CSE 421: Intro Algorithms

2: Analysis

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Defining Efficiency

"Runs fast on typical real problem instances"

Pro:

- sensible, bottom-line-oriented

Con:

- moving target (diff computers, compilers, Moore's law)
- highly subjective (how fast is "fast"? what's "typical"?)

Efficiency

Our correct TSP algorithm was incredibly slow

Basically slow no matter what computer you have

We want a general theory of "efficiency" that is

- Simple
- Objective
- Relatively independent of changing technology
- But still predictive - "theoretically bad" algorithms should be bad in practice and vice versa (usually)

Measuring efficiency

Time ≈ # of instructions executed in a simple programming language

- only simple operations (+,*,-,=,if,call,…)
- each operation takes one time step
- each memory access takes one time step
- no fancy stuff (add these two matrices, copy this long string,…) built in; write it/charge for it as above

No fixed bound on the memory size

We left out things but...

- Things we've dropped
  - memory hierarchy
    - disk, caches, registers have many orders of magnitude differences in access time
  - not all instructions take the same time in practice
  - different computers have different primitive instructions
- However,
  - the RAM model is useful for designing algorithms and measuring their efficiency
  - one can usually tune implementations so that the hierarchy etc. is not a huge factor

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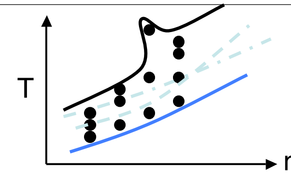
Complexity analysis



- Problem size n
  - Worst-case complexity: max # steps algorithm takes on any input of size n
  - Best-case complexity: min # steps algorithm takes on any input of size n
  - Average-case complexity: avg # steps algorithm takes on inputs of size n

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Pros and cons:

- Best-case
  - unrealistic oversell
- Average-case
  - over what probability distribution? (different people may have different "average" problems)
  - analysis often hard
- Worst-case
  - a fast algorithm has a comforting guarantee
  - maybe too pessimistic

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Why Worst-Case Analysis?

- Appropriate for time-critical applications, e.g. avionics
- Unlike Average-Case, no debate about what the right definition is
  - If worst >> average, then (a) alg is doing something pretty subtle, & (b) are hard instances really that rare?
- Analysis often easier
- Result is often representative of "typical" problem instances
- Of course there are exceptions…

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General Goals

Characterize growth rate of (worst-case) run time as a function of problem size, up to a constant factor

Why not try to be more precise?

- Technological variations (computer, compiler, OS, …) easily 10x or more
- Being more precise is a ton of work
- A key question is "scale up": if I can afford to do it today, how much longer will it take when my business problems are twice as large? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

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Complexity

The complexity of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

Mathematically,

- $T: N+ \rightarrow R+$
- that is T is a function that maps positive integers (giving problem sizes) to positive real numbers (giving number of steps).

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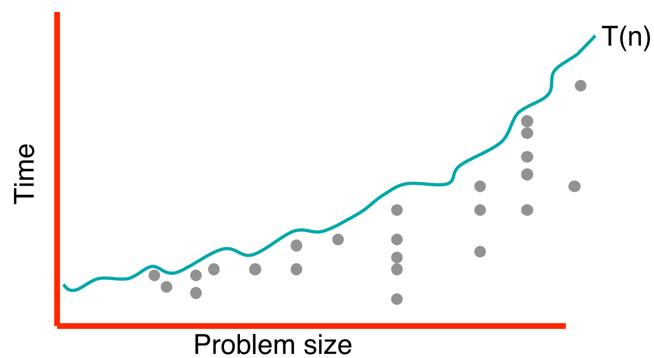
Complexity



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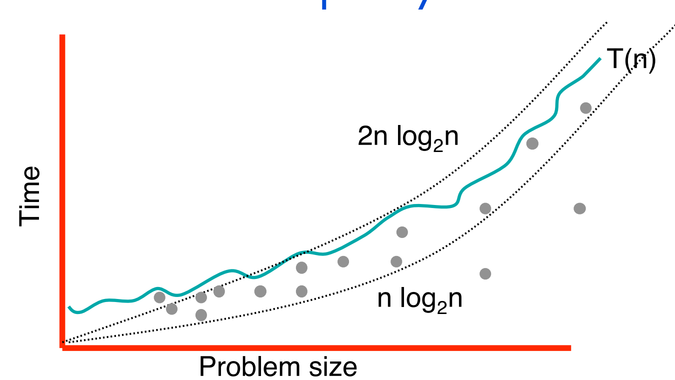
Complexity



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O-notation etc

Given two functions f and $g:N \to R$

f(n) is O(g(n)) iff there is a constant $c>0$ so that
f(n) is eventually always $\leq c\, g(n)$

f(n) is Ω (g(n)) iff there is a constant $c>0$ so that
f(n) is eventually always $\geq c\, g(n)$

f(n) is Θ (g(n)) iff there is are constants c_1, $c_2>0$ so that
eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

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Examples

$10n^2\text{-}16n+100$ is $O(n^2)$ also $O(n^3)$

$10n^2\text{-}16n+100 \leq 11n^2$ for all $n \geq 10$

$10n^2\text{-}16n+100$ is Ω (n^2) also Ω (n)

$10n^2\text{-}16n+100 \geq 9n^2$ for all $n \geq 16$

Therefore also $10n^2\text{-}16n+100$ is Θ (n^2)

$10n^2\text{-}16n+100$ is not O(n) also not Ω (n^3)

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Properties

Transitivity.

