to **R**

5

Generic Traversal Always Works

- Claim: At termination **R** is the set of nodes reachable from **s**
- Proof
  - ⊆: For every node **v**∈**R** there is a path from s to v
  - ⊇: Suppose there is a node **w**∉**R** reachable from s via a path **P**
    - Take first node **v** on **P** such that **v**∉**R**
    - Predecessor **u** of **v** in **P** satisfies
      - **u ∈ R**
      - **(u,v)∈E**
    - But this contradicts the fact that the algorithm exited the while loop.

6

Breadth-First Search

- Completely explore the vertices in order of their distance from s
- Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(**s**)
    mark **s** "visited"; R←{s}; layer L_0←{s}
    while $\mathbf{L_i}$ not empty
        $\mathbf{L_{i+1} \leftarrow \varnothing}$
        For each $\mathbf{u \in L_i}$
            for each edge {**u**,**v**}
                if (**v** is "unvisited")
                    mark **v** "visited"
                    Add **v** to set **R** and to layer $\mathbf{L_{i+1}}$
            mark **u** "fully-explored"
        i ← i+1

Properties of BFS(v)

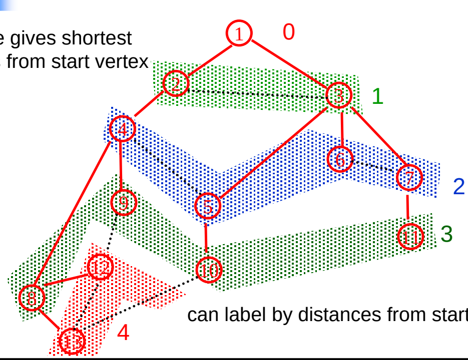
- BFS(**s**) visits **x** if and only if there is a path in **G** from **s** to **x**.
- Edges followed to undiscovered vertices define a "breadth first spanning tree" of **G**
- Layer **i** in this tree, $\mathbf{L_i}$
  - those vertices **u** such that the shortest path in **G** from the root **s** is of length **i**.
- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers

Properties of BFS

- On undirected graphs
  - All non-tree edges join vertices on the same or adjacent layers
  - Suppose not
    - Then there would be vertices (**x**,**y**) such that $\mathbf{x \in L_i}$ and $\mathbf{y \in L_j}$ and $\mathbf{j > i+1}$
    - Then, when vertices incident to **x** are considered in BFS **y** would be added to $\mathbf{L_{i+1}}$ and not to $\mathbf{L_j}$

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - Given: vertices **u** and **v** in **G**
  - Is there a path from **u** to **v**?
- Idea: create array **A** such that
  A[u] = smallest numbered vertex that is connected to **u**
  - question reduces to whether A[u]=A[v]?

Q: Why not create an array **Path[u,v]**?

Graph Search Application: Connected Components

- initial state: all **v** unvisited
  for **s**←**1** to **n** do
      if state(**s**) ≠ "fully-explored" then
          BFS(**s**): setting **A**[**u**] ←**s** for each **u** found
              (and marking **u** visited/fully-explored)
    endif
  endfor
- Total cost: **O(n+m)**
  - each vertex is touched once in this outer procedure and the edges examined in the different BFS runs are disjoint
  - works also with Depth First Search

13

DFS(u) – Recursive version

Global Initialization: mark all vertices "unvisited"
DFS(**u**)
  mark **u** "visited" and add **u** to **R**
  for each edge {**u**,**v**}
    if (**v** is "unvisited")
      DFS(**v**)
  end for
  mark **u** "fully-explored"

14

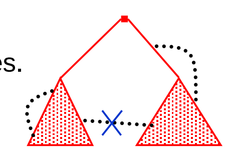
Properties of DFS(s)

- Like BFS(**s**):
  - DFS(**s**) visits **x** if and only if there is a path in **G** from **s** to **x**
  - Edges into undiscovered vertices define a "depth first spanning tree" of **G**
- Unlike the BFS tree:
  - the DFS spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

15

Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- No cross edges.



