CSE 421: Introduction to Algorithms

Divide and Conquer

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Algorithm Design Techniques

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Typically, each sub-problem is **at most a constant fraction** of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

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Fast exponentiation

- Power(**a**,**n**)
  - **Input:** integer **n** and number **a**
  - **Output:** a^n
- Obvious algorithm
  - **n-1** multiplications
- Observation:
  - if **n** is even, **n=2m**, then $a^n=a^m \cdot a^m$

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Divide & Conquer Algorithm

- Power(**a**,**n**)

```
if n=0 then return(1)
else if n=1 then return(a)
else
    x ←Power(a,⌊n/2⌋)
    if n is even then
        return(x•x)
    else
        return(a•x•x)
```

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Analysis

- Worst-case recurrence
  - $T(n)=T(\lfloor n/2 \rfloor)+2$ for $n \geq 1$
  - $T(1)=0$
- Time
  - $T(n)=T(\lfloor n/2 \rfloor)+2 \leq T(\lfloor n/4 \rfloor)+2+2 \leq \ldots$
    $\leq T(1)+\underbrace{2+\ldots+2}_{\log_2 n \text{ copies}} = 2 \log_2 n$
- More precise analysis:
  - $T(n)= \lceil \log_2 n \rceil$ + # of **1**'s in **n**'s binary representation

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A Practical Application- RSA

- Instead of a^n want a^n **mod N**
  - a^{i+j} mod N = ((a^i mod N)•(a^j mod N)) mod N
  - same algorithm applies with each **x•y** replaced by
    - ((**x** mod **N**)•(**y** mod **N**)) mod **N**
- In RSA cryptosystem (widely used for security)
  - need a^n **mod N** where **a**, **n**, **N** each typically have **1024** bits
  - Power: at most **2048** multiplies of 1024 bit numbers
    - relatively easy for modern machines
  - Naive algorithm: 2^{1024} multiplies

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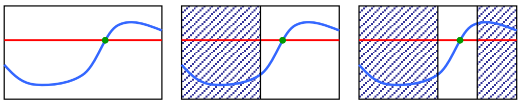
Binary search for roots (bisection method)



- Given:
  - continuous function **f** and two points **a<b** with **f(a) ≤ 0** and **f(b) > 0**
- Find:
  - approximation to **c** s.t. **f(c)=0** and **a<c<b**

Bisection method

Bisection(**a**,**b**, **ε**)
  if (**a**-**b**) < **ε** then
    return(**a**)
  else
    c ←(**a**+**b**)/2
    if **f**(**c**) ≤ **0** then
      return(Bisection(**c**,**b**,**ε**))
    else
      return(Bisection(**a**,**c**,**ε**))

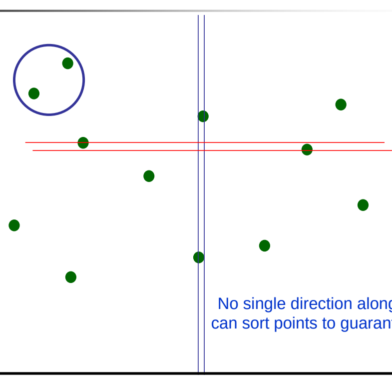
Time Analysis

- At each step we halved the size of the interval
- It started at size **b-a**
- It ended at size **ε**
- # of calls to f is $\log_2((b-a)/\varepsilon)$

Euclidean Closest Pair

- Given a set **P** of n points $p_1,\ldots,p_n$ with real-valued coordinates
- Find the pair of points $p_i,p_j \in P$ such that the Euclidean distance $d(p_i,p_j)$ is minimized
- $\Theta(n^2)$ possible pairs
- In one dimension there is an easy $O(n \log n)$ algorithm
  - Sort the points
  - Compare consecutive elements in the sorted list
- What about points in the plane?