If f = O(g) and g = O(h) then f = O(h).
If f = Ω(g) and g = Ω(h) then f = Ω(h).
If f = Θ(g) and g = Θ(h) then f = Θ(h).

Additivity.

If f = O(h) and g = O(h) then f + g = O(h).
If f = Ω(h) and g = Ω(h) then f + g = Ω(h).
If f = Θ(h) and g = O(h) then f + g = Θ(h).

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"One-Way Equalities"

2 + 2 is 4	$2n^2 + 5n$ is $O(n^3)$
2 + 2 = 4	$2n^2 + 5n = O(n^3)$
4 = 2 + 2	~~$O(n^3) = 2n^2 + 5n$~~
All dogs are mammals	~~All mammals are dogs~~

Bottom line:

OK to put big-O in R.H.S. of equality, but not left.
[Better, but uncommon, notation: $T(n) \in O(f(n))$.]

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Working with O-Ω-Θ notation

Claim: For any a, and any b>0, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \le (2n)^b$ for $n \ge |a|$
$= 2^b n^b$
$= cn^b$ for $c = 2^b$
so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if a <0)
$= 2^{-b} n^b$
$= c'n$ for $c' = 2^{-b}$
so $(n+a)^b$ is $\Omega(n^b)$

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Working with O-Ω-Θ notation

Claim: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$

$\log_a b = x$ means $a^x = b$

$a^{\log_a b} = b$

$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$

$(\log_a b)(\log_b n) = \log_a n$

$c \log_b n = \log_a n$ for the constant $c = \log_a b$

So:

$\log_b n = \Theta(\log_a n) = \Theta(\log n)$

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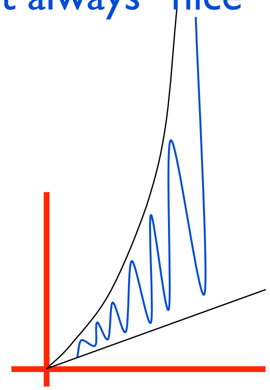
Big-Theta, etc. not always "nice"

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

f(n) ≠ $\Theta(n^a)$ for any *a*.

Fortunately, such nasty cases are rare



f(n log n) ≠ $\Theta(n^a)$ for any *a, either, but at least it's simpler.*

A Possible Misunderstanding?

We have looked at

- type of complexity analysis
  - worst-, best-, average-case
- types of function bounds
  - O, Ω, Θ

These two considerations are independent of each other

- one can do any type of function bound with any type of complexity analysis - measuring different things with same yardstick

Insertion Sort:
- $\Omega(n^2)$ (worst case)
- $O(n)$ (best case)

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Asymptotic Bounds for Some Common Functions

Polynomials:
$a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Logarithms:
$O(\log_a n) = O(\log_b n)$ for any constants $a,b > 0$

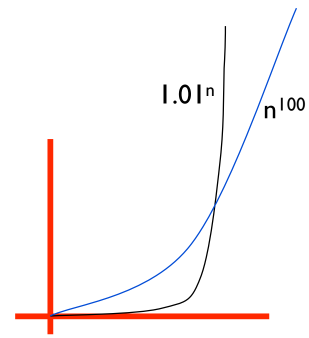
Logarithms:
For all $x > 0$, $\log n = O(n^x)$

log grows slower than every polynomial

Asymptotic Bounds for Some Common Functions

Exponentials.
For all $r > 1$
and all $d > 0$,
$n^d = O(r^n)$.

every exponential grows faster than every polynomial



Polynomial time

Running time is $O(n^d)$ for some constant d independent of the input size n.

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Geek-speak Faux Pas du Jour

"Any comparison-based sorting algorithm requires at least O(n log n) comparisons."

- Statement doesn't "type-check."
- Use Ω for lower bounds.

Domination

f(n) is o(g(n)) iff $\lim_{n\to\infty} f(n)/g(n)=0$

- that is g(n) dominates f(n)

If $a \le b$ then n^a is $O(n^b)$

If $a < b$ then n^a is $o(n^b)$

Note:
if f(n) is Θ (g(n)) then it cannot be o(g(n))

Working with little-o

$n^2 = o(n^3)$ [Use algebra]:

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$n^3 = o(e^n)$ [Use L'Hospital's rule 3 times]:

$$\lim_{n\to\infty} \frac{n^3}{e^n} = \lim_{n\to\infty} \frac{3n^2}{e^n} = \lim_{n\to\infty} \frac{6n}{e^n} = \lim_{n\to\infty} \frac{6}{e^n} = 0$$

Summary

Typical initial goal for algorithm analysis is to find a

- reasonably tight ← i.e., Θ if possible
- *asymptotic* ← i.e., O or Θ
- bound on ← usually upper bound
- *worst case* running time
- as a function of problem *size*

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - so you can concentrate on the good ones!