16

No cross edges in DFS on undirected graphs

- Claim: During **DFS(x)** every vertex marked visited is a descendant of **x** in the DFS tree **T**
- Claim: For every **x,y** in the DFS tree **T**, if **(x,y)** is an edge not in **T** then one of **x** or **y** is an ancestor of the other in **T**
- Proof:
  - One of **x** or **y** is visited first, suppose WLOG that **x** is visited first and therefore **DFS(x)** was called before **DFS(y)**
    - During **DFS(x)**, the edge **(x,y)** is examined
  - Since **(x,y)** is a not an edge of **T**, **y** was visited when the edge **(x,y)** was examined during **DFS(x)**
  - Therefore **y** was visited during the call to **DFS(x)** so **y** is a descendant of **x**.

17

Applications of Graph Traversal: Bipartiteness Testing

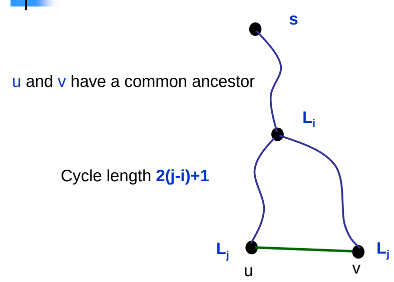
- Easy: A graph G is not bipartite if it contains an odd length cycle
- WLOG: G is connected
  - Otherwise run on each component
- Simple idea: start coloring nodes starting at a given node s
  - Color s red
  - Color all neighbors of s blue
  - Color all their neighbors red
  - If you ever hit a node that was already colored
    - the same color as you want to color it, ignore it
    - the opposite color, output error

18

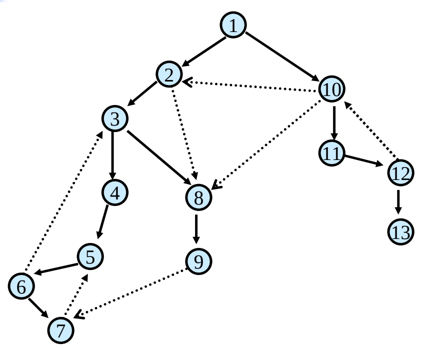
BFS gives Bipartiteness

- Run BFS assigning all vertices from layer L_i the color i mod 2
  - i.e. red if they are in an even layer, blue if in an odd layer
- If there is an edge joining two vertices from the same layer then output "Not Bipartite"

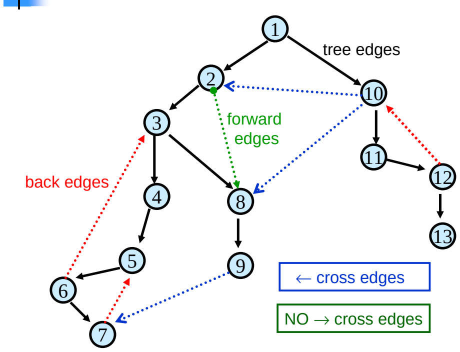
Why does it work?



DFS(v) for a directed graph



DFS(v)



Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s
- Every cycle contains a back edge in the DFS tree

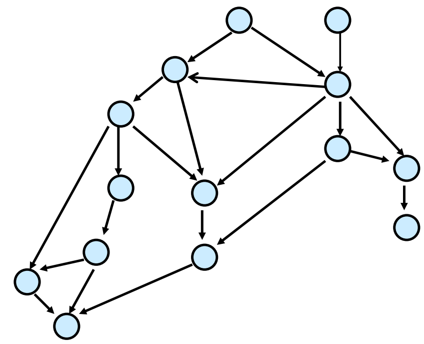
Directed Acyclic Graphs

- A directed graph G=(V,E) is acyclic if it has no directed cycles
- Terminology: A directed acyclic graph is also called a DAG

Topological Sort

- **Given:** a directed acyclic graph (DAG) **G=(V,E)**
- **Output:** numbering of the vertices of **G** with distinct numbers from **1** to **n** so edges only go from lower number to higher numbered vertices
- Applications
  - nodes represent tasks
  - edges represent precedence between tasks
  - topological sort gives a sequential schedule for solving them

Directed Acyclic Graph



In-degree 0 vertices

- Every DAG has a vertex of in-degree 0
- Proof: By contradiction
  - Suppose every vertex has some incoming edge
  - Consider following procedure:

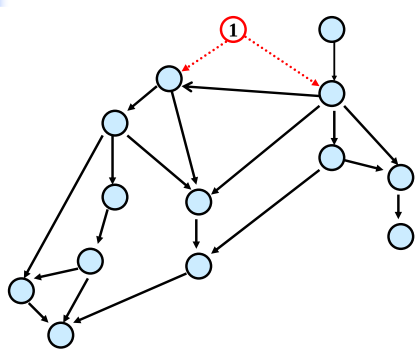
    while (true) do

        v←some predecessor of **v**
  - After **n+1** steps where **n**=|**V**| there will be a repeated vertex
    - This yields a cycle, contradicting that it is a DAG

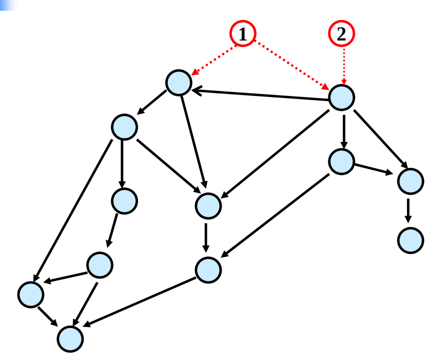
Topological Sort

- Can do using DFS
- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph and then give a vertex of in-degree 0 number 2, etc.

