CSci 421

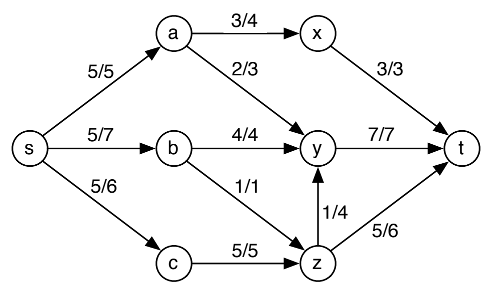
Introduction to Algorithms

Homework Assignment 6

Due: Wednesday, 8 Aug 2007

1. (a) Draw the residual graph corresponding to the flow in the figure. Is this flow maximum? If so, give a corresponding min cut; if not give an augmenting path.

   (b) Repeat part (a) assuming $c(b, y) = 6$ (instead of 4, as shown in the figure).



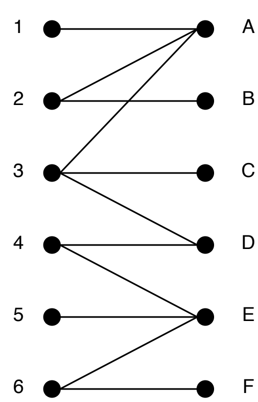
2. Let $G = (V, E)$ be a directed graph with edge capacities given by $c : E \rightarrow \Re^+$ (the non-negative reals), $f : V \times V \rightarrow \Re$ be a flow on G. (Note: the definition of a flow function I used on the lecture slides is different from the one used in the book. I think you'll find mine simpler to work with than the book's, but you may use either. But be sure to *say which definition you are using*.) Let G_f be the residual graph induced by f. Finally let $g : V \times V \rightarrow \Re$ be a flow function on G_f (not G), and define $h : V \times V \rightarrow \Re$ to be $f + g$, i.e. for all $u, v \in V$, $h(u, v) = f(u, v) + g(u, v)$.

   Prove or disprove: h is a flow on G.

   Note: I showed in lecture that this result is true in the special case where g sends a non-zero flow only along a single s–t path, so the question here is whether that generalizes to an arbitrary augmenting flow.

3. *Note: In this prob. and the next, I use the terms* alternating *and* augmenting *path slightly differently from the book. A path is* alternating *with respect to a given matching M if its edges alternate between M and $E - M$. An* augmenting *path is an alternating path whose end points are both unmatched.*

   Let G be the bipartite graph shown in the figure. Let M be the (non-maximum) matching $\{ \{3, A\}, \{4, E\}, \{6, F\}\}$.



   (a) List 3 alternating paths that are *not* augmenting paths.

   (b) List *all* augmenting paths in G (with respect to M).

   (c) What is the smallest maximal set of pairwise vertex-disjoint augmenting paths? What is the largest?

   Terminology: two paths are "vertex disjoint" if they don't have any vertices in common. A set of paths is "pairwise vertex disjoint" if no two of the paths in the set have any vertices in common. The full set of augmenting paths is not pairwise vertex disjoint, but various subsets of it are. Such a subset is "maximal" if it can't be enlarged without destroying the "pairwise vertex disjoint" property. So the question is asking you to give the largest (most paths) and smallest such sets.

(d) Let P be the augmenting path of length 3 containing $\{4, E\}$. Considering M and P to be sets of edges, $M \oplus P$ is their set theoretic *symmetric difference*: $(M \cup P) - (M \cap P)$. What set of edges is $M' = M \oplus P$? Is it a matching?

4. Let G be any bipartite graph, M any matching in G, and P any augmenting path (with respect to M).

   (a) Prove that $M' = M \oplus P$ is a matching.

   (b) Show $|M'| = |M| + 1$. How is the set of matched vertices in M' related to the set of matched vertices in M and the set of vertices (incident to edges) in P?

   (c) Give a counterexample to 4a if P is an arbitrary path, i.e. show that there is a graph G, matching M and path P such that $M \oplus P$ is not a matching. Is it true or false if P is an alternating path that is not an augmenting path? Prove or give a counterexample.

   (d) Suppose that there are *two* augmenting paths P and P' with respect to M, and that P and P' are vertex-disjoint. Show that P' also is an augmenting path with respect to the *augmented* matching $(M \oplus P)$, and similarly that P is augmenting with respect to $(M \oplus P')$. What could you say about a case where there were, say, 17 pairwise disjoint paths $P_1, \ldots, P_{17}$, all augmenting paths with respect to M? What, and how big, is $M \oplus P_1 \oplus \cdots \oplus P_{17}$?

5. (Optional Extra Credit:) The above ideas suggest an approach to bipartite matching that is a little more direct than using the flow reduction given in the book: Find a set of vertex-disjoint augmenting paths, update accordingly, and repeat. Flesh out this sketch, argue that it is correct and analyze the running time of your solution.

6. (Optional Extra Credit:) A graph is *d-regular* if every vertex has degree exactly d. Prove that a d-regular bipartite graph always has a perfect matching.