# CSE 421: Introduction to Algorithms

## NP-completeness

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## Computational Complexity

- Classify problems according to the amount of computational resources used by the best algorithms that solve them
- Recall:
  - worst-case running time of an algorithm
    - **max** # steps algorithm takes on any input of size **n**



## Decision problems

- Computational complexity usually analyzed using decision problems
  - answer is just 1 or 0 (yes or no).
- Why?
  - much simpler to deal with
  - *deciding* whether **G** has a path from s to t, is certainly no harder than *finding* a path from s to t in G, so a *lower* bound on deciding is also a lower bound on finding
  - Less important, but if you have a good decider, you can often use it to get a good finder.



## Polynomial time

- Define **P** (polynomial-time) to be
  - the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.



## Beyond P?

- There are many natural, practical problems for which we don't know any polynomial-time algorithms
- e.g. decisionTSP:
  - Given a weighted graph **G** and an integer **k**, does there exist a tour that visits all vertices in **G** having total weight at most **k**?



## Relative Complexity of Problems

- Want a notion that allows us to compare the complexity of problems
  - Want to be able to make statements of the form
    - "If we could solve problem **B** in polynomial time then we can solve problem **A** in polynomial time"
    - "Problem **B** is at least as hard as problem **A**"

## Polynomial Time Reduction

- $A \le_P B$ if there is an algorithm for $A$ using a 'black box' (subroutine) that solves $B$ that
  - Uses only a polynomial number of steps
  - Makes only a polynomial number of calls to a subroutine for $B$
- Thus, poly time algorithm for $B$ implies poly time algorithm for $A$
  - Not only is the number of calls polynomial but the size of the inputs on which the calls are made is polynomial!
- If you can prove there is no fast algorithm for $A$, then that proves there is no fast algorithm for $B$

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## A Special kind of Polynomial-Time Reduction

- We will always use a restricted form of polynomial-time reduction often called Karp or many-one reduction
- $A \le_P^1 B$ if and only if there is an algorithm for $A$ given a black box solving $B$ that on input $x$
  - Runs for polynomial time computing an input $f(x)$
  - Makes one call to the black box for $B$
  - Returns the answer that the black box gave

We say that the function $f$ is the reduction

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## Why the name reduction?

- Weird: it maps an easier problem into a harder one
- Same sense as saying Maxwell reduced the problem of analyzing electricity & magnetism to solving partial differential equations
  - solving partial differential equations in general is a much harder problem than solving E&M problems

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## A geek joke

- An engineer
  - is placed in a kitchen with an empty kettle on the table and told to boil water; she fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - she is next confronted with a kettle full of water sitting on the counter and told to boil water; she puts it on the stove, turns on the gas and boils water.
- A mathematician
  - is placed in a kitchen with an empty kettle on the table and told to boil water; he fills the kettle with water, puts it on the stove, turns on the gas and boils water.
  - he is next confronted with a kettle full of water sitting on the counter and told to boil water: he empties the kettle in the sink, places the empty kettle on the table and says, "I've reduced this to an already solved problem".

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## Reductions from a Special Case to a General Case

- Show: Vertex-Cover $\le_P$ Set-Cover
- Vertex-Cover:
  - Given an undirected graph $G=(V,E)$ and an integer $k$ is there a subset $W$ of $V$ of size at most $k$ such that every edge of $G$ has at least one endpoint in $W$? (i.e. $W$ covers all edges of $G$).
- Set-Cover:
  - Given a set $U$ of $n$ elements, a collection $S_1,\dots,S_m$ of subsets of $U$, and an integer $k$, does there exist a collection of at most $k$ sets whose union is equal to $U$?

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## The Simple Reduction

- Transformation $f$ maps $(G=(V,E),k)$ to $(U,S_1,\dots,S_m,k')$
  - $U \leftarrow E$
  - For each vertex $v \in V$ create a set $S_v$ containing all edges that touch $v$
  - $k' \leftarrow k$
- Reduction $f$ is clearly polynomial-time to compute
- We need to prove that the resulting algorithm gives the right answer!

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## Proof of Correctness

- Two directions:
  - If the answer to Vertex-Cover on (G,k) is YES then the answer for Set-Cover on f(G,k) is YES
    - If a set W of k vertices covers all edges then the collection $\{S_v \mid v \in W\}$ of k sets covers all of U
  - If the answer to Set-Cover on f(G,k) is YES then the answer for Vertex-Cover on (G,k) is YES
    - If a subcollection $S_{v_1},\ldots,S_{v_k}$ covers all of U then the set $\{v_1,\ldots,v_k\}$ is a vertex cover in G.



## Reductions by Simple Equivalence

- Show: Independent-Set $\le_P$ Clique
- Independent-Set:
  - Given a graph G=(V,E) and an integer k, is there a subset U of V with $|U| \ge k$ such that no two vertices in U are joined by an edge.
- Clique:
  - Given a graph G=(V,E) and an integer k, is there a subset U of V with $|U| \ge k$ such that every pair of vertices in U is joined by an edge.



## Independent-Set $\le_P$ Clique

- Given (G,k) as input to Independent-Set where G=(V,E)
- Transform to (G',k) where G'=(V,E') has the same vertices as G but E' consists of **precisely** those edges that are not edges of G
- U is an independent set in G
- $\Leftrightarrow$ U is a clique in G'



## More Reductions

- Show: Independent Set $\le_P$ Vertex-Cover
- Vertex-Cover:
  - Given an undirected graph G=(V,E) and an integer k is there a subset W of V of size at most k such that every edge of G has at least one endpoint in W? (i.e. W covers all edges of G).
- Independent-Set:
  - Given a graph G=(V,E) and an integer k, is there a subset U of V with $|U| \ge k$ such that no two vertices in U are joined by an edge.



## Reduction Idea

- Claim: In a graph G=(V,E), S is an independent set iff V-S is a vertex cover
- Proof:
  - $\Rightarrow$ Let S be an independent set in G
    - Then S contains at most one endpoint of each edge of G
    - At least one endpoint must be in V-S
    - V-S is a vertex cover
  - $\Leftarrow$Let W=V-S be a vertex cover of G
    - Then S does not contain both endpoints of any edge (else W would miss that edge)
    - S is an independent set



## Reduction

- Map (G,k) to (G,n-k)
  - Previous lemma proves correctness
- Clearly polynomial time
- We also get that
  - Vertex-Cover $\le_P$ Independent Set

## Satisfiability

- Boolean variables $x_1,\ldots,x_n$
  - taking values in {0,1}. 0=false, 1=true
- Literals
  - $x_i$ or $\neg x_i$ for $i=1,\ldots,n$
- Clause
  - a logical OR of one or more literals
  - e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$
- CNF formula
  - a logical AND of a bunch of clauses
- **k**-CNF formula
  - All clauses have exactly **k** variables



## Satisfiability

- CNF formula example
  $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
- If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - the one above is, the following isn't
  - $x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$
- 3-SAT: Given a CNF formula F with 3 variables per clause, is it satisfiable?



## Common property of these problems

- There is a special piece of information, a **short certificate** or proof, that allows you to **efficiently verify** (in polynomial-time) that the YES answer is correct. This certificate might be very hard to find
- e.g.
  - **DecisionTSP**: the tour itself,
  - **Independent-Set**, **Clique**: the set **U**
  - **3-SAT**: an assignment that makes **F** true.



## The complexity class NP

**NP** consists of all decision problems where

- You can **verify** the **YES** answers efficiently (in polynomial time) given a short (polynomial-size) **certificate**

And

- **No certificate** can fool your polynomial time verifier into saying **YES** for a **NO** instance



## More Precise Definition of NP

- A decision problem is in NP iff there is a polynomial time procedure **verify**(.,.), and an integer k such that
  - for every input **x** to the problem that is a **YES** instance there is a certificate **t** with $|t| \le |x|^k$ such that **verify**(**x**,**t**) = **YES**

  and
  - for every input **x** to the problem that is a **NO** instance there does **not** exist a certificate **t** with $|t| \le |x|^k$ such that **verify**(**x**,**t**) = **YES**



## Example: CLIQUE is in NP

procedure **verify**(**x**,**t**)
  if
    **x** is a well-formed representation of a graph **G** = (**V**, **E**) and an integer **k**,
  and
    **t** is a well-formed representation of a vertex subset **U** of **V** of size **k**,
  and
    **U** is a clique in **G**,
  then output "**YES**"
  else output "**I'm unconvinced**"

## Is it correct?

For every **x** = (**G**,**k**) such that **G** contains a **k**-clique, there is a certificate **t** that will cause **verify**(**x**,**t**) to say **YES**,

- **t** = a list of the vertices in such a **k**-clique

And no certificate can fool **verify**(**x**,·) into saying **YES** if either

- **x** isn't well-formed (the uninteresting case)
- **x** = (**G**,**k**) but **G** does not have any cliques of size **k** (the interesting case)



## Keys to showing that a problem is in NP

- What's the output? (must be **YES**/**NO**)
- What must the input look like?
- Which inputs need a **YES** answer?
  - Call such inputs **YES** inputs/**YES** instances
- For every given **YES** input, is there a certificate that would help?
  - OK if some inputs need no certificate
- For any given **NO** input, is there a fake certificate that would trick you?



## Solving NP problems without hints

- The only **obvious algorithm** for most of these problems is **brute force**:
  - try all possible certificates and check each one to see if it works.
  - *Exponential* time:
    - $2^n$ truth assignments for **n** variables
    - **n!** possible TSP tours of **n** vertices
    - $\binom{n}{k}$ possible **k** element subsets of **n** vertices
    - etc.



## What We Know

- Nobody knows if all problems in **NP** can be done in polynomial time, i.e. does **P**=**NP**?
  - one of the most important open questions in all of science.
  - huge practical implications
- Every problem in **P** is in **NP**
  - one doesn't even need a certificate for problems in **P** so just ignore any hint you are given
- Every problem in **NP** is in exponential time



## NP-hardness & NP-completeness

- Some problems in **NP** seem hard
  - people have looked for efficient algorithms for them for hundreds of years without success
- However
  - nobody knows how to **prove** that they are really hard to solve, i.e. **P≠ NP**



## Problems in NP that seem hard

- Some Examples in **NP**
  - 3-SAT
  - Independent-Set
  - Clique
  - Vertex Cover
- All hard to solve; certificates seem to help on all
- Fast solution to *any* gives fast solution to *all!*

## NP-hardness & NP-completeness

- Alternative approach to proving problems not in **P**
  - show that they are at least as hard as any problem in **NP**
- Rough definition:
  - A problem is NP-hard iff it is at least as hard as any problem in **NP**
  - A problem is NP-complete iff it is both
    - **NP**-hard
    - in **NP**

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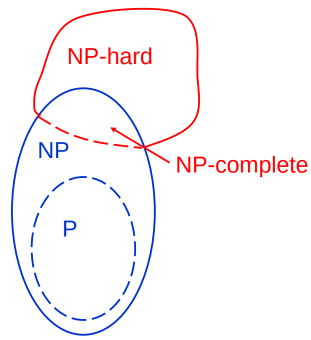
## P and NP

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## NP-hardness & NP-completeness

- Definition: A problem **B** is NP-hard iff every problem **A**$\in$**NP** satisfies **A** $\le_P$**B**
- Definition: A problem **B** is NP-complete iff **A** is NP-hard and **A** $\in$**NP**
- Even though we seem to have lots of hard problems in **NP** it is not obvious that such super-hard problems even exist!

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## Cook-Levin Theorem

- Theorem (Cook 1971, Levin 1973):
  **3-SAT** is **NP**-complete
- Recall
  - CNF formula
    - $(x_1 \vee \neg x_3 \vee x_4) \wedge (x_2 \vee \neg x_4 \vee x_3) \wedge (x_2 \vee \neg x_1 \vee x_3)$
  - If there is some assignment of **0**'s and **1**'s to the variables that makes it true then we say the formula is satisfiable
  - **3-SAT:** Given a 3-CNF formula F, is it satisfiable?

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## Implications of Cook-Levin Theorem?

- There is at least one interesting super-hard problem in **NP**
- Is that such a big deal?
- YES!
  - There are lots of other problems that can be solved if we had a polynomial-time algorithm for **3-SAT**
  - Many of these problems are exactly as hard as **3-SAT**

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## A useful property of polynomial-time reductions

- Theorem: If **A** $\le_P$**B** and **B** $\le_P$**C** then **A** $\le_P$**C**
- Proof idea: (Using $\le_P^1$ )
  - Compose the reduction **f** from **A** to **B** with the reduction **g** from **B** to **C** to get a new reduction **h**(**x**)=**g**(**f**(**x**)) from **A** to **C**.
  - The general case is similar and uses the fact that the composition of two polynomials is also a polynomial

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## Cook-Levin Theorem & Implications

- Theorem (Cook 1971, Levin 1973):
  **3-SAT** is **NP**-complete
  For proof see CSE 431
- Corollary: **B** is **NP**-hard $\Leftrightarrow$ **3-SAT** $\le_P$ **B**
  - (or **A** $\le_P$ **B** for any **NP**-complete problem **A**)
- Proof:
  - If **B** is **NP**-hard then every problem in **NP** polynomial-time reduces to **B**, in particular **3-SAT** does since it is in **NP**
  - For any problem **A** in **NP**, **A** $\le_P$ **3-SAT** and so if **3-SAT** $\le_P$ **B** we have **A** $\le_P$ **B**.
    - therefore **A** is **NP**-hard if **3-SAT** $\le_P$ **B**



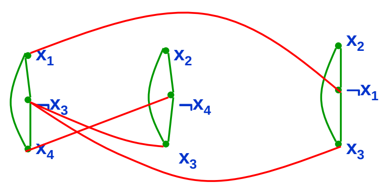
## Another NP-complete problem: 3-SAT $\le_P$ Independent-Set

- A Tricky Reduction:
  - mapping CNF formula **F** to a pair **<G,k>**
  - Let **m** be the number of clauses of **F**
  - Create a vertex in **G** for each literal in **F**
  - Join two vertices **u**, **v** in **G** by an edge iff
    - **u** and **v** correspond to literals in the same clause of **F**, (green edges) or
    - **u** and **v** correspond to literals **x** and **¬x** (or vice versa) for some variable **x**. (red edges).
  - Set **k=m**
  - Clearly polynomial-time



## 3-SAT $\le_P$ Independent-Set

**F:** $(x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_4 \lor x_3) \land (x_2 \lor \neg x_1 \lor x_3)$






## 3-SAT $\le_P$ Independent-Set

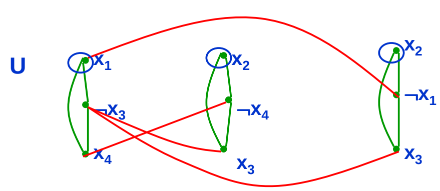
- Correctness:
  - If **F** is **satisfiable** then there is some assignment that satisfies at least one literal in each clause.
  - Consider the set **U** in **G** corresponding to the **first satisfied literal in each clause**.
    - **|U|=m**
    - Since **U** has only one vertex per clause, no two vertices in **U** are joined by green edges
    - Since a truth assignment never satisfies both **x** and **¬x**, **U** doesn't contain vertices labeled both **x** and **¬x** and so no vertices in **U** are joined by red edges
    - Therefore **G** has an independent set, **U**, of size at least **m**
  - Therefore **(G,m)** is a **YES** for independent set.



## 3-SAT $\le_P$ Independent-Set

1 0 1 1 0 1 1 0 1

**F:** $(x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_4 \lor x_3) \land (x_2 \lor \neg x_1 \lor x_3)$




Given assignment $x_1=x_2=x_3=x_4=1$,
**U** is as circled



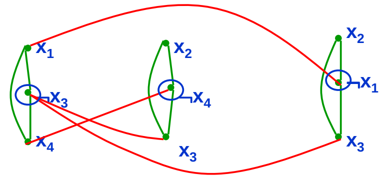
## 3-SAT $\le_P$ Independent-Set

- Correctness continued:
  - If **(G,m)** is a **YES** for Independent-Set then there is a set **U** of **m** vertices in **G** containing no edge.
    - Therefore **U** has precisely one vertex per clause because of the green edges in **G**.
    - Because of the red edges in **G**, **U** does not contain vertices labeled both **x** and **¬x**
    - Build a truth assignment **A** that makes all literals labeling vertices in **U** true and for any variable not labeling a vertex in **U**, assigns its truth value arbitrarily.
    - By construction, **A** satisfies **F**
  - Therefore **F** is a **YES** for 3-SAT.

## 3-SAT $\leq_P$ Independent-Set

0 1 0 ? 1 0 ? 1 0

F: $(x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_4 \lor x_3) \land (x_2 \lor \neg x_1 \lor x_3)$




Given **U**, satisfying assignment is $x_1=x_3=x_4=0$, $x_2=0$ or $1$

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## Independent-Set is NP-complete

- We just showed that **Independent-Set** is NP-hard and we already knew **Independent-Set** is in **NP**.
- Corollary: **Clique** is **NP**-complete
  - We showed already that **Independent-Set** $\leq_P$ **Clique** and **Clique** is in **NP**.

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## Problems we already know are NP-complete

- 3-SAT
- Independent-Set
- Clique
- Vertex-Cover
- There are 1000's of practical problems that are NP-complete, e.g. scheduling, optimal VLSI layout etc.

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## Steps to Proving Problem B is NP-complete

- Show **B** is **NP**-hard:
  - State:"Reduction is from **NP**-hard Problem **B**"
  - Show what the map **f** is
  - Argue that **f** is polynomial time
  - Argue correctness: two directions Yes for **A** implies Yes for **B** and vice versa.
- Show **B** is in **NP**
  - State what hint is and why it works
  - Argue that it is polynomial-time to check.

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## Some other NP-complete examples you should know

- **Hamiltonian-Cycle** Given a directed graph G is there a cycle in G that visits each vertex in G exactly once?
- **Hamiltonian-Path** Given a directed graph G is there a path in G that visits each vertex in G exactly once?
  - Both are also **NP**-complete when G is an undirected graph
- Note that deciding the similar questions for **Eulerian-Cycle** and **Eulerian-Path** (which require that each edge be visited exactly once rather than each vertex) can be done in polynomial time.
  - How?

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## Travelling-Salesman Problem (TSP)

- Given a set of **n** cities $v_1,\ldots,v_n$ and distances between each pair of cities $d(v_i,v_j)$ what is the shortest tour that visits all the cities?
  - Not a decision problem
- **DecisionTSP**:
  - Given a set of distances given by **d** for each pair of cities in $v_1,\ldots,v_n$ and an integer **D**, does there exist a tour that visits all cities having total weight at most **D**?

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## Hamiltonian-Cycle$\leq_P$DecisionTSP

- Define the reduction
  - Vertices **V** of **G=(V,E)** become cities
  - Set $\mathbf{d(v_i,v_j)}$ to **1** if $\mathbf{(v_i,v_j)\in E}$
    **2** if not
  - Set **D=|V|**
- Claim: There is a Hamiltonian cycle in **G** iff there is a tour of length **|V|**



## Graph Colorability

- Defn: Given a graph G=(V,E), and an integer k, a k-coloring of G is
  - an assignment of up to k different colors to the vertices of G so that the endpoints of each edge have different colors.
- 3-Color: Given a graph G=(V,E), does G have a 3-coloring?
- Claim: 3-Color is NP-complete
- Proof: 3-Color is in NP:
  - Hint is an assignment of red,green,blue to the vertices of G
  - Easy to check that each edge is colored correctly

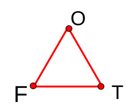


## 3-SAT $\leq_P$3-Color

- Reduction:
  - We want to map a 3-CNF formula **F** to a graph **G** so that
    - **G** is 3-colorable iff **F** is satisfiable



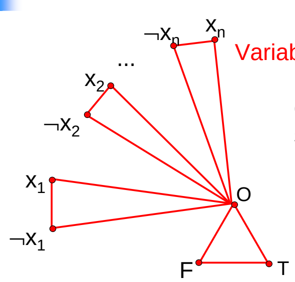
## 3-SAT $\leq_P$3-Color




Base Triangle



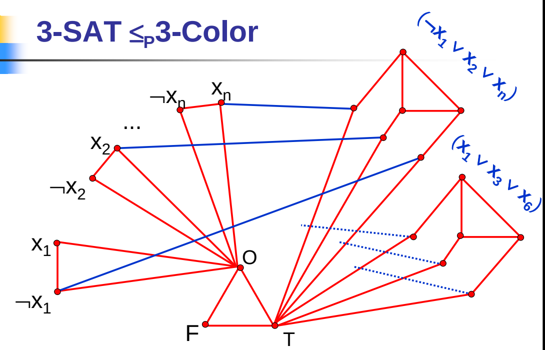
## 3-SAT $\leq_P$3-Color




Variable Part:
in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)

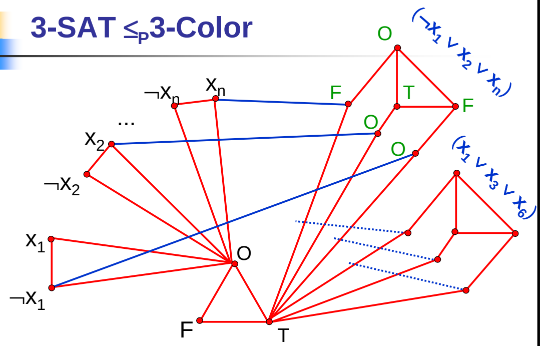


## 3-SAT $\leq_P$3-Color




Clause Part:
Add one 6 vertex gadget per clause connecting its 'outer vertices' to the literals in the clause

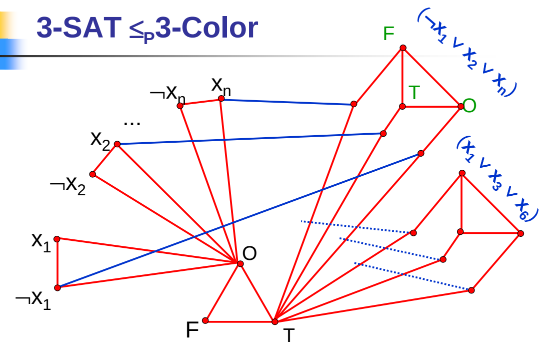
## 3-SAT $\leq_P$ 3-Color




Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph



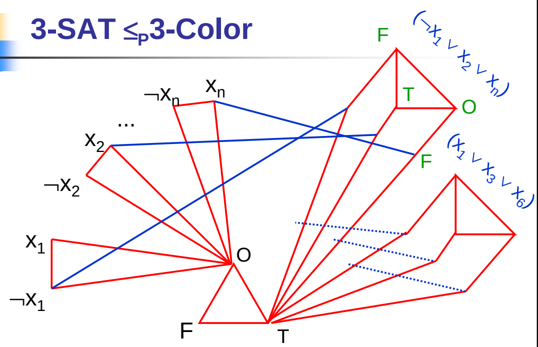
## 3-SAT $\leq_P$ 3-Color




Any 3-coloring of the graph colors each gadget triangle using each color



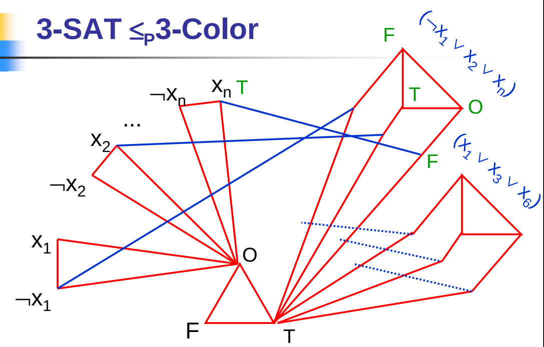
## 3-SAT $\leq_P$ 3-Color




Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget



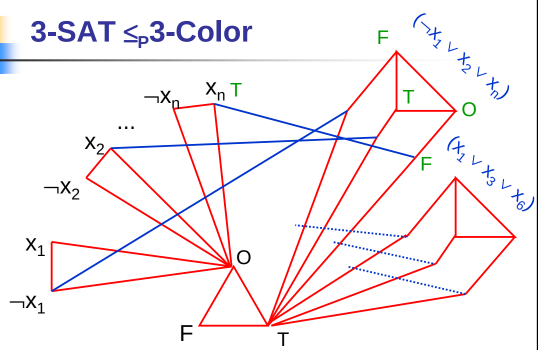
## 3-SAT $\leq_P$ 3-Color




Any 3-coloring of the graph has T at the other end of the blue edge connected to the F



## 3-SAT $\leq_P$ 3-Color




Any 3-coloring of the graph yields a satisfying assignment to the formula



## More NP-completeness

- Subset-Sum problem
  - Given **n** integers $\mathbf{w_1,\ldots,w_n}$ and integer **W**
  - Is there a subset of the **n** input integers that adds up to exactly **W**?

- **O**(**nW**) solution from dynamic programming but if **W** and each $\mathbf{w_i}$ can be **n** bits long then this is exponential time

## 3-SAT $\leq_P$ Subset-Sum

- Given a 3-CNF formula with **m** clauses and **n** variables
- Will create **2m+2n** numbers that are **m+n** digits long
  - Two numbers for each variable $x_i$
    - $t_i$ and $f_i$ (corresponding to $x_i$ being true or $x_i$ being false)
  - Two extra numbers for each clause
    - $u_j$ and $v_j$ (filler variables to handle number of false literals in clause $C_j$)



## 3-SAT $\leq_P$ Subset-Sum

| | i | j |
|---|---|---|
| | 1 2 3 4 … n | 1 2 3 4 … m |
| $t_1$ | 1 0 0 0 … 0 | 0 0 1 0 … 1 |
| $f_1$ | 1 0 0 0 … 0 | 1 0 0 1 … 0 |
| $t_2$ | 0 1 0 0 … 0 | 0 1 0 0 … 1 |
| $f_2$ | 0 1 0 0 … 0 | 0 0 1 1 … 0 |
| | … | …. |
| $u_1=v_1$ | 0 0 0 0 … 0 | 1 0 0 0 … 0 |
| $u_2=v_2$ | 0 0 0 0 … 0 | 0 1 0 0 … 0 |
| | … | …. |
| W | 1 1 1 1 … 1 | 3 3 3 3 … 3 |

$C_4=(x_1 \vee \neg x_2 \vee x_5)$



## P vs NP

- Theory
  - **P = NP?**
  - Open Problem!
  - Bet against it
- Practice
  - Many interesting, useful, natural, well-studied problems known to be **NP**-complete
  - With rare exceptions, no one routinely succeeds in finding exact solutions to large, arbitrary instances



## Is NP as bad as it gets?

- NO! **NP**-complete problems are frequently encountered, but there's worse:
  - Some problems provably require exponential time.
    - Ex: Does **M** halt on input **x** in $2^{|x|}$ steps?
  - Some require $2^n$, $2^{2^n}$, $2^{2^{2^n}}$, … steps
  - And some are just plain uncomputable