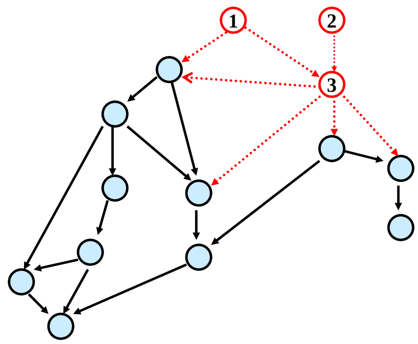
Topological Sort



Topological Sort

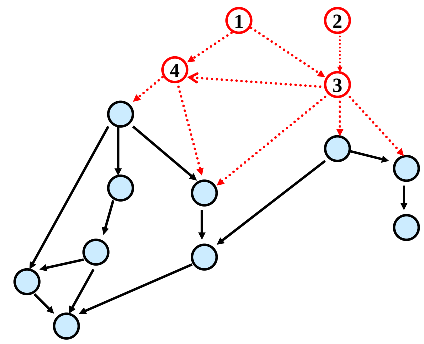


Topological Sort



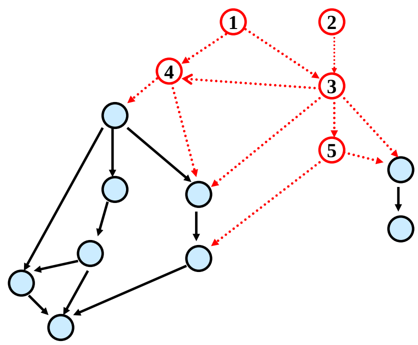
31

Topological Sort



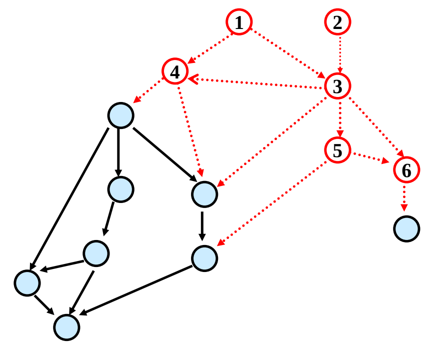
32

Topological Sort



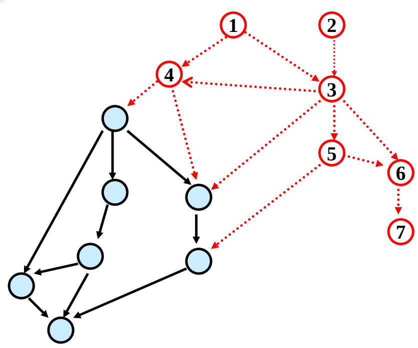
33

Topological Sort



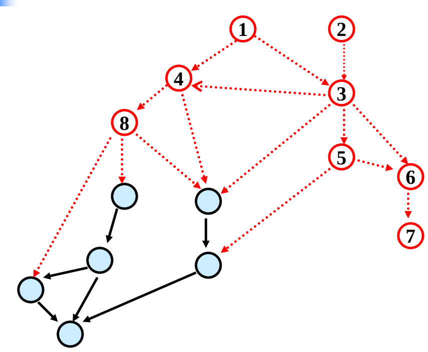
34

Topological Sort



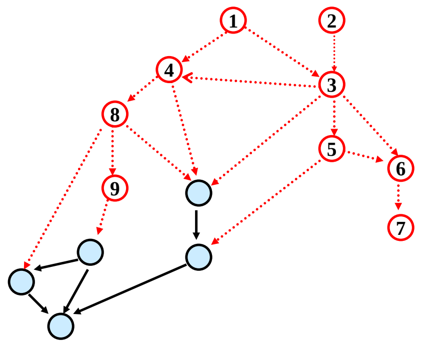
35

Topological Sort



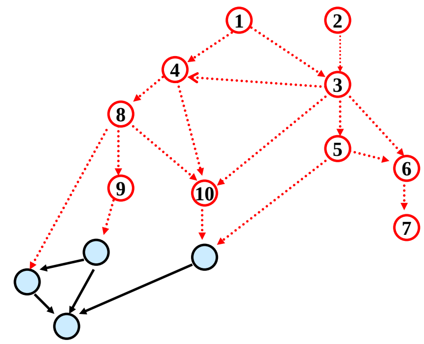
36

Topological Sort



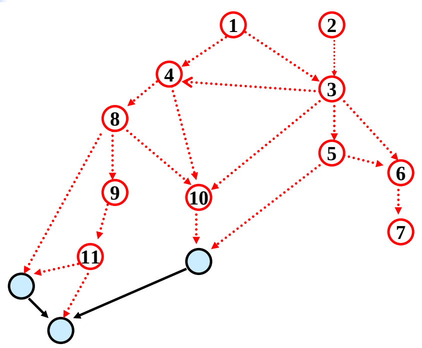
37

Topological Sort



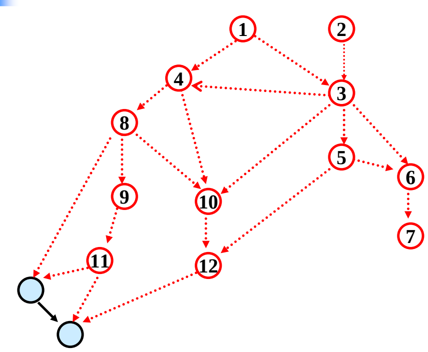
38

Topological Sort



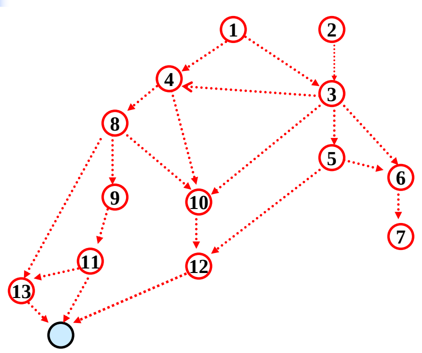
39

Topological Sort



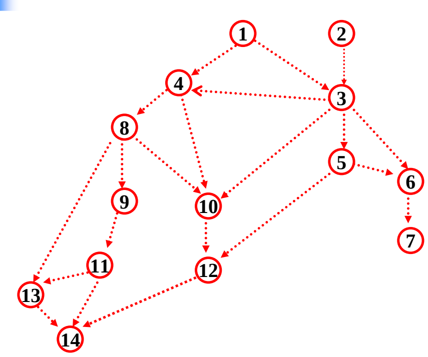
40

Topological Sort



41

Topological Sort



42

Implementing Topological Sort

- Go through all edges, computing in-degree for each vertex **O(m+n)**
- Maintain a queue (or stack) of vertices of in-degree **0**
- Remove any vertex in queue and number it
- When a vertex is removed, decrease in-degree of each of its neighbors by **1** and add them to the queue if their degree drops to **0**
- Total cost **O(m+n)**

43