Closest Pair in the Plane

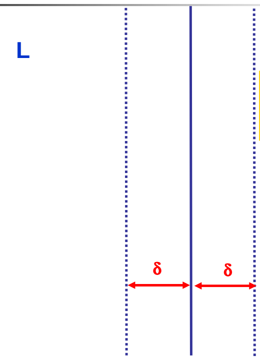


No single direction along which one can sort points to guarantee success!

Closest Pair In the Plane: Divide and Conquer

- Sort the points by their **x** coordinates
- Split the points into two sets of **n**/2 points **L** and **R** by **x** coordinate
- Recursively compute
  - closest pair of points in **L**, (p_L,q_L)
  - closest pair of points in **R**, (p_R,q_R)
- Let $\delta=\min\{d(p_L,q_L),d(p_R,q_R)\}$ and let (p,q) be the pair of points that has distance δ
- This may not be enough!
  - Closest pair of points may involve one point from **L** and the other from **R**!

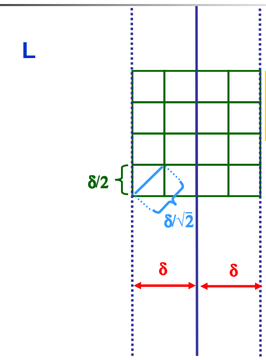
A clever geometric idea



R

Any pair of points $p \in L$ and $q \in R$ with $d(p,q) < \delta$ must lie in band

A clever geometric idea

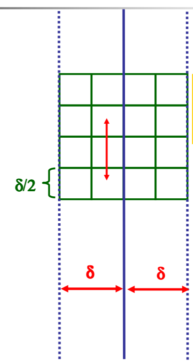


R

Any pair of points $p \in L$ and $q \in R$ with $d(p,q) < \delta$ must lie in band

No two points can be in the same green box

A clever geometric idea



R

Any pair of points $p \in L$ and $q \in R$ with $d(p,q) < \delta$ must lie in band

No two points can be in the same green box

Only need to check pairs of points up to 2 rows above and below - At most **15** other points!

Closest Pair Recombining

- Sort points by **y** coordinate ahead of time
- On recombination only compares each point in **L∪R** to the **12** points above it in the **y** sorted order
- If any of those distances is better than **δ** replace **(p,q)** by the best of those pairs
- **O(n log n)** for **x** and **y** sorting at start
- Two recursive calls on problems on half size
- **O(n)** recombination
- Total **O(n log n)**

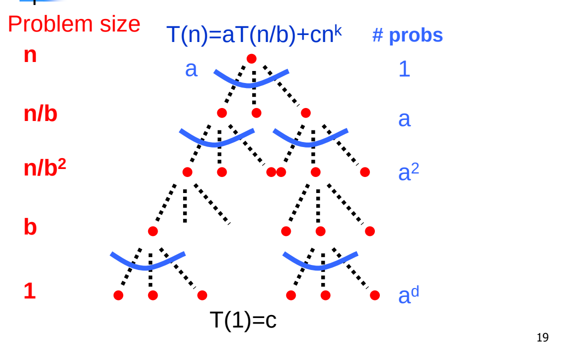
Sometimes two sub-problems aren't enough

- More general divide and conquer
  - You've broken the problem into **a** different sub-problems
  - Each has size at most **n/b**
  - The cost of the break-up and recombining the sub-problem solutions is $O(n^k)$
- Recurrence
  - $T(n) \le a \cdot T(n/b) + c \cdot n^k$

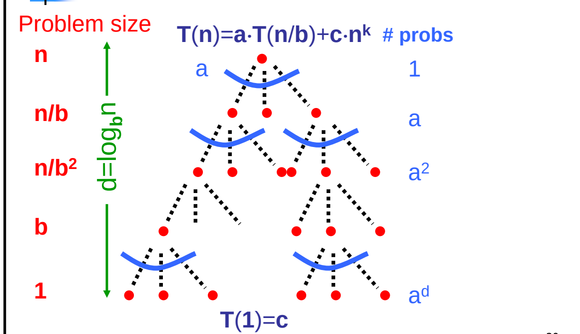
Master Divide and Conquer Recurrence

- If $T(n) \le a \cdot T(n/b) + c \cdot n^k$ for $n > b$ then
  - if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$
  - if $a < b^k$ then $T(n)$ is $\Theta(n^k)$
  - if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

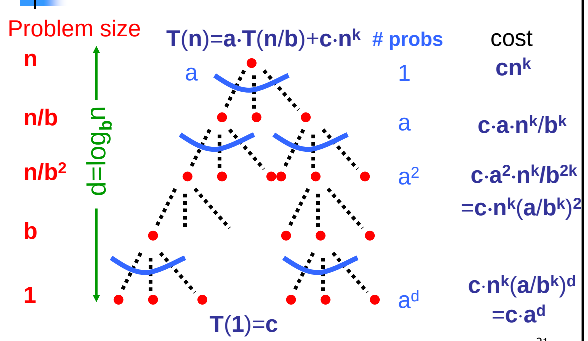
Proving Master recurrence



Proving Master recurrence



Proving Master recurrence



Geometric Series

- $S \;= t + tr + tr^2 + \ldots + tr^{n-1}$
- $r\cdot S = \quad\; tr + tr^2 + \ldots + tr^{n-1} + tr^n$
- $(r-1)S = tr^n - t$
- so $S=t(r^n-1)/(r-1)$ if $r\neq 1$.

- Simple rule
  - If $r \neq 1$ then **S** is a constant times largest term in series

Total Cost

- Geometric series
  - ratio a/b^k
  - $d+1=\log_b n+1$ terms
  - first term cn^k, last term ca^d
- If $a/b^k=1$
  - all terms are equal $T(n)$ is $\Theta(n^k \log n)$
- If $a/b^k<1$
  - first term is largest $T(n)$ is $\Theta(n^k)$
- If $a/b^k>1$
  - last term is largest $T(n)$ is $\Theta(a^d)=\Theta(a^{\log_b n})=\Theta(n^{\log_b a})$
  (To see this take $\log_b$ of both sides)

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

- n^3 multiplications, n^3-n^2 additions

Multiplying Matrices

```
for i=1 to n
    for j=1 to n
        C[i,j]←0
        for k=1 to n
            C[i,j]=C[i,j]+A[i,k]·B[k,j]
        endfor
    endfor
endfor
```

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Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

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Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

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Multiplying Matrices

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$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \bullet \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

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Simple Divide and Conquer

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

- $T(n)=8T(n/2)+4(n/2)^2=8T(n/2)+n^2$
  - $8>2^2$ so $T(n)$ is $\Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$

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Strassen's Divide and Conquer Algorithm

- Strassen's algorithm
  - Multiply 2×2 matrices using **7** instead of **8** multiplications (and lots more than **4** additions)
  - $T(n)=7\,T(n/2)+cn^2$
    - $7>2^2$ so $T(n)$ is $\Theta(n^{\log_2 7})$ which is $O(n^{2.81\ldots})$
  - Fastest algorithms theoretically use $O(n^{2.376})$ time
    - not practical but Strassen's is practical **provided calculations are exact** and we stop recursion when matrix has size about **100** (maybe **10**)

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The algorithm

$P_1 \leftarrow A_{12}(B_{11}+B_{21})$; $P_2 \leftarrow A_{21}(B_{12}+B_{22})$

$P_3 \leftarrow (A_{11} - A_{12})B_{11}$; $P_4 \leftarrow (A_{22} - A_{21})B_{22}$

$P_5 \leftarrow (A_{22} - A_{12})(B_{21} - B_{22})$

$P_6 \leftarrow (A_{11} - A_{21})(B_{12} - B_{11})$

$P_7 \leftarrow (A_{21} - A_{12})(B_{11}+B_{22})$

$C_{11} \leftarrow P_1+P_3$; $C_{12} \leftarrow P_2+P_3+P_6 - P_7$

$C_{21} \leftarrow P_1+P_4+P_5+P_7$; $C_{22} \leftarrow P_2+P_4$

Another Divide &Conquer Example: Multiplying Faster

- If you analyze our usual grade school algorithm for multiplying numbers
  - $\Theta(n^2)$ time
  - On real machines each "digit" is, e.g., **32** bits long but still get $\Theta(n^2)$ running time with this algorithm when run on **n**-bit multiplication
- We can do better!
  - We'll describe the basic ideas by multiplying polynomials rather than integers
  - Advantage is we don't get confused by worrying about carries at first

Notes on Polynomials

- These are just formal sequences of coefficients
  - when we show something multiplied by x^k it just means shifted **k** places to the left – basically no work

Usual polynomial multiplication

$$\begin{array}{r} 4x^2 + 2x + 2 \\ x^2 - 3x + 1 \\ \hline 4x^2 + 2x + 2 \\ -12x^3 - 6x^2 - 6x \quad \\ 4x^4 + 2x^3 + 2x^2 \qquad\qquad \\ \hline 4x^4 - 10x^3 + 0x^2 - 4x + 2 \end{array}$$

Polynomial Multiplication

- **Given:**
  - Degree **n-1** polynomials **P** and **Q**
    - $P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$
    - $Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-2}x^{n-2} + b_{n-1}x^{n-1}$
- **Compute:**
  - Degree **2n-2** Polynomial **P Q**
  - $PQ = a_0b_0 + (a_0b_1+a_1b_0)\,x + (a_0b_2+a_1b_1+a_2b_0)\,x^2 + \ldots + (a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\,x^{2n-3} + a_{n-1}b_{n-1}\,x^{2n-2}$
- **Obvious Algorithm:**
  - Compute all a_ib_j and collect terms
  - $\Theta(n^2)$ time

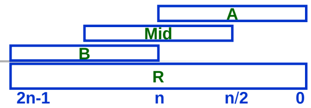
Naive Divide and Conquer

- Assume **n=2k**
  - $P = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-2} x^{k-2} + a_{k-1} x^{k-1}) + (a_k + a_{k+1} x + \ldots + a_{n-2}x^{k-2} + a_{n-1}x^{k-1})\, x^k$ $= P_0 + P_1 x^k$ where P_0 and P_1 are degree **k-1** polynomials
  - Similarly $Q = Q_0 + Q_1 x^k$
- $PQ = (P_0+P_1x^k)(Q_0+Q_1x^k)$ $= P_0Q_0 + (P_1Q_0+P_0Q_1)x^k + P_1Q_1x^{2k}$
- **4** sub-problems of size **k=n/2** plus linear combining
  - $T(n)=4 \cdot T(n/2)+cn$ Solution $T(n) = \Theta(n^2)$

Karatsuba's Algorithm

- A better way to compute the terms
  - Compute
    - $A \leftarrow P_0Q_0$
    - $B \leftarrow P_1Q_1$
    - $C \leftarrow (P_0+P_1)(Q_0+Q_1) = P_0Q_0+P_1Q_0+P_0Q_1+P_1Q_1$
  - Then
    - $P_0Q_1+P_1Q_0 = C-A-B$
    - So $PQ=A+(C-A-B)x^k+Bx^{2k}$
  - **3** sub-problems of size **n/2** plus **O(n)** work
    - $T(n) = 3\,T(n/2) + cn$
    - $T(n) = O(n^{\alpha})$ where $\alpha = \log_2 3 = 1.59\ldots$

Karatsuba: Details



PolyMul(**P**, **Q**):

// **P**, **Q** are length **n** =**2k** vectors, with **P[i]**, **Q[i]** being
// the coefficient of x^i in polynomials **P**, **Q** respectively.
// Let **Pzero** be elements **0..k-1** of **P**; **Pone** be elements **k..n-1**
// **Qzero**, **Qone** : similar

If **n=1** then Return(**P[0]*****Q[0]**) else
 A ← PolyMul(**Pzero**, **Qzero**); // result is a (**2k-1**)-vector
 B ← PolyMul(**Pone**, **Qone**); // ditto
 Psum ← **Pzero** + **Pone**; // add corresponding elements
 Qsum ← **Qzero** + **Qone**; // ditto
 C ← polyMul(**Psum**, **Qsum**); // another (**2k-1**)-vector
 Mid ← **C** – **A** – **B**; // subtract correspond elements
 R ← **A** + Shift(**Mid**, **n/2**) +Shift(**B**,**n**) // a (**2n-1**)-vector
 Return(**R**);

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Multiplication

- Polynomials
  - Naïve: $\Theta(n^2)$
  - Karatsuba: $\Theta(n^{1.59\ldots})$
  - Best known: Θ(**n log n**)
    - "Fast Fourier Transform"
    - FFT widely used for signal processing
- Integers
  - Similar, but some ugly details re: carries, etc. gives Θ(**n** log **n** loglog **n**),
    - mostly unused in practice except for symbolic manipulation systems like Maple

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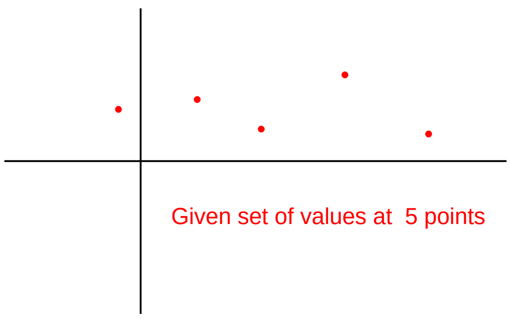
Hints towards FFT: Interpolation



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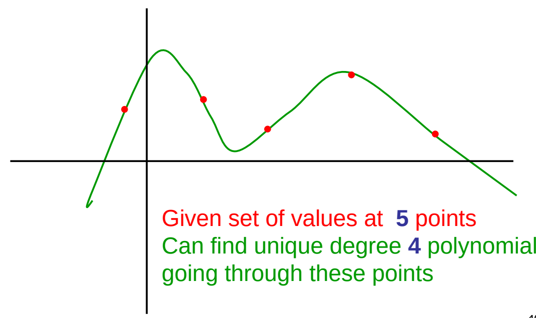
Hints towards FFT: Interpolation



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Interpolation

- Given values of degree **n-1** polynomial **R** at **n** distinct points $y_1,\ldots,y_n$
  - $R(y_1),\ldots,R(y_n)$
- Compute coefficients $c_0,\ldots,c_{n-1}$ such that
  - $R(x)=c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}$
- System of linear equations in $c_0,\ldots,c_{n-1}$

$c_0 + c_1y_1 + c_2y_1^2 + \ldots + c_{n-1}y_1^{n-1} = R(y_1)$ known
$c_0 + c_1y_2 + c_2y_2^2 + \ldots + c_{n-1}y_2^{n-1} = R(y_2)$
…
$c_0 + c_1y_n + c_2y_n^2 + \ldots + c_{n-1}y_n^{n-1} = R(y_n)$ unknown

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Interpolation: n equations in n unknowns

- Matrix form of the linear system

$$\begin{pmatrix} 1 & y_1 & y_1^2 & \ldots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \ldots & y_2^{n-1} \\ & \ldots & & & \\ & \ldots & & & \\ 1 & y_n & y_n^2 & \ldots & y_n^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} R(y_1) \\ R(y_2) \\ . \\ . \\ R(y_n) \end{pmatrix}$$

- Fact: Determinant of the matrix is $\prod_{i<j}(y_i-y_j)$ which is not **0** since points are distinct
  - System has a unique solution $c_0,\ldots,c_{n-1}$

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Hints towards FFT: Evaluation & Interpolation

P: $a_0,a_1,\ldots,a_{n-1}$
Q: $b_0,b_1,\ldots,b_{n-1}$

ordinary polynomial multiplication $\Theta(n^2)$

$c_k \leftarrow \sum_{i+j=k} a_i b_j$

R: $c_0,c_1,\ldots,c_{2n-1}$

evaluation at $y_0,\ldots,y_{2n-1}$ O(?)

interpolation from $y_0,\ldots,y_{2n-1}$ O(?)

$P(y_0),Q(y_0)$
$P(y_1),Q(y_1)$
...
$P(y_{2n-1}),Q(y_{2n-1})$

point-wise multiplication of numbers $O(n)$

$R(y_0)\leftarrow P(y_0)\cdot Q(y_0)$
$R(y_1)\leftarrow P(y_1)\cdot Q(y_1)$
...
$R(y_{2n-1})\leftarrow P(y_{2n-1})\cdot Q(y_{2n-1})$

Karatsuba's algorithm and evaluation and interpolation

- Strassen gave a way of doing 2x2 matrix multiplies with fewer multiplications
- Karatsuba's algorithm can be thought of as a way of multiplying degree 1 polynomials (which have 2 coefficients) using fewer multiplications
  - $PQ=(P_0+P_1z)(Q_0+Q_1z)$
    $= P_0Q_0 + (P_1Q_0+P_0Q_1)z + P_1Q_1z^2$
  - Evaluate at **0,1,-1** (Could also use other points)
    - $A = P(0)Q(0)= P_0Q_0$
    - $C = P(1)Q(1)=(P_0+P_1)(Q_0+Q_1)$
    - $D = P(-1)Q(-1)=(P_0-P_1)(Q_0-Q_1)$
  - Interpolating, Karatsuba's **Mid=(C-D)/2** and **B=(C+D)/2-A**

Hints towards FFT: Evaluation at Special Points

- Evaluation of polynomial at 1 point takes $O(n)$
  - So $2n$ points (naively) takes $O(n^2)$—no savings
- Key trick:
  - use carefully chosen points where there's some sharing of work for several points, namely various powers of $\omega = e^{2\pi i/n},\ i=\sqrt{-1}$
- Plus more Divide & Conquer.
- Result:
  - both evaluation and interpolation in $O(n \log n)$ time

Fun facts about $\omega=e^{2\pi i/n}$ for even n

- $\omega^n = 1$
- $\omega^{n/2} = -1$
- $\omega^{n/2+k} = -\omega^k$ for all values of k
- $\omega^2 = e^{2\pi i/m}$ where $m=n/2$
- $\omega^k = \cos(2k\pi/n)+i\sin(2k\pi/n)$ so can compute with powers of ω

The key idea for n even

- $P(\omega) = a_0+a_1\omega+a_2\omega^2+a_3\omega^3+a_4\omega^4+\ldots+a_{n-1}\omega^{n-1}$
  $= a_0 +a_2\omega^2 +a_4\omega^4 +\ldots+ a_{n-2}\omega^{n-2}$
  $+ a_1\omega+a_3\omega^3 +a_5\omega^5 +\ldots+a_{n-1}\omega^{n-1}$
  $= P_{even}(\omega^2) + \omega P_{odd}(\omega^2)$
- $P(-\omega)=a_0 -a_1\omega+a_2\omega^2 -a_3\omega^3+a_4\omega^4-\ldots\ -a_{n-1}\omega^{n-1}$
  $= a_0 +a_2\omega^2 +a_4\omega^4 +\ldots+ a_{n-2}\omega^{n-2}$
  $- (a_1\omega+a_3\omega^3 +a_5\omega^5 +\ldots+a_{n-1}\omega^{n-1})$
  $= P_{even}(\omega^2) - \omega P_{odd}(\omega^2)$

where $P_{even}(x) = a_0 +a_2x +a_4x^2 +\ldots+ a_{n-2}x^{n/2-1}$

and $P_{odd}(x) = a_1+a_3x +a_5x^2 +\ldots+a_{n-1}x^{n/2-1}$

The recursive idea for n a power of 2

- Also
  - P_{even} and P_{odd} have degree $n/2$ where
  - $P(\omega^k)=P_{even}(\omega^{2k})+\omega^kP_{odd}(\omega^{2k})$
  - $P(-\omega^k)=P_{even}(\omega^{2k})-\omega^kP_{odd}(\omega^{2k})$
- Recursive Algorithm
  - Evaluate P_{even} at $1,\omega^2,\omega^4,\ldots,\omega^{n-2}$
  - Evaluate P_{odd} at $1,\omega^2,\omega^4,\ldots,\omega^{n-2}$
  - Combine to compute P at $1,\omega,\omega^2,\ldots,\omega^{n/2-1}$
  - Combine to compute P at $-1,-\omega,-\omega^2,\ldots,-\omega^{n/2-1}$ (i.e. at $\omega^{n/2},\ \omega^{n/2+1},\ \omega^{n/2+2},\ldots,\ \omega^{n-1}$)

ω^2 is $e^{2\pi i/m}$ where $m=n/2$ so problems are of same type but smaller size

Analysis and more

- Run-time
  - $T(n)=2 \cdot T(n/2)+cn$ so $T(n)=O(n \log n)$
- So much for evaluation ... what about interpolation?
  - Given
    - $r_0=R(1)$, $r_1=R(\omega)$, $r_2=R(\omega^2)$,..., $r_{n-1}=R(\omega^{n-1})$
  - Compute
    - $c_0, c_1,\ldots,c_{n-1}$ s.t. $R(x)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}$

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Interpolation ≈ Evaluation: strange but true

- Weird fact:
  - If we define a new polynomial $S(x) = r_0 + r_1x + r_2x^2 +\ldots+ r_{n-1}x^{n-1}$ where $r_0, r_1, \ldots, r_{n-1}$ are the evaluations of R at $1, \omega, \ldots, \omega^{n-1}$
  - Then $c_k=S(\omega^{-k})/n$ for $k=0,\ldots,n-1$
- So...
  - evaluate S at $1,\omega^{-1},\omega^{-2},\ldots,\omega^{-(n-1)}$ then divide each answer by n to get the $c_0,\ldots,c_{n-1}$
  - ω^{-1} behaves just like ω did so the same $O(n \log n)$ evaluation algorithm applies !

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Divide and Conquer Summary

- Powerful technique, when applicable
- Divide large problem into a few smaller problems of the same type
- Choosing sub-problems of roughly equal size is usually critical
- Examples:
  - Merge sort, quicksort (sort of), polynomial multiplication, FFT, Strassen's matrix multiplication algorithm, powering, binary search, root finding by bisection, …

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Why this is called the discrete Fourier transform

- Real Fourier series
  - Given a real valued function f defined on $[0,2\pi]$ the Fourier series for f is given by $f(x)=a_0+a_1 \cos(x) + a_2 \cos(2x) +\ldots+ a_m \cos(mx) +\ldots$ where

$$\boxed{a_m=\frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx}$$

  - is the component of f of frequency m
  - In signal processing and data compression one ignores all but the components with large a_m and there aren't many since

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Why this is called the discrete Fourier transform

- Complex Fourier series
  - Given a function f defined on $[0,2\pi]$ the complex Fourier series for f is given by $f(z)=b_0+b_1 e^{iz} + b_2 e^{2iz} +\ldots+ b_m e^{miz} +\ldots$ where

$$\boxed{b_m=\frac{1}{2\pi}\int_0^{2\pi} f(z)\,e^{-miz}\,dz}$$

    is the component of f of frequency m
  - If we **discretize** this integral using values at n **2π/n apart** equally spaced points between 0 and 2π we get

$$\boxed{\bar{b}_m = \frac{1}{n}\sum_{k=0}^{n-1} f_k\, e^{-2km\pi i/n} = \frac{1}{n}\sum_{k=0}^{n-1} f_k\, \omega^{-km} \quad \text{where } f_k=f(2k\pi/n)}$$

just like interpolation